Sudoku Colorings of a 16-cell Pre-Fractal

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Abstract. We study coloring problems of the third-level approximation of a 16-cell fractal. This four-dimensional object is projected to a cube in eight different ways, after which it forms an $8 \times 8 \times 8$ grid of cubes. On each such grid, we can consider two Sudoku-like colorings. Our question is whether it is possible to assign colors to the 8^3 pieces of this pre-fractal object in such a manner that all of its eight cubic projections form Sudoku-like colorings. We analyzed this problem and its variants and constructed solution patterns to the cases they exist. We also enumerated the number of solutions with computer programs for some of the cases.

1 Introduction

An imaginary cube is a three-dimensional object that has square projections in three orthogonal directions, just as a cube has [3, 4]. Among imaginary cubes, a hexagonal bipyramid imaginary cube (simply called an H, Fig. 1) is a double imaginary cube, i.e., it is an imaginary cube of two different cubes. Therefore, it has square projections in six ways. In addition, from an H, a double imaginary cube fractal with the similarity dimension two is generated. When the first author designed a sculpture based on the second-level approximation of this fractal, 81 pieces were colored with nine colors so that the colors form a Sudoku solution pattern in each of the six square projections, which form 9×9 grids (Fig. 2) [1]. As the upper-middle picture of Fig. 2 indicates, this coloring pattern is based on simple rules. In [2], he studied this Sudoku coloring problem and showed that it has 140 solutions modulo change of colors and 30 solutions modulo change of

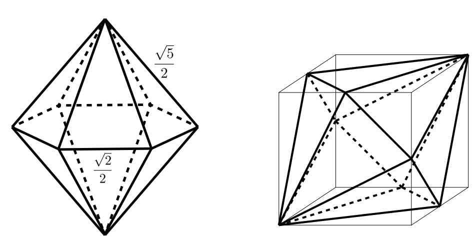


Fig. 1. H (Hexagonal bipyramid imaginary cube). This object is projected to a square in six different ways.

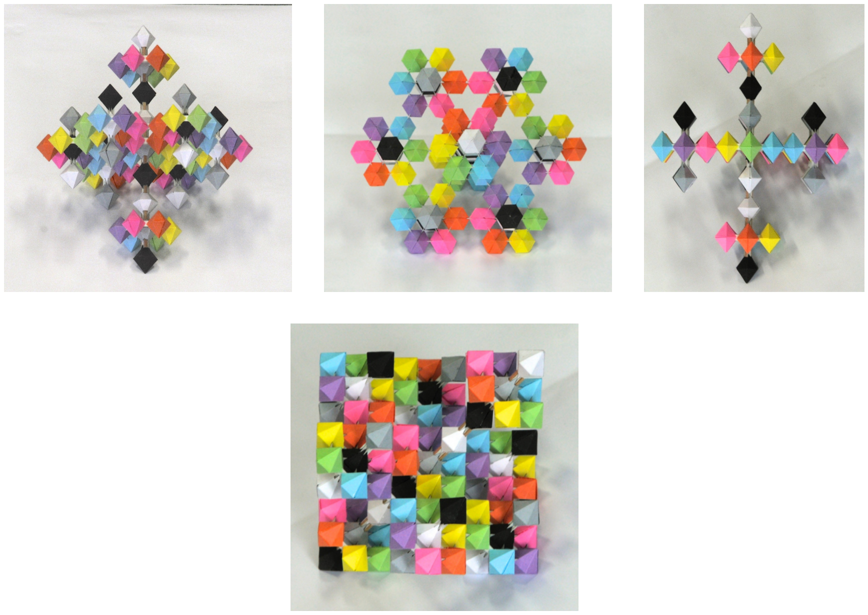


Fig. 2. Fractal Sudoku Sculpture [1, 2] (reassembled by Y. Tsukamoto in 2013).

colors and congruences of the object. This calculation was first done with a computer program and then performed manually, i.e., it was shown mathematically as a proof. Tsuiki and Yokota also studied this Sudoku coloring problem only for three orthogonal square projections, and they enumerated their solutions using computer programs [5].

In this paper, we report our study of Sudoku colorings of the third level approximation of the 16-cell fractal. This four-dimensional object is projected to a cube in eight different ways, after which it forms an $8 \times 8 \times 8$ grid of cubes. On each such grid of cubes, we can consider two Sudoku-like coloring problems indicated in Figure 3(a,b). Our question is whether it is possible to assign colors to the 8^3 pieces of this pre-fractal object in such a manner that all of its eight cubic projections form Sudoku-like colorings. We analyzed this problem and its variants and constructed solution patterns to the cases they exist. We also enumerated the number of solutions with computer programs for some of the cases.

In the next section, we explain a 16-cell and its pre-fractals, and explain Puzzle A and Puzzle B which are the two Sudoku-like coloring problems which we study in this paper. In Section 3, we study properties of cubic projections of a 16-cell. We study Puzzle A and its variants in Section 4, and Puzzle B and its variants in Section 5.

