Computational Dimension of Topological Spaces

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Abstract. When a topological space X can be embedded into the space $\Sigma^{\omega}_{\perp,n}$ of $n\perp$-sequences of Σ, then we can define the corresponding computational notion over X because a machine with $n+1$ heads on each tape can input/output sequences in $\Sigma^{\omega}_{\perp,n}$. This means that the least number n such that X can be topologically embedded into $\Sigma^{\omega}_{\perp,n}$ serves as a degree of complexity of the space. We prove that this number, which we call the computational dimension of the space, is equal to the topological dimension for separable metric spaces. First, we show that the weak inductive dimension of $\Sigma^{\omega}_{\perp,n}$ is n, and thus the computational dimension is at least as large as the weak inductive dimension for all spaces. Then, we show that the Nöbeling's universal n-dimensional space can be embedded into $\Sigma^{\omega}_{\perp,n}$ and thus the computational dimension is at most as large as the weak inductive dimension for separable metric spaces. As a corollary, the 2-dimensional Euclidean space $\mathbb{R}^2$ can be embedded in $\{0,1\}^{\omega}_{\perp,2}$ but not in $\Sigma^{\omega}_{\perp,1}$ for any character set Σ, and infinite dimensional spaces like the set of closed/open/compact subsets of $\mathbb{R}^m$ and the set of continuous functions from $\mathbb{R}^l$ to $\mathbb{R}^m$ can be embedded in $\Sigma^{\omega}_{\perp}$ but not in $\Sigma^{\omega}_{\perp,n}$ for any n.

1 Introduction

In order to perform computation over the set of reals, we need to represent them as (infinite) sequences of characters. However, it is known that there is no one-to-one representation, or equivalently, no embedding of real numbers into Σ^{ω} which induces reasonable notion of computation over reals, and thus redundant representations are commonly used [Wei00][BH00]. In a previous paper [Tsu01b], the author used, instead of Σ^{ω}, the set $\Sigma^{\omega}_{\perp,1}$ of $1\perp$-sequences of Σ. Here, an $n\perp$-sequence of Σ is an infinite sequence of Σ in which at most n cells are allowed to be left undefined (denoted by $\perp$). He proposed a machine, called an IM2-machine, which input/output $1\perp$-sequences with two heads on each input/output tape, composed a topological embedding of the set of real numbers into $\Sigma^{\omega}_{\perp,1}$, and thus induced a notion of computation over reals, which is shown to be equivalent to the standard notion of computation over reals.

It is easy to extend this input/output mechanism of an IM2-machine to the set $\Sigma^{\omega}_{\perp,n}$ of $n\perp$-sequences of Σ by putting $n+1$ heads on each input/output tape, and therefore, we can obtain a computational notion over a topological space X when X can be embedded into $\Sigma^{\omega}_{\perp,n}$. Thus, we define the least number n such that a space X can be embedded into $\Sigma^{\omega}_{\perp,n}$ as the computational dimension of X. The computational dimension is a degree of computational complexity of the space in that it is equal to the number of extra heads required to define computation over the space by an IM2-machine.

The main theorem of this paper is that the computational dimension and the usual topological dimension coincide for separable metric spaces. First, we show that the weak inductive dimension of $\Sigma^{\omega}_{\perp,n}$ is n, and therefore the computational dimension is at least as large as the weak inductive dimension for all spaces. Then, we show that the Nöbeling's universal n-dimensional space can be embedded into $\Sigma^{\omega}_{\perp,n}$ and thus the computational dimension is at most as large as the weak inductive dimension for separable metric spaces.

From this theorem, the 2-dimensional Euclidean space can be embedded into $\Sigma^{\omega}_{\perp,2}$, but not in $\Sigma^{\omega}_{\perp,1}$ for any character set Σ, and infinite dimensional spaces like the spaces of closed/open/compact subsets of $\mathbb{R}^l$ and the space of continuous functions from $\mathbb{R}^l$ to $\mathbb{R}^m$ cannot be embedded in $\Sigma^{\omega}_{\perp,n}$ for any n.

In the next section, we review the notion of Gray code embedding and IM2-machines following [Tsu01b]. Then, we introduce the computational dimension in Section 3. In Section 4, we show that the weak inductive dimension of $\Sigma^{\omega}_{\perp,n}$ is n. In Section 5, we study the embedding of the 2-dimensional Euclidean space to $\Sigma^{\omega}_{\perp,2}$, and in Section 6, we prove that the computational dimension and the weak inductive dimension coincide for separable metric spaces. In Section 7, we study the embeddings of infinite dimensional spaces in $\Sigma^{\omega}_{\perp}$.

2 Gray Code Embedding and Computability by IM2-machines

We write $\Sigma^{\omega}_{\perp}$ for the set of infinite sequences of Σ in which undefined cells ($\perp$) are allowed to exist. That is, $\Sigma^{\omega}_{\perp}$ is the set of infinite sequences of $\Sigma \cup \{\perp\}$. We write $\Sigma^{\omega}_{\perp,n}$ for the set of infinite sequences of Σ in which at most n undefined cells are allowed to exist, Gray code embedding G (Definition 1 below) is an embedding of the unit open interval $\mathcal{I} = (0, 1)$ (or the whole real line $\mathbb{R}$) to the set $\{0, 1\}^{\omega}_{\perp,1}$. It is based on the Gray code expansion, which is another expansion of real numbers.

