

**DYADIC SUBBASES AND