2 A 16-cell and Sudoku-like coloring problems

A 16-cell is a four-dimensional regular polytope with eight vertices and sixteen regular tetrahedron facets. We first review properties of this object and (pre-)fractals generated by it based on [4]. Then, we explain our Sudoku-like coloring problems.

A 16-cell is a four-dimensional counterpart of a regular octahedron in that it is a cross-polytope. That is, $V = \{(\pm 2, 0, 0, 0), (0, \pm 2, 0, 0), (0, 0, \pm 2, 0), (0, 0, 0, \pm 2)\}$ is the set of vertices of a 16-cell. A 16-cell is also obtained by selecting eight non-adjacent vertices of a hypercube. That is, let V_1 and V_2 be the subsets of $(\pm 1, \pm 1, \pm 1, \pm 1)$ with even and odd number of $+1$ coordinates, respectively. Then, V_1 and V_2 are sets of vertices of 16-cells. If these two 16-cells are projected along each of the four axis of coordinates, then we have cubes. Therefore, a 16-cell is an imaginary 4-cube. Here, an *imaginary n-cube* is an n-dimensional object that has $(n-1)$-dimensional hypercube projections in n orthogonal directions just as an n-dimensional hypercube has. One can see that these cubic projections are projections from four pairs of facets of a 16-cell. Therefore, by symmetry, it also has cubic projections from the other four pairs of facets. Thus, a 16-cel has cubic projections in eight directions and the eight directions are divided into two sets of four mutually orthogonal directions. We call such an object a *double imaginary* 4*-cube.*

More interestingly, not only a 16-cell but also a fractal based on a 16-cell is a double imaginary 4-cube. Let F_i $(1 \leq i \leq 8)$ be the homothetic transformations with centers at the eight vertices of a 16-cell S and with scales 1/2. Here, a homothetic transformation is a similitude that performs no rotations. We define a map G on the space $\mathcal{H}^4$ of non-empty compact subsets of $\mathbb{R}^4$ as $G(X) = \bigcup_{i=1}^{8} F_i(X)$. Since G is a contraction map in $\mathcal{H}^4$, the sequence $S_0 = S, S_1 = G(S_0), S_2 = G(S_1), S_3 = G(S_2), \ldots$ converges to the unique fixedpoint S_∞ of G, which is called the fractal generated by the iterative function system $\{F_i \mid 1 \leq i \leq 8\}$. One can easily see that S_∞ has the similarity dimension 3, and it is also a double imaginary 4-cube. In addition, not only S_∞ but also S_n, which is a n-th level approximation of S_∞, is a double imaginary hypercube for every $n \geq 1$. We call such an approximation of a fractal a *pre-fractal*.

Since each cubic projection image of S_n consists of a set of 8^n cubes forming a $2^n \times 2^n \times 2^n$ grid, if there is a coloring notion on a $2^n \times 2^n \times 2^n$ grid of cubes, then we have a corresponding coloring notion on the pre-fractal S_n that all the eight cubic projections satisfy the coloring. We consider the case $n = 3$ and consider two Sudoku-like coloring puzzles on a $8 \times 8 \times 8$-grid of cubes.

Puzzle A_0**:** Assign 64 colors to an $8 \times 8 \times 8$ grid of cubes so that each 8×8-plane (3×8 exist) and each $4 \times 4 \times 4$-block (8 exist) contains all 64 colors (Fig. 3(a)).

Puzzle B_0**:** Assign 8 colors to an $8 \times 8 \times 8$ grid of cubes so that each 8-sequence (3×64 exist) and each $2 \times 2 \times 2$-block (64 exist) contains all 8 colors (Fig. 3(b)).

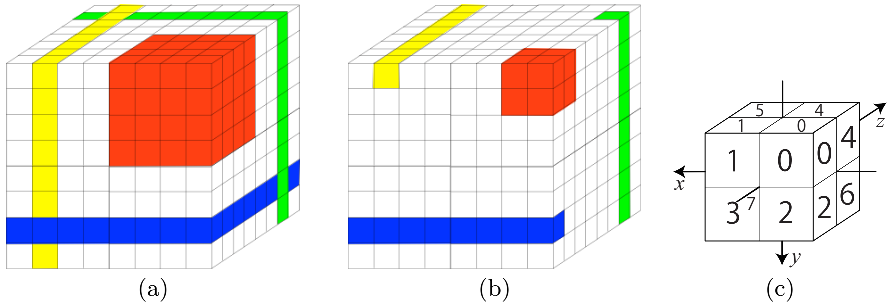


Fig. 3. (a) Sets of pieces with different colors in Puzzle A_0, (b) that of Puzzle B_0, (c) address of cubes in a lattice.