Figure 1 shows the usual binary expansion and the Gray code expansion of the unit open interval. Here, a horizontal line means that the corresponding bit has value 1 on the line and value 0 otherwise. In this way, Gray code expansion of (0,1) is composed from that of $(0, 1/2)$ by taking the mirroring image on (1/2, 1) with the first bit on. As is the case for the usual binary expansion, we have two expansions to dyadic numbers. Here, a dyadic number is a rational number of the form $m \times 2^{-n}$ for integers m and n. For example, we have two Gray code expansions 111000... and 101000... for $3/4$, corresponding to the two binary expansions 11000... and 10111.... However, the two expansions are different only at one digit (in this case the second). This means that the second digit does not contribute to the fact that this number is $3/4$. Therefore, by defining the value

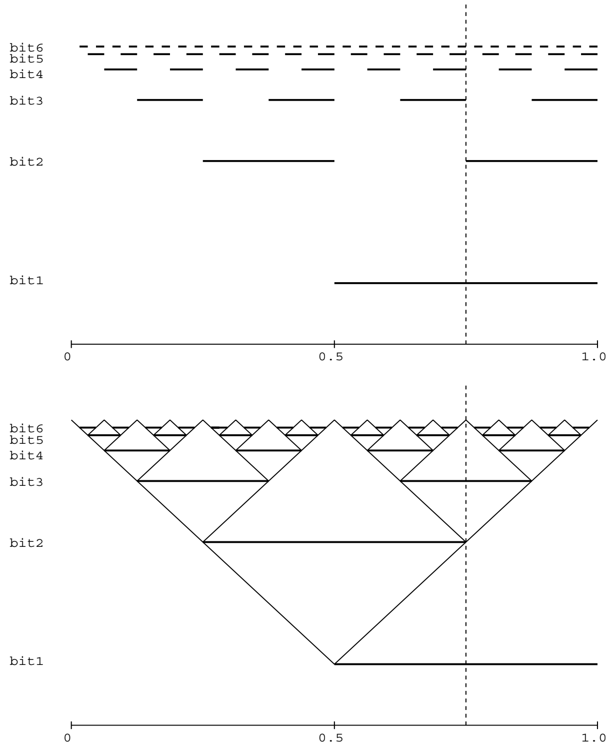

Fig. 1. Binary Expansion and Gray code Expansion of real numbers

of such a digit as $\perp$, we define the Gray code embedding G of $\mathcal{I}$ to $\{0,1\}^{\omega}_{\perp,1}$ as follows.

Definition 1. The Gray code embedding G is a function from $\mathcal{I}$ to $\{0,1\}^{\omega}_{\perp,1}$ defined coinductively as follows. Let $G(x) = a_0 a_1 \ldots$. The first character a_0 is $1, \perp$, or 0, according as x is bigger than, equal to, or less than $1/2$. The rest $a_1 a_2 \ldots$ is defined to be $G(f(x))$ where

$$f(x) = \begin{cases} 2x & (x \leq 1/2) \\ 2(1-x) & (x > 1/2) \end{cases} .$$

G can be extended to a function from $(-1,1)$ to $\{0,1\}^{\omega}_{\perp,1}$ by adding the sign bit as the first bit, and to the whole real line by composing it with some computable embedding of $\mathbb{R}$ into $(-1,1)$ such as the function $f(x) = 2{*}arctan(x)/\pi$.

Note that G comes to be an injective function to $\{0,1\}^{\omega}_{\perp,1}$. Moreover, we can show that G maps $\mathcal{I}$ homeomorphically into $\{0,1\}^{\omega}_{\perp,1}$, and therefore, G is actually a topological embedding. Here, the topology on $\Sigma^{\omega}_{\perp,1}$ is the subspace topology of the Scott topology on the cpo $\Sigma^{\omega}_{\perp}$ (see Section 4).

Now, we define a machine which inputs/outputs sequences in $\Sigma^{\omega}_{\perp,1}$. First, we consider outputs. We consider that our machine calculates a real number x ($0 < x < 1$) by producing shrinking intervals $(a_1, b_1), (a_2, b_2), \ldots$ infinitely so that their intersection is $\{x\}$, and output $G(x)$ on a tape based on this approximation information. We consider that the tape is filled with $\perp$ at the beginning. When we know that $x < 1/2$ or $x > 1/2$, we can write 0 or 1 as the first digit, respectively. However, when $x = 1/2$, neither will happen and therefore we cannot fill the first cell. However, if we know that $1/4 < x < 3/4$, we can fill the second cell with 1. After that, we can fill the first cell with 0 or 1 when we know that $1/4 < x < 1/2$ or $1/2 < x < 3/4$, respectively, and the third cell with 0 when we know that $3/8 < x < 5/8$. In particular, when $x = 1/2$, the first cell is unfilled eternally, and the sequence $1000\ldots$ is output from the second cell. This kind of output can be expressed naturally if we consider two heads $\mathrm{H}_1(O)$ and $\mathrm{H}_2(O)$ on the output tape O. They are placed at the first and the second cell at the beginning, and move automatically when a character is output: after an output from $\mathrm{H}_1(O)$, $\mathrm{H}_1(O)$ is moved to the place of $\mathrm{H}_2(O)$ and $\mathrm{H}_2(O)$ is moved to the next cell, and after an output from $\mathrm{H}_2(O)$, $\mathrm{H}_2(O)$ is moved to the next cell. In this way, $\mathrm{H}_1(O)$ and $\mathrm{H}_2(O)$ are always located at the first and the second unfilled cells, respectively.

Next, we consider an input of a sequence in $\Sigma^{\omega}_{\perp,1}$. When we have only one head on an input tape, the machine stucks when the head comes to the $\perp$ cell because it will never be filled. Therefore, our machine has two heads $\mathrm{H}_1(I_i)$ and $\mathrm{H}_2(I_i)$ on each input tape I_i, which move the same way as the heads of an output tape, and the machine proceeds depending on an input from one of them. This means that when both cells under $\mathrm{H}_1(I_i)$ and $\mathrm{H}_2(I_i)$ are filled, we have two applicable rules. Thus, our machine has many computational paths to an input and all the paths should produce valid results. This property is called indeterminism. In this way, we define an IM2-machine (Indeterministic Multihead Type-2 machine), which inputs/outputs sequences in $\Sigma^{\omega}_{\perp,1}$. See [Tsu01b] for the detailed definition of the machine.