For each of them, there is a Sudoku-like coloring Puzzle A (resp. Puzzle B) of S_3 to assign 64 (resp. 8) colors to the components of S_3 so that each of the eight cubic projections is a solution of Puzzle A_0 (resp. Puzzle B_0). We can also consider four projection variants of these puzzles. That is, fix a set of four orthogonal cubic projections of a 16-cell and consider the condition that each of them is a solution of Puzzle A_0 (or Puzzle B_0). We call them Puzzle A_S and Puzzle B_S, respectively.

Remark: In [4] and [3], it is shown that (1) all the convex double imaginary 3-cubes are variants of H, (2) a 16-cell is the only convex double imaginary 4-cube, (3) there is no double imaginary n-cube for $n \geq 5$, and (4) H is the only convex double imaginary 3-cube from which one can generate a double imaginary 3-cube fractal with the similarity dimension 2. Therefore, a 16-cell fractal is the only object in three- and higher-dimensional spaces on which one can consider a coloring problem similar to the one on the H pre-fractal.

3 Projections of a 16-cell pre-fractal

We study how different cubic projections of S_3 are related.

We first study how vertices of a 16-cell are mapped by cubic projections. Let $\boldsymbol{v}_0 = (2,0,0,0)$, $\boldsymbol{v}_1 = (0,2,0,0)$, $\boldsymbol{v}_2 = (0,0,2,0)$, $\boldsymbol{v}_3 = (0,0,0,2)$ and let S be the 16-cell with the set of vertices $\{\pm\boldsymbol{v}_0, \pm\boldsymbol{v}_1, \pm\boldsymbol{v}_2, \pm\boldsymbol{v}_3\}$. We consider a cube C with vertices $(\pm1, \pm1, \pm1)$ and assign a number in $D = \{i \mid 0 \leq i \leq 7\}$ to the vertices of C so that (x, y, z) is given the number $b(z)b(y)b(x)$ in binary notation with $b(-1) = 0$ and $b(1) = 1$ (c.f. Fig. 3(c)). We sometime use binary notation for elements of D. We define $\mathrm{inv}(i) = 7 - i$ so that i and $7 - i$ specify space diagonal vertices.

For each tuple $(a_0, a_1, a_2, a_3) \in \{-1, 1\}^4$, there is a regular tetrahedron facet F of S with the set of vertices $\{a_0\boldsymbol{v}_0, a_1\boldsymbol{v}_1, a_2\boldsymbol{v}_2, a_3\boldsymbol{v}_3\}$. S is projected to a cube when it is projected from F, that is, projected along the vector $a_0\boldsymbol{v}_0 + a_1\boldsymbol{v}_1 + a_2\boldsymbol{v}_2 + a_3\boldsymbol{v}_3$. We fix $a_0 = 1$ and denote by $P_{(a_1,a_2,a_3)}$ this projection.

By $P_{(a_1,a_2,a_3)}$, the four space diagonals of S are projected to the four space diagonals of a cube. We transfer cubes obtained by projections to the cube C through rotations and reflections so that $\boldsymbol{v}_0$ is mapped to vertex 0 and the space diagonal between $\pm\boldsymbol{v}_i$ is mapped to the space diagonal between the vertices i and $\mathrm{inv}(i)$. We redefine this map from S to C as the projection $P_{(a_1,a_2,a_3)}$.

By $P_{(a_1,a_2,a_3)}$, the two regular tetrahedron facets with the vertices $(\boldsymbol{v}_0, a_1\boldsymbol{v}_1, a_2\boldsymbol{v}_2, a_3\boldsymbol{v}_3)$ and $(-\boldsymbol{v}_0, -a_1\boldsymbol{v}_1, -a_2\boldsymbol{v}_2, -a_3\boldsymbol{v}_3)$ preserve their shapes and these lists of vertices are mapped to lists of vertices of regular tetrahedrons in C, that is, vertices $(0, 6, 5, 3)$ and $(7, 1, 2, 4)$ in C. Since v_0 and $-v_0$ are mapped to vertices 0 and 7, respectively, it determines how $P_{(a_1,a_2,a_3)}$ maps vertices of S to vertices of C. That is, $\boldsymbol{v}_0$ is always mapped to the vertex 0 and $\boldsymbol{v}_i$ $(i = 1, 2, 3)$ is mapped to the vertex 2^{i-1} if $a_i = -1$ and to $\mathrm{inv}(2^{i-1})$ if $a_i = 1$.

Instead of studying colorings on S_3, we consider colorings of the $8\times8\times8$ grid of cubes obtained by projection $P_{(1,1,1)}$. In order to express the constraints caused by other projections, it is important to know how the same piece of S_3 is mapped by different projections. For this, we first study action of $P_{(a_1,a_2,a_3)} \circ P^{-1}_{(1,1,1)}$ on vertices of C, which can be expressed as a permutation on D. The above observation shows that this action is generated by the three transpositions $\alpha = (1, 6)$, $\beta = (2, 5)$, and $\gamma = (3, 4)$. We denote by U the subgroup of the symmetric group S_8 generated by these three transpositions. The order of U is 8 and it is isomorphic to the group $2 \times 2 \times 2$.