This notion of an IM2-machine can easily be generalized to a machine with $n+1$ heads on each input/output tape. The heads $H_1(T), H_2(T), \ldots, H_{n+1}(T)$ of a tape T move as follows: when $H_i(T)$ $(1 \leq i \leq n+1)$ is used, $H_j(T)$ $(i \leq j \leq n)$ move to the position of $H_{j+1}(T)$ and H_{n+1} moves to the next position. Note that an $n+1$-head machine can manipulate sequences in $\Sigma^{\omega}_{\perp,n}$. Therefore, we define an IM2-machine of type $(\Sigma^{\omega}_{\perp,n_1}, \ldots, \Sigma^{\omega}_{\perp,n_k}, \Sigma^{\omega}_{\perp,n_0})$ as a machine which has k input tapes with $n_i + 1$ heads $(i = 1, \ldots, k)$ and one output tape with $n_0 + 1$ heads.

Since an IM2-machine has indeterministic behavior, it defines a partial multi-valued function $F :\subseteq \Sigma^{\omega}_{\perp,n_1} \times \ldots \times \Sigma^{\omega}_{\perp,n_k} \rightrightarrows \Sigma^{\omega}_{\perp,n_0}$, which is a subset of $\Sigma^{\omega}_{\perp,n_1} \times \ldots \times \Sigma^{\omega}_{\perp,n_k} \times \Sigma^{\omega}_{\perp,n_0}$ considered as a partial function from $\Sigma^{\omega}_{\perp,n_1} \times \ldots \times \Sigma^{\omega}_{\perp,n_k}$ to (the power set of $\Sigma^{\omega}_{\perp,n_0}$) $-\ \emptyset$.

Definition 2. An IM2-machine M with k inputs realizes a partial multi-valued function $F :\subseteq \Sigma^{\omega}_{\perp,n_1} \times \ldots \times \Sigma^{\omega}_{\perp,n_k} \rightrightarrows \Sigma^{\omega}_{\perp,n_0}$ if all the computational paths M have with the input tapes filled with $(p_1, \ldots, p_k) \in dom(F)$ produce infinite outputs, and the set of outputs forms a subset of $F(p_1, \ldots, p_k)$. We say that F is IM2-computable when it is realized by an IM2-machine.

Definition 3. Let H_i be embeddings of X_i to $\Sigma^\omega_{\perp,n_i}$ $(i = 0, 1, \ldots, k)$. A multi-valued function $F :\subseteq X_1 \times \ldots \times X_k \rightrightarrows X_0$ is realized by an IM2-machine M if $H_0 \circ F \circ (H_1^{-1}, \ldots, H_k^{-1})$ is realized by M. In this case, we say that F is $(H_1, H_2, \ldots, H_k, H_0)_\perp$-computable.

When H_i are the Gray code embedding G of $\mathbb{R}$ in $\{0,1\}^\omega_{\perp,1}$, we say that $F :\subseteq \mathbb{R}^k \to \mathbb{R}$ is Gray code computable. In this definition, we add the suffix $_\perp$ to the type of the computability to distinguish it from the usual representation-based computability notion by a Type-2 machine [Wei00].

Definition 4. A partial function is $(H_1, H_2, \ldots, H_k, H_0)_\perp$-computable if it is $(H_1, H_2, \ldots, H_k, H_0)_\perp$-computable as a multi-valued function.

By generalizing the proofs in [Tsu01b], one has the followings:

Theorem 1. *If a partial function is $(H_1, H_2, \ldots, H_k, H_0)_\perp$-computable, then it is continuous.*

Theorem 2. *A multi-valued function $F :\subseteq \mathcal{I}^k \to \mathcal{I}$ is Gray-code-computable iff it is $((\rho)^k, \rho)$-computable in the sense of [Wei00]. Here, ρ is the signed digit representation or some equivalent ones.*

In [Tsu01b], the author gave some basic algorithms by IM2-machines like addition with respect to the Gray code embedding. In [Tsu01a], he showed that the behavior of an IM2-machine can be naturally expressed in a parallel logic programming language GHC.

3 The Computational Dimension of a Topological Space

Now, we study whether topological spaces other than $\mathbb{R}$ have similar embeddings in $\Sigma^\omega_{\perp,n}$. As we have shown, if a space X can be embedded into $\Sigma^\omega_{\perp,n}$, then we can define computational notion on X based on $n+1$-head IM2-machines. Thus, the least number n such that X can be embedded into $\Sigma^\omega_{\perp,n}$ for some Σ has a meaning as a computational complexity of the space.

Definition 5. The computational dimension of a space X is the least number n such that X can be embedded into $\Sigma^\omega_{\perp,n}$. If X can be embedded into $\Sigma^\omega_\perp$ and X cannot be embedded into $\Sigma^\omega_{\perp,n}$ for every n, then we define the computational dimension of X as ∞.

We show that this computational dimension coincides with the usual topological dimension for separable metric spaces. There are several definitions of the topological dimension of a space: the covering dimension, the strong inductive dimension, and the weak inductive dimension [HW48,Nag65]. It is known that these three dimensions are equivalent and have many good properties for a separable metric space. However, we need to develop dimension theory also to T_0 spaces and only the weak inductive dimension has the properties we need for such a general space.

We write $B_P(O)$ for the boundary of O in a topological space P, or $B(O)$ when it is not ambiguous.

Definition 6. The weak inductive dimension ind of a topological space X is defined to be
i) $\mathrm{ind}\, X = -1$ if $X = \emptyset$,
ii) $\mathrm{ind}\, X \leq n$ if for every neighborhood U of a point $p \in X$ there exists an open set V such that $x \in V \subset U$ and $\mathrm{ind}\, B(V) \leq n-1$.
If $\mathrm{ind}\, X \leq n$ and $\mathrm{ind}\, X \not\leq n-1$, then we define $\mathrm{ind}\, X = n$. If $\mathrm{ind}\, X \not\leq n$ for every n, then $\mathrm{ind}\, X = \infty$.

The following proposition is straightforward and we use this in calculating the dimension.

Proposition 1. *If X has an open base $\mathcal{O}$ such that every element $U \in \mathcal{O}$ satisfies* $\mathrm{ind}\, B(U) \leq n-1$, *then* $\mathrm{ind}\, X \leq n$.

Lemma 1. *Let P be a subspace of a topological space X and $O \subset X$. Then, $B_P(O \cap P) \subset B_X(O) \cap P$.*

For a counter example to $B_P(O \cap P) = B_X(O) \cap P$, consider the Scott topology of three point poset $a < b > c$ for X, $\{a, c\}$ for P, and $\{b, c\}$ for O.

Proposition 2 (heredity of ind). *If* $\mathrm{ind}\, X \leq n$ *and P is a subspace of X, then* $\mathrm{ind}\, P \leq n$.

Proof. By induction on n. It is trivial for the case $n = -1$. Assume it for $n-1$. Since $\mathrm{ind}\, X \leq n$, for all $x \in P$ and $O \ni x$, there exists $x \in O' \subset O$ such that $\mathrm{ind}\, B(O') \leq n-1$. Since $B_P(O' \cap P) \subset B(O')$, by induction hypothesis, we have $\mathrm{ind}\, B_P(O' \cap P) \leq n-1$.

This heredity property does not hold for T_0 spaces when we consider the covering dimension and the strong inductive dimension. See Appendix of [HW48] for the detail. Since this heredity holds and that dimension is preserved by homeomorphisms, we have the following:

Proposition 3. *If* $\mathrm{ind}\, X > \mathrm{ind}\, Y$, *then there is no topological embedding of X in Y.*

4 The Weak Inductive Dimension of $n\perp$-sequence Spaces

We write $\Sigma^{\omega}_{\perp,n,m}$ for the subspace $\Sigma^{\omega}_{\perp,n} - \Sigma^{\omega}_{\perp,m-1}$ of $\Sigma^{\omega}_{\perp}$ $(n \geq m)$, and $\Sigma^{\omega}_{\perp,-,m}$ for the subspace $\Sigma^{\omega}_{\perp} - \Sigma^{\omega}_{\perp,m-1}$ of $\Sigma^{\omega}_{\perp}$. When $\alpha \in \Sigma^{\omega}_{\perp}$, we write $\alpha[j]$ for the j-th component of α, and we write $\alpha|_k$ for the compact element of $\Sigma^{\omega}_{\perp}$ such that $\alpha|_k[n] = \alpha[n]$ for $n \leq k$ and $\alpha[n] = \perp$ for $n > k$.

The Scott topology of a complete partial order (cpo) P is defined so that a subset O is open iff it is upward closed and for each directed subset S of P with $\sqcup S \in O$, $s \in O$ for some $s \in S$. We say that an element x of P is compact if for each directed subset S of P with $x \leq \sqcup S$, $x \leq s$ for some $s \in S$. In the cpo $\Sigma_\perp^\omega$, d is a compact element iff $d[k] \neq \perp$ for finitely many k. $\Sigma_\perp^\omega$ has the base $\{d\uparrow \mid d \in P \text{ is compact}\}$. Here, we write $d\uparrow$ for the subset $\{x \in P | d \leq x\}$ of P.

Definition 7. The length of a poset P is the maximal length n of a strictly increasing chain $a_0 < a_1 < \ldots < a_n$ in P. If there is an arbitrary long chain in P, then we define that the length of P is infinite.

Proposition 4. *1) The length of Σ^ω is 0.*
2) The length of $\Sigma_{\perp,n}^\omega$ is n.
3) The length of $\Sigma_\perp^\omega$ is infinite.

It is easy to prove that the dimension of a cpo Q with the Scott topology is equal to the length of Q; it is a finite number n only when Q consists only of compact elements. However, when we consider the Scott topology on a cpo Q, and consider its subspace topology on a subposet P, then the length of P and the dimension of P do not coincide. For example, one can consider the image $im(G) \subset \Sigma_\perp^\omega$ of the Gray code embedding G. It has length 0 because there is no order relation among elements of $im(G)$, whereas it has dimension 1 because it is homeomorphic to $\mathcal{I}$.

It does not hold generally that the closure of an open set O is $\{x \mid x \leq y \text{ for } \exists y \in O\}$. As a counter example, consider the case $X = \{0,1\}_\perp^\omega$, $O = \cup\{d\uparrow \mid d = 0^n 1 \perp^\omega \text{ for some } n\}$. Since O includes $0^n 1 0^\omega$ for all n, $Cl(O)$ includes the increasing sequence $0\perp^\omega < 00\perp^\omega < 000\perp^\omega < \ldots$, and therefore includes its limit 0^ω. However, this property holds when O is a base element $d\uparrow$. We prove the following stronger statement:

Lemma 2. *Suppose that P is a closed subspace of $\Sigma_\perp^\omega$ and d is a compact element of P. Then, $B_P(d\uparrow \cap P) \ni \alpha$ iff $d \sqcup \alpha$ exists in P and $d \not\leq \alpha$.*

Proof. The if part is trivial. $B_P(d\uparrow \cap P) \ni \alpha$ means that all the open sets containing α intersect with $d\uparrow$ in P. Therefore, $\alpha|_k\uparrow$ intersects with $d\uparrow$, and thus $d \sqcup \alpha|_k$ is a member of P for all k. In this way, we have an increasing sequence $d \sqcup \alpha|_1 \leq d \sqcup \alpha|_2 \leq \ldots$ whose limit $d \sqcup \alpha$ exists because $\Sigma_\perp^\omega$ is a cpo. From the closedness of P, $d \sqcup \alpha$ is in P.