Among the eight projections, the projection lines of $P_{(1,1,1)}$, $P_{(-1,-1,1)}$, $P_{(1,-1,-1)}$, and $P_{(-1,1,-1)}$ are mutually orthogonal. One can see that $P \circ P^{-1}_{(1,1,1)}$ for P these four projections cause the identity permutation, $\alpha\beta$, $\beta\gamma$, and $\gamma\alpha$, respectively. They are even permutations and they form the Klein four-group. We denote by U_S this subgroup of U.

Now, we study how each piece of S_3 is mapped to a piece of an $8 \times 8 \times 8$ grid of cubes. As in Fig. 3(c), we assign numbers in D to a $2 \times 2 \times 2$ grid of cubes. We give addresses (i, j, k) for $i, j, k \in D$ to an $8 \times 8 \times 8$ grid of cubes so that i specifies the big block, j specifies the small block, and k specifies the address in the small block. Therefore, a piece of S_3 that is mapped to the cube (i, j, k) by the projection $P_{(1,1,1)}$ is mapped by $P_{(a_1,a_2,a_3)}$ to the cube $(\delta(i), \delta(j), \delta(k))$. Here, $\delta \in U$ is $\alpha^{b_1}\beta^{b_2}\gamma^{b_3}$ where b_i is 0 or 1 depending on whether a_i is 1 or -1.

4 Solutions of Puzzle A

Based on the observation in the previous section, we formalize Puzzle A as a three-dimensional puzzle on a cube.

Let $c : D \times D \times D \to D \times D$ be a coloring of an $8 \times 8 \times 8$ grid of cubes with $D \times D$. The condition that all of the $4 \times 4 \times 4$-blocks contain all the 64 colors can be expressed as follows.

$$\text{For each } i \in D, \text{ the cardinality of } \{c(i, j, k) \mid j, k \in D\} \text{ is } 64. \tag{1}$$

Let $\widetilde{\mathcal{F}}_0 = \{\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3\}$ for

$$\begin{aligned}
\mathcal{F}_1 &= \{\{0,1,2,3\},\{4,5,6,7\}\}\,(=\{\{b_z b_y b_x \mid b_z = 0\},\{b_z b_y b_x \mid b_z = 1\}\}),\\
\mathcal{F}_2 &= \{\{0,1,4,5\},\{2,3,6,7\}\}\,(=\{\{b_z b_y b_x \mid b_y = 0\},\{b_z b_y b_x \mid b_y = 1\}\}),\\
\mathcal{F}_3 &= \{\{0,2,4,6\},\{1,3,5,7\}\}\,(=\{\{b_z b_y b_x \mid b_x = 0\},\{b_z b_y b_x \mid b_x = 1\}\}).
\end{aligned}$$

The condition of Puzzle A_0 that all of the 8×8-planes contain all the 64 colors can be expressed as the requirement that the following condition holds for every $\mathcal{F} \in \widetilde{\mathcal{F}}_0$.

$$\begin{aligned}
&\text{For each } (F_1, F_2, F_3) \in \mathcal{F} \times \mathcal{F} \times \mathcal{F},\\
&\qquad \text{the cardinality of } \{c(i,j,k) \mid i \in F_1, j \in F_2, k \in F_3\} \text{ is } 64.
\end{aligned} \tag{2}$$

For Puzzle A_S, we have the condition that $\mathcal{F} = \delta(\mathcal{F}')$ for $\mathcal{F}' \in \widetilde{\mathcal{F}}_0$ and $\delta \in U_S$ also satisfy (2). Here, $\delta(\{F_1, F_2\}) = \{\delta(F_1), \delta(F_2)\}$ and $\delta(\{i,j,k,l\}) = \{\delta(i), \delta(j), \delta(k), \delta(l)\}$. One can see that $\alpha(\beta(\mathcal{F}_1)) = \mathcal{F}_4$ for

$$\mathcal{F}_4 = \{\{0,3,5,6\},\{1,2,4,7\}\}. \tag{3}$$

In addition, the set $\widetilde{\mathcal{F}}_S = \{\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4\}$ is closed under the action of U_S. Therefore, we can restate this condition to the requirement that (2) is satisfied for every $\mathcal{F} \in \widetilde{\mathcal{F}}_S$. Note that, when $\mathcal{F} = \mathcal{F}_4$, the pieces for the cases $F_1 = F_2 = F_3 = \{0,3,5,6\}$ and $F_1 = F_2 = F_3 = \{1,2,4,7\}$ are third-level cubic approximations of the Sierpinski Tetrahedron.