Proposition 5. *If $P \subset \Sigma_{\perp,-,m}^\omega$ is a closed subset of $\Sigma_\perp^\omega$ and d is a compact element of P, then $B_P(d\uparrow \cap P) \subset \Sigma_{\perp,-,m+1}^\omega$.*

Proof. If all the elements of $B_P(d\uparrow \cap P)$ have infinite number of bottom components, then we have nothing to prove. Suppose that $\alpha \in B_P(d\uparrow \cap P)$ has finite number of bottom components. Then, from Lemma 2, $d \not\leq \alpha$ and $d \sqcup \alpha$ exists in P. Since $d \sqcup \alpha$ is strictly greater than α, $d \sqcup \alpha$ has fewer bottom components than α has. At the same time, $d \sqcup \alpha$ has at least m bottom components. This means that α has at least $m+1$ bottom components and therefore $\alpha \in \Sigma_{\perp,-,m+1}^\omega$.

Lemma 3. *Suppose that P is a closed subset of $\Sigma^\omega_\perp$. $B_{P\cap\Sigma^\omega_{\perp,n}}(d\!\uparrow \cap P \cap \Sigma^\omega_{\perp,n}) = B_P(d\!\uparrow \cap P) \cap \Sigma^\omega_{\perp,n}$.*

Proof. Use Lemma 1 for $\subset$ and Lemma 2 for $\supset$.

Proposition 6. *Let $P \subset \Sigma^\omega_{\perp,-,m}$ be a closed subset of $\Sigma^\omega_\perp$ and $n \geq m$. Then,* ind $(P \cap \Sigma^\omega_{\perp,n}) \leq n - m$.

Proof. By induction on $n-m$. First, consider the case that $n = m$. We have $B_{P\cap\Sigma^\omega_{\perp,n}}(d \uparrow \cap P \cap \Sigma^\omega_{\perp,n}) = B_P(d \uparrow \cap P) \cap \Sigma^\omega_{\perp,n}$ by Lemma 3 and $B_P(d \uparrow \cap P) \cap \Sigma^\omega_{\perp,n} = \emptyset$ since $B_P(d\!\uparrow \cap P) \subset \Sigma^\omega_{\perp,-,m+1}$ by Proposition 5. Therefore, ind $B_{P\cap\Sigma^\omega_{\perp,n}}(d\!\uparrow \cap P \cap \Sigma^\omega_{\perp,n}) = -1$ for all compact element d. By Proposition 1, we have ind $(P \cap \Sigma^\omega_{\perp,n}) \leq n - m$.

Next, consider the case $n > m$. We need to show ind $B_{P\cap\Sigma^\omega_{\perp,n}}(d \uparrow \cap P \cap \Sigma^\omega_{\perp,n}) = B_P(d\!\uparrow \cap P) \cap \Sigma^\omega_{\perp,n} \leq n - m - 1$ for all compact element d. We have $B_P(d\!\uparrow \cap P) \subset \Sigma^\omega_{\perp,-,m+1}$ by Proposition 5. Since $B_P(d\!\uparrow \cap P)$ is closed, we have ind $B_P(d\!\uparrow \cap P) \leq n - (m+1)$ by the induction hypothesis.

Theorem 3. ind $\Sigma^\omega_{\perp,n} = n$.

Proof. ind $\Sigma^\omega_{\perp,n} \leq n$ by applying Proposition 6 to the case $P = \Sigma^\omega_\perp$ and $m = 0$. For ind $\Sigma^\omega_{\perp,n} \geq n$, when $|\Sigma| \geq 2$, we use the embedding of $\mathbb{R}^n$ in $\Sigma^\omega_{\perp,n}$ constructed in Section 5. It is well known that ind $\mathbb{R}^n = n$ ([Eng78]). Therefore, by Proposition 2, we have ind $\Sigma^\omega_{\perp,n} \geq n$. When $|\Sigma| = 1$, we prove it in proposition 7.

When $\Sigma = \{1\}$, $\Sigma^\omega_\perp$ is isomorphic to $P_\omega = \{a \mid a \subset N\}$ for $N = \{0, 1, 2, 3, \ldots\}$ by identifying $\alpha \in \Sigma^\omega$ with $a \in P_\omega$ when $\alpha[k] = 1$ iff $k \in a$. With this correspondence, $\Sigma^\omega_{\perp,n}$ corresponds to $P^{(n)}_\omega = \{a \in P_\omega \mid |N - a| \leq n\}$. Since there is the top element N in P_ω, every non-empty open set U includes N, and therefore, its closure is the whole space P_ω. This means that the boundary $B(U)$ is the complement $P_\omega - U$. This is also the case for $P^{(n)}_\omega$. That is, $B_X(U) = X - U$ for $X = P^{(n)}_\omega$.

Proposition 7. ind $P^{(n)}_\omega = n$.

Proof. Since we have already shown ind $P^{(n)}_\omega \leq n$, we need to prove ind $P^{(n)}_\omega \not\leq n - 1$. We prove it by induction on n.

It is meediate when $n = 0$. When $n > 0$, let $X = P^{(n)}_\omega$ and show that there exists an open set $U \subset X$ such that ind $B(U) \not\leq n-1$. Then, for each non-empty open subset $V \subset U$, we have $B(V) \supset B(U)$ because $B(V) = X - V$, and by heredity (Proposition 2), ind $B(V) \not\leq n - 1$.

Take $U = \{0\} \uparrow \cap X$. Then, $B(U) = X - U = \{a \in X \mid 0 \not\in a\}$. This set is isomorphic to $\{a \mid a \subset N', |N' - a| \leq n - 1\}$ for $N' = \{1, 2, 3, \ldots\}$. Therefore, $B(U)$ is isomorphic to $P^{(n-1)}_\omega$, and thus ind $B(U) \not\leq n - 1$. Note that the topological structure of $B(U)$ as a subspace of X and that of $P^{(n-1)}_\omega$ are the same.

From Theorem 3 and Proposition 3, we have the followings.

Corollary 1. *The computational dimension of a space is at least as large as its weak inductive dimension.*

Corollary 2. *There are no embeddings of* $\mathbb{R}^n$ *and* $\mathcal{I}^n$ *in* $\Sigma^\omega_{\perp,n-1}$ *for any character set* Σ.

Proof. $\mathrm{ind}\ \mathbb{R}^n = \mathrm{ind}\ \mathcal{I}^n = n$ ([Eng78]).

5 Embeddings of Finite Dimensional Spaces

In this section, we construct an embedding of $\mathbb{R}^2$ in $\Sigma^\omega_{\perp,2}$ for $\Sigma = \{0,1\}$, namely, interleaving $G(x)$ and $G(y)$ to define the name of $(x,y) \in \mathbb{R}^2$. Thus, the computational dimension of $\mathbb{R}^2$ is 2.

Let F be the function from $\Sigma^\omega_{\perp,1} \times \Sigma^\omega_{\perp,1}$ to $\Sigma^\omega_{\perp,2}$ which maps $(a_1 a_2 \ldots, b_1 b_2 \ldots)$ to $a_1 b_1 a_2 b_2 \ldots$. Then, F is a topological homeomorphism from $\Sigma^\omega_{\perp,1} \times \Sigma^\omega_{\perp,1}$ into $\Sigma^\omega_{\perp,2}$. Since $\mathbb{R}$ can be embedded in $\Sigma^\omega_{\perp,1}$ by the Gray code embedding G, we can topologically embed $\mathbb{R}^2$ into $\Sigma^\omega_{\perp,2}$ by $F \circ (G,G)$. In the same way, $\mathbb{R}^n$ can be embedded into $\Sigma^\omega_{\perp,n}$. Combining this fact with Corollary 1, we have

Theorem 4. $\mathbb{R}^n$ *has the computational dimension* n.

In order to show that the computability notion induced on $\mathbb{R}^2$ by $F \circ (G,G)$ in Definition 3 and 4 is the standard one, we show that the encoding F and the decoding F^{-1} can be expressed by an IM2-machine.

First, we consider encoding. That is, we construct an IM2-machine of type $(\Sigma^\omega_{\perp,1}, \Sigma^\omega_{\perp,1}, \Sigma^\omega_{\perp,2})$ which inputs two sequences in $\Sigma^\omega_{\perp,1}$ and outputs its interleaving in $\Sigma^\omega_{\perp,2}$.

The machine has two input tapes I_i $(i = 1,2)$ with two heads $\mathrm{H}_1(I_i)$ and $\mathrm{H}_2(I_i)$ on each I_i $(i = 1,2)$, and one output tape O with three heads $\mathrm{H}_1(O), \mathrm{H}_2(O)$, and $\mathrm{H}_3(O)$. It has 4 states (c,s) for $c \in \{1,2\}$ and $s \in \{1,2\}$ with (1,1) the initial state. c indicates the tape to input the next character from, and s indicates the pair of heads used to output the next character: $s = 1$ means to output from $\mathrm{H}_1(O)$ and $\mathrm{H}_3(O)$, $s = 2$ means to output from $\mathrm{H}_2(O)$ and $\mathrm{H}_3(O)$. We need 16 rules corresponding to the combination of 4 states, 2 heads and 2 input characters. We abbreviate them into two rules with variables by allowing pattern matching on the left hand side of a rule, and using the *not* function defined as *not* $1 = 2$ and *not* $2 = 1$ on the right hand side.

$$\begin{aligned} &(c,s)\ \mathrm{H}_1(I_c)(x)\ \Rightarrow (\mathit{not}\ c, 1)\ \mathrm{H}_s(O)(x) \\ &(c,s)\ \mathrm{H}_2(I_c)(x)\ \Rightarrow (\mathit{not}\ c, \mathit{not}\ s)\ \mathrm{H}_3(O)(x) \end{aligned}$$

The first rule is read as when the state is (c,s) and it inputs the character x from the first head, then it changes its state to $(\mathit{not}\ c, 1)$ and output the character x

from the s-th head. For the decoding function to the first component, we consider a machine of type $(\Sigma^{\omega}_{\perp,2}, \Sigma^{\omega}_{\perp,1})$ with the following rules:

$$\begin{array}{ll}
(1,1)\ \mathrm{H}_1(I)(x) & \Rightarrow (2,1)\ \mathrm{H}_1(O)(x) \\
(1,2)\ \mathrm{H}_2(I)(x) & \Rightarrow (2,1)\ \mathrm{H}_1(O)(x) \\
(1,s)\ \mathrm{H}_3(I)(x) & \Rightarrow (2,\mathit{not}\ s)\ \mathrm{H}_2(O)(x) \\
(2,1)\ \mathrm{H}_1(I)(x) & \Rightarrow (1,1) \\
(2,2)\ \mathrm{H}_2(I)(x) & \Rightarrow (1,1) \\
(2,s)\ \mathrm{H}_3(I)(x) & \Rightarrow (1,\mathit{not}\ s)
\end{array}$$

Thus, $f : \mathbb{R}^2 \to \mathbb{R}$ is $(F \circ (G,G), G)_\perp$-computable iff it is $(G,G,G)_\perp$-computable as a function in two arguments, and thus it is (ρ,ρ,ρ)-computable, where ρ is a representation equivalent to the Cauchy representation by Theorem 2.