For Puzzle A, we have the condition that $\mathcal{F} = \delta(\mathcal{F}')$ satisfies (2) for $\mathcal{F}' \in \widetilde{\mathcal{F}}_0$ and $\delta \in U$. In this case, $\mathcal{F}$ ranges over all the eight divisions of D into two sets that do not contain i and $\mathrm{inv}(i)$ for every $i \in D$. The cardinality of this set is 8 and we denote this set by $\widetilde{\mathcal{F}}$. We summarize these results.

Proposition 1. *Let $c : D \times D \times D \to D \times D$ be a coloring.*

(a) c is a solution of Puzzle A_0 if and only if (1) is satisfied and (2) is satisfied for $\mathcal{F} \in \widetilde{\mathcal{F}}_0$.

(b) c is a solution of Puzzle A_S if and only if (1) is satisfied and (2) is satisfied for $\mathcal{F} \in \widetilde{\mathcal{F}}_S$.

(c) c is a solution of Puzzle A if and only if (1) is satisfied and (2) is satisfied for $\mathcal{F} \in \widetilde{\mathcal{F}}$.

Our goal is to see whether there exists a solution to these puzzles and to present a solution if it exists.

Theorem 2. *The following is a solution of Puzzle A (and therefore is a solution of Puzzle A_0 and Puzzle A_S).*

$$\begin{aligned}
c(i,j,k) &= (j,k) && (i = 0,7)\\
c(i,j,k) &= (\mathrm{inv}(j),k) && (i = 1,6)\\
c(i,j,k) &= (j,\mathrm{inv}(k)) && (i = 2,5)\\
c(i,j,k) &= (\mathrm{inv}(j),\mathrm{inv}(k)) && (i = 3,4)
\end{aligned}$$

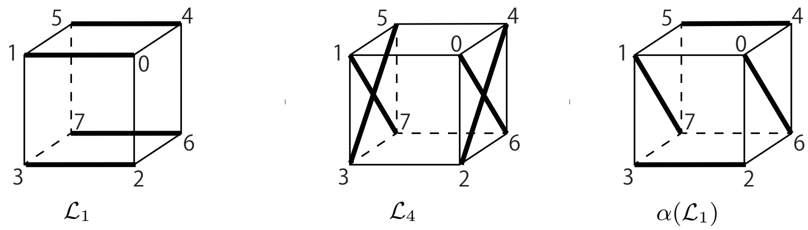


Fig. 4. $\mathcal{L}_1 \in \widetilde{\mathcal{L}}_0$, $\mathcal{L}_4 \in \widetilde{\mathcal{L}}_S$ and $\alpha(\mathcal{L}_1) \in \widetilde{\mathcal{L}}$ considered as relations between vertices of a cube.

Proof. Condition (1) is obviously satisfied. Let $\mathcal{F} \in \widetilde{\mathcal{F}}$ and $F_1, F_2, F_3 \in \mathcal{F}$. We show that the cardinality of $\{c(i,j,k) \mid i \in F_1, j \in F_2, k \in F_3\}$ is 64. It holds because F_1 contains one element of each of $\{0,7\}$, $\{1,6\}$, $\{2,5\}$, $\{3,4\}$, and the four sets $\{(j,k) \mid j \in F_2, k \in F_3\}$, $\{(\mathrm{inv}(j),k) \mid j \in F_2, k \in F_3\}$, $\{(j,\mathrm{inv}(k)) \mid j \in F_2, k \in F_3\}$, $\{(\mathrm{inv}(j),\mathrm{inv}(k)) \mid j \in F_2, k \in F_3\}$ are all disjoint. $\square$

We formalized the condition of Puzzle A as a conjunctive normal form Boolean formula and put it into a SAT solver miniSAT version 2.2.0 to obtain some more solutions. Enumeration of all of the solutions of each puzzle is an open problem.

5 Solutions of Puzzle B

We study Puzzle B and its variants. Let $c : D \times D \times D \to D$ be a coloring of an $8 \times 8 \times 8$ grid of cubes with D. The condition that each $2 \times 2 \times 2$-block contains all the 8 colors can be expressed as follows.

$$\text{For each } (i,j) \in D \times D, \text{ the cardinality of } \{c(i,j,k) \mid k \in D\} \text{ is } 8. \tag{4}$$

Let $\widetilde{\mathcal{L}}_0 = \{\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3\}$ for

$$\begin{aligned}
\mathcal{L}_1 &= \{\{0,1\},\{2,3\},\{4,5\},\{6,7\}\},\\
\mathcal{L}_2 &= \{\{0,2\},\{1,3\},\{4,6\},\{5,7\}\},\\
\mathcal{L}_3 &= \{\{0,4\},\{1,5\},\{2,6\},\{3,7\}\}.
\end{aligned}$$

See Fig. 4 for the meaning of $\mathcal{L}_1$. The condition of Puzzle B_0 that all the 64×3 sequences contain all the 8 colors can be expressed by stating that the following condition holds for every $\mathcal{L} \in \widetilde{\mathcal{L}}_0$.