For other finite dimensional spaces, it is immediate to show that S^1 ($= [0,1]/\sim$ with $\sim$ defined as $0 \sim 1$) can be embedded into $\Sigma^{\omega}_{\perp,1}$. From Figure 1, one can find that G can be modified to a function from S^1 by assigning $\perp 000000$ as $\mathrm{G}(0) = \mathrm{G}(1)$. Since ind $S^1 = 1$, we have the following.

Proposition 8. *The computational dimension of S^1 is* 1.

More generally, we show in the next section that the computational dimension and the topological dimension coincide for all the separable metric spaces.

6 The Coincidence of the Computational Dimension and the Topological Dimension

Definition 8. *Let $\mathcal{I}$ be the unit open interval* $(0,1)$ *and Q^n_i be the subspace of $\mathcal{I}^n$ with i dyadic coordinates, where a dyadic number is a rational number of the form $m \times 2^{-n}$ for integers m and n. We define*

$$N^n_k = Q^n_0 \cup \dots Q^n_k.$$

That is, N^n_k is the subspace of $\mathcal{I}^n$ consisting of all points which have at most k dyadic coordinates.

It is known that N^n_k has dimension k ([Eng78]). The space N^{2n+1}_n is essentially the same as the Nöbeling's universal n-dimensional space, and it is universal for the class of all n-dimensional separable metric spaces in the following sense.

Proposition 9. *For any n-dimensional separable metric space X, there is a topological embedding of X in N^{2n+1}_n.*

Proof. See [Eng78], for example.

Thus, if we can embed N^{2n+1}_n in $\Sigma^{\omega}_{\perp,n}$, it means that we can embed any n-dimensional topological space in $\Sigma^{\omega}_{\perp,n}$, and therefore the computational dimension and the weak inductive dimension coincide for all the separable metric spaces.

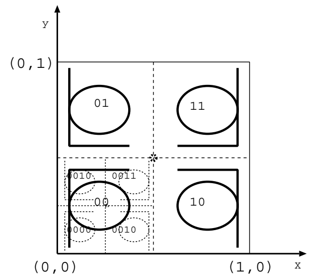

Fig. 2. The coding of N_1^2.

Theorem 5. *There is an embedding from N_n^m to $\Sigma_{\perp,n}^{\omega}$ for $m \geq n$ and $\Sigma = \{0,1\}$.*

Proof. We start with the embedding G^m of $\mathcal{I}^m$ in $\Sigma_{\perp,m}^{\omega}$ we constructed in Theorem 4. Since it is an interleaving of the Gray code, the number of $\perp$ which appear in $G^m(x)$ is equal to the number of dyadic coordinates of $x \in \mathcal{I}^m$. Therefore, when we restrict it to N_n^m, then the image has at most n dyadic coordinates. That is, $G^m(N_n^m) \subset \Sigma_{\perp,n}^{\omega}$. Thus, this restriction of G^m to N_n^m is a homeomorphic embedding to $\Sigma_{\perp,n}^{\omega}$.

Corollary 3. *The computational dimension and the weak inductive dimension coincide for all the separable metric spaces.*

Proof. It is immediate from Theorem 5 when ind X is finite. When ind X is infinite, separable metric spaces are second countable T_0 spaces and as we note in the next section, every second countable T_0 spaces can be embedded into $\Sigma_{\perp}^{\omega}$. Therefore, the computational dimension of X is also infinite.

Corollary 4. *If a separable metric space can be embedded in $\Sigma_{\perp,n}^{\omega}$ for some alphabet Σ, then it can be embedded in $\{0,1\}_{\perp,n}^{\omega}$.*

Figure 2 depicts how the code G^2 of N_1^2 is composed. We split N_1^2 into four sub-areas, and assign the first two bits of $G^2(x)$ as 00, 01, 10, and 11 in each sub-areas. Note that N_1^2 is the unit square $\mathcal{I}^2$ minus points whose coordinates are both dyadic numbers. On the boundaries, we assign $\perp 0$ or $\perp 1$ on $x = 1/2$, and $0\perp$ or $1\perp$ on $y = 1/2$. Note that the center point $(1/2, 1/2)$ is not included in N_1^2. The rest of the bits are coded coinductively; we have the $1/2$ reduction of the code of N_1^2 on the right lower subsquare, which is flipped horizontally and vertically to fill all the subsquares so that they agree on the boundaries $x = 1/2$ and $y = 1/2$.

7 Embedding of Infinite Dimensional Objects

We consider infinite dimensional spaces which appear in computable analysis, like the set $\mathcal{A}^{(n)}$ of the closed subsets of $\mathbb{R}^n$, $\mathcal{O}^{(n)}$ of the open subsets of $\mathbb{R}^n$,

$\mathcal{K}^{(n)}$ of the compact subsets of $\mathbb{R}^n$, and $C(A, \mathbb{R}^n)$ of the continuous functions from a subset $A \subset \mathbb{R}^m$ to $\mathbb{R}^n$. As for the topological structures on them, we have some possibilities. In this section, we consider $\mathcal{A}^{(n)}$ with the following three topological structures and discuss how they are embedded into $\Sigma_\perp^\omega$.