$$\begin{aligned}
&\text{For each } (L_1, L_2, L_3) \in \mathcal{L} \times \mathcal{L} \times \mathcal{L},\\
&\quad \text{the cardinality of } \{c(i,j,k) \mid i \in L_1, j \in L_2, k \in L_3\} \text{ is } 8.
\end{aligned} \tag{5}$$

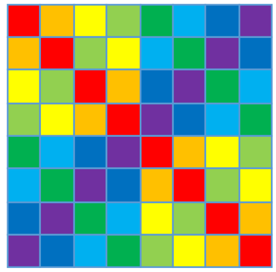


Fig. 5. Conditions of Puzzle B_S expressed as colors.

For Puzzle B_S, we have the condition that $\mathcal{L} = \delta(\mathcal{L}')$ satisfies (5) for $\mathcal{L}' \in \widetilde{\mathcal{L}}_0$ and $\delta \in U_S$. Here, $\delta(\{L_1, L_2, L_3, L_4\}) = \{\delta(L_1), \delta(L_2), \delta(L_3), \delta(L_4)\}$ and $\delta(\{i, j\}) = \{\delta(i), \delta(j)\}$. We define $\widetilde{\mathcal{L}}_S = \{\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \mathcal{L}_4, \mathcal{L}_5, \mathcal{L}_6\}$ for

$$\begin{aligned} \mathcal{L}_4 &= \{\{0,6\},\{2,4\},\{1,7\},\{3,5\}\}, \\ \mathcal{L}_5 &= \{\{0,5\},\{1,4\},\{2,7\},\{3,6\}\}, \\ \mathcal{L}_6 &= \{\{0,3\},\{1,2\},\{4,7\},\{5,6\}\}. \end{aligned}$$

See Fig. 4 for the meaning of $\mathcal{L}_4$. We have $\alpha(\beta(\mathcal{L}_1)) = \mathcal{L}_4$, $\beta(\gamma(\mathcal{L}_2)) = \mathcal{L}_5$, and $\gamma(\alpha(\mathcal{L}_3)) = \mathcal{L}_6$. In addition, $\widetilde{\mathcal{L}}_S$ is closed under the action of U_S. Therefore, the condition can be restated as the requirement that (5) is satisfied for $\mathcal{L} \in \widetilde{\mathcal{L}}_S$. The condition that (5) is satisfied for $\mathcal{L}_4$, $\mathcal{L}_5$, and $\mathcal{L}_6$ says that on each of the 24 8×8-planes, different D-colors are assigned to those cubes with the same color in Figure 5.

For Puzzle B, we have the condition that $\mathcal{L} = \delta(\mathcal{L}')$ satisfies (2) for $\mathcal{L}' \in \widetilde{\mathcal{L}}_0$ and $\delta \in U$. In this case, $\mathcal{L}$ ranges over all the 12 divisions of D into four pairs that do not contain i and $\mathrm{inv}(i)$ for $i \in D$ and that if i and j are paired, then $\mathrm{inv}(i)$ and $\mathrm{inv}(j)$ are also paired. We will denote this set by $\widetilde{\mathcal{L}}$. One can see by Fig. 4 that most of the sets of eight cubes which ought to have different colors by condition (5) for $\mathcal{L} = \alpha(\mathcal{L}_1)$ are not on 8×8-planes.

We summarize these results.

Proposition 3. *Let $c : D \times D \times D \to D$ be a coloring.*

(a) c is a solution of Puzzle B_0 if and only if (4) is satisfied and (5) is satisfied for $\mathcal{L} \in \widetilde{\mathcal{L}}_0$.

(b) c is a solution of Puzzle B_S if and only if (4) is satisfied and (5) is satisfied for $\mathcal{L} \in \widetilde{\mathcal{L}}_S$.

(c) c is a solution of Puzzle B if and only if (4) is satisfied and (5) is satisfied for $\mathcal{L} \in \widetilde{\mathcal{L}}$.

We obtained the result that Puzzle B has no solution using a computer program, and this fact was verified using the SAT solver miniSAT version 2.2.0.

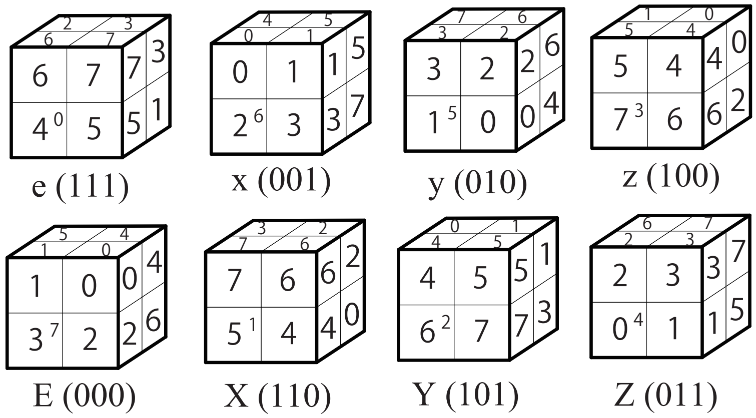


Fig. 6. Coloring $d(k) = a \oplus k$ of the unit cube for $a \in D$. The names E, X, Y, Z, e, x, y, z of the colorings are used in Fig. 7.