Let $Cb^{(n)}$ be the set of all open rational cubes of $\mathbb{R}^n$. $\tau_<^{\mathcal{A}}$ is the topology of $\mathcal{A}^{(n)}$ which has $\{A \subset \mathbb{R}^n \mid A \cap J \neq \emptyset\}$ as a subbase element for $J \in Cb^{(n)}$. In the same way, $\tau_>^{\mathcal{A}}$ is the topology which has $\{A \subset \mathbb{R}^n \mid A \cap \overline{J} = \emptyset\}$ as a subbase element for $J \in Cb^{(n)}$, and $\tau^{\mathcal{A}}$ is the topology with the union of these two as the subbase. These three topologies are the final topologies of the three representations $\psi_<$, $\psi_>$ and ψ in [Wei00].

Let M be a homeomorphism from $\mathbb{R}$ to $(0,1)$. Then, since $\mathbb{R}$ includes infinite number of disjoint open intervals $(0,1), (1,2), (2,3), \ldots$, we can assign to $(a_0, a_1, \ldots) \in \mathbb{R}^\omega$ the closed set $\{M(a_0), M(a_1) + 1, M(a_2) + 2, \ldots\}$. It is easy to show that this map from $\mathbb{R}^\omega$ to $\mathcal{A}^{(1)}$ is a topological embedding with respect to $\tau_<^{\mathcal{A}}$, $\tau_>^{\mathcal{A}}$, and $\tau^{\mathcal{A}}$ on $\mathcal{A}^{(1)}$. In the same way, we can construct embeddings from $\mathbb{R}^\omega$ to $\mathcal{A}^{(n)}$ for every n. Therefore, $(\mathcal{A}^{(n)}, \tau_<^{\mathcal{A}})$, $(\mathcal{A}^{(n)}, \tau_>^{\mathcal{A}})$, and $(\mathcal{A}^{(n)}, \tau^{\mathcal{A}})$ are infinite dimensional spaces, and they have the computational dimension ∞ if they have embedding in $\Sigma_\perp^\omega$.

On the other hand, these spaces have natural embeddings into $P_\omega = \{1\}_\perp^\omega$. More generally, one can define an embedding of a second countable T_0-space X into P_ω, by fixing a subbase $O = \{O_i \mid i \in N\}$ with numbering, assigning i-th cell to O_i, and defining the embedding E of X as $E(x)[i] = 1$ iff $x \in O_i$ [Eng68]. This corresponds, in the standard representation theory, to considering the standard representation of S restricted to complete names where S is an effective topological space $S = (X, O, \mu)$ with $\mu(i) = O_i$, or when $\{(i,j) \mid O_i = O_j\}$ is r.e., considering the standard representation in a computable topological space $S = (X, O, \mu)$ [Wei00]. Applying this to the above-mentioned subbase, we define embeddings $E_<$, $E_>$, and E of $(\mathcal{A}, \tau_<^{\mathcal{A}})$, $(\mathcal{A}, \tau_>^{\mathcal{A}})$, and $(\mathcal{A}, \tau^{\mathcal{A}})$ to P_ω, respectively. Thus, these spaces have the computational dimension ∞.

When we use $\{0,1\}_\perp^\omega$ instead of $\{1\}_\perp^\omega$, we can not only express positive properties like $n \in O_i$ but also negative properties like $n \notin Cl(O_i)$. Therefore, we can define a new embbeding $\hat{E}$ of $(\mathcal{A}^{(n)}, \tau^{\mathcal{A}})$ to $\{0,1\}_\perp^\omega$ which is based on the subbase of $(\mathcal{A}^{(n)}, \tau_<^{\mathcal{A}})$. For the embedding E, it is obvious that $E(x)$ includes infinite number of $\perp$-cells for each $x \in \mathcal{A}^{(n)}$. Our result that the computational dimension of $(\mathcal{A}^{(n)}, \tau^{\mathcal{A}})$ is infinite shows that $\hat{E}(x)$ also includes infinite number of $\perp$-cells for some $x \in \mathcal{A}^{(n)}$, and also says that one cannot think of an embbeding with finite number of $\perp$-cells for each element, even if we use bigger alphabet as Σ.

8 Conclusion

We have proposed a way of defining computation over a topological space X by considering an embedding of X in the sets $\Sigma_{\perp,n}^\omega$ $(n = 0, 1, \ldots)$ of sequences of Σ in which bottoms are allowed to exist. We have shown that the computational dimension, which is the number of bottoms required in the name space, and the usual topological dimension coincide for separable metric spaces.

Since our theory is based on an embedding of a space into the domain $\Sigma_\perp^\omega$, it can be considered as a variant of domain theoretic approaches ([ES98], [Gia99], [SHT99],[Bla97]). On the other hand, each element of $\Sigma_{\perp,n}^\omega$ has a unique textual representation as an $n\perp$-sequence and we have the notion of a machine which operates on such extended sequences. In this sense, it can also be considered as a variant of more concrete representation-based approaches [Wei00].

One of the benefits of our embedding-based approach is that, since we consider embeddings in name spaces, we can study properties of computable functions over $\Sigma_{\perp,n}^\omega$ to study properties of computable functions over topological spaces in general. For example, multi-valued functions over X play an important role in computable analysis. They are represented by multi-valued functions over name spaces in our approach, whereas they are represented by single-valued functions over Σ^ω which are given multi-valued meaning through redundant representations in the standard representation theory. Our result relating the dimension of a space and the number of heads required to perform computation over the space is another example.

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