6 Constructions of solutions of Puzzle B_0 and Puzzle B_S

We consider D as a linear space over the finite field $F_2 = \{0, 1\}$ and use $\oplus$ for addition in D, which is the bitwise "exclusive or" operation. We construct solutions of Puzzle B_0 and Puzzle B_S by considering the address space $D \times D \times D$ and the color space D as linear spaces and restricting the coloring function $c : D \times D \times D \to D$ to linear functions. Thus,

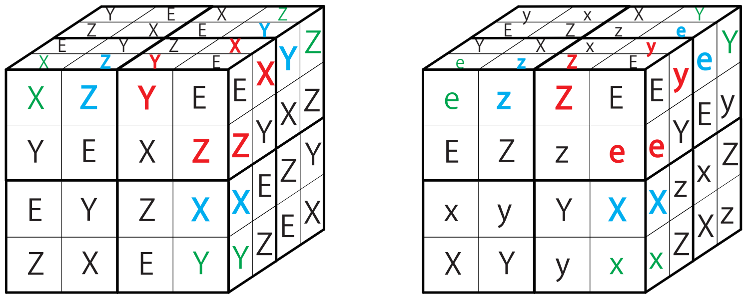
$$c(i, j, k) = c(i, 0, 0) \oplus c(0, j, 0) \oplus c(0, 0, k). \tag{6}$$

We call a $2 \times 2 \times 2$ grid of cubes a unit cube and give address to the set of unit cubes with $D \times D$. Through change of colors, we fix the coloring of the unit cube at $(0, 0)$ as $c(0, 0, k) = k$. We define $\varphi(i) = c(i, 0, 0)$ and $\psi(j) = c(0, j, 0)$. Thus, we have

$$c(i, j, k) = \varphi(i) \oplus \psi(j) \oplus k. \tag{7}$$

Note that the coloring $d : D \to D$ of the unit cube at (i, j) is $d(k) = a \oplus k$ for $a = \varphi(i) \oplus \psi(j)$. We list such colorings in Figure 7. They are rotations and reflections of the coloring of the unit cube at (0,0). When $a = 001, 010, 100$, it is the image of reflection through the yz-, zx-, xy-coordinate plane , respectively; when $a = 110, 101, 011$, it is the image of a 180-degree rotation along x-, y-, z-coordinate axis, respectively, and when $a = 111$, it is the image of the antipodal map. Note that these maps form an Abelian group of order 8.

The linear map φ is determined by $\varphi(001), \varphi(010), \varphi(100)$, and ψ is determined by $\psi(001), \psi(010), \psi(100)$. Therefore, the coloring is determined by these six elements of D. We consider conditions on φ and ψ so that (7) forms a solution of Puzzle B_0 and Puzzle B_S. Condition (4) is automatically satisfied. As a part of condition (5) for $\mathcal{L} = \mathcal{L}_1$, it says that $c(i, j, k)$ for $i, j, k \in \{000, 001\}$



(a) (b)

Fig. 7. (a)Solution of Puzzle B_0 in Example 5. (b) Solution of Puzzle B_S in Example 6. Meanings of the names E, X, Y, Z, e, x, y, z are given in Fig. 6. Red names are values of φ at 001, 010, 110 and cyan ones are values of ψ at 001, 010, 110. They determine green ones as $\varphi(a) \oplus \psi(a)$ for $a \in \{001, 010, 100\}$ and the rest just as a three-dimensional group multiplication table.

are all different and therefore the cardinality of $\{\varphi(i) \oplus \psi(j) \oplus k \mid i, j, k \in \{000, 001\}\}$ is 8. That is, $\{\varphi(001), \psi(001), 001\}$ is linearly independent in the linear space D over F_2. Similarly, condition (5) for $\mathcal{L} = \mathcal{L}_2$ and $\mathcal{L} = \mathcal{L}_3$ imply that $\{\varphi(010), \psi(010), 010\}$ and $\{\varphi(100), \psi(100), 100\}$ are linearly independent in D, respectively. We show in Theorem 4(a) that they form a necessary and sufficient condition for Puzzle B_0.

Theorem 4. *(a) Coloring (7) is a solution of Puzzle* B_0 *if and only if each of the sets of vectors* $\{\varphi(001), \psi(001), 001\}$, $\{\varphi(010), \psi(010), 010\}$, $\{\varphi(100), \psi(100), 100\}$ *is linearly independent.*

(b) Coloring (7) is a solution of Puzzle B_S *if and only if, in addition to the three sets of vectors in (a), each of the sets of vectors* $\{\varphi(110), \psi(110), 110\}$, $\{\varphi(101), \psi(101), 101\}$, $\{\varphi(011), \psi(011), 011\}$ *is linearly independent.*

Note that $\{\varphi(a), \psi(a), a\}$ is linearly independent in D if and only if $\varphi(a), \psi(a)$, a and $\varphi(a)\psi(a)$ are all different.

Proof. Let $a \in \{001, 010, 100, 110, 101, 011\}$. For any pair of elements (b, c) such that $\{a, b, c\}$ is linearly independent, the eight elements $\{000, a, b, c, a \oplus b, b \oplus c, c \oplus a, a \oplus b \oplus c\}$ are all different and the set $\mathcal{L}_a = \{\{000, a\}, \{b, b \oplus a\}, \{c, c \oplus a\}, \{b \oplus c, a \oplus b \oplus c\}\}$ is uniquely determined by a. Since $\mathcal{L}_i (1 \leq i \leq 6)$ is $\mathcal{L}_a$ for $a = 001, 010, 100, 110, 101, 011$, respectively, in order to show (a) and (b), we prove that $\{\varphi(a), \psi(a), a\}$ is linearly independent if and only if, the cardinality of $\{\varphi(i) \oplus \psi(j) \oplus k \mid i \in L_1, j \in L_2, k \in L_3\}$ is 8 for each $(L_1, L_2, L_3) \in \mathcal{L}_a^3$.

For the if part, consider the case $L_1 = L_2 = L_3 = \{000, a\}$. Since the cardinality of $\{\varphi(i) \oplus \psi(j) \oplus k \mid i, j, k \in \{000, a\}\}$ is eight, $\{\varphi(a), \psi(a), a\}$ is linearly independent.

For the only-if part, since $\{\varphi(a), \psi(a), a\}$ is linearly independent, the cardinality of $X = \{\varphi(i) \oplus \psi(j) \oplus k \mid i, j, k \in \{000, a\}\}$ is 8. Let $d_1, d_2, d_3 \in \{000, b, c, b \oplus c\}$. By adding $\varphi(d_1) \oplus \psi(d_2) \oplus d_3$ to each element of X, we have the set $Y = \{\varphi(d_1 \oplus i) \oplus \psi(d_2 \oplus j) \oplus (d_3 \oplus k) \mid i, j, k \in \{000, a\}\}$ whose cardinality is also 8. Let $L_1 = \{d_1, d_1 \oplus a\}$, $L_2 = \{d_2, d_2 \oplus a\}$, $L_3 = \{d_3, d_3 \oplus a\}$. One can see that Y is equal to $\{\varphi(i) \oplus \psi(j) \oplus k \mid i \in L_1, j \in L_2, k \in L_3\}$. Since $\{d, d \oplus a\}$ takes all elements of $\mathcal{L}_a$ if d ranges over $\{000, b, c, b \oplus c\}$, we have the result. $\square$

Example 5. We present a solution of Puzzle B_0 (see Fig. 7(a)). φ and ψ defined as $\varphi(001) = 011$, $\psi(001) = 101$, $\varphi(010) = 110$, $\psi(010) = 011$, $\varphi(100) = 101$, $\psi(100) = 110$ satisfies the condition of Theorem 4(a). This solution consists of only four colorings E, X, Y, Z of the unit cube in Figure 6, which are the identity map and 180-degree rotations around the three axes. This is not a solution of Puzzle B_S, because $\varphi(110) = 011$, $\psi(110) = 101$, and 110 are linearly dependent.

Example 6. We give a solution of Puzzle B_S. See Fig. 7(b). In order to show its symmetric structure, we present $\mathrm{seq}(i) = (\varphi(i), \psi(i), \varphi(i)\psi(i))$ instead of $(\varphi(i), \psi(i))$ for $i \in \{001, 010, 100\}$.

$$\begin{aligned}
\mathrm{seq}(001) &= (100, 011, 111),\\
\mathrm{seq}(010) &= (110, 111, 001),\\
\mathrm{seq}(100) &= (111, 010, 101).
\end{aligned}$$

We can calculate the followings

$$\begin{aligned}
\mathrm{seq}(110) &= (001, 101, 100),\\
\mathrm{seq}(101) &= (011, 001, 010),\\
\mathrm{seq}(011) &= (010, 100, 110),
\end{aligned}$$

and see that it satisfies the condition of Theorem 4(b). Using a computer program, we found 480 solutions of Puzzle B_S that satisfy the condition of Theorem 4(b).

Through computer calculation, we have obtained 1148928 solutions of Puzzle B_S modulo change of colors, and this number is verified by a #SAT solver sharpSAT version 1.1 [6]. The enumeration of the solutions of Puzzle B_0 is an open problem.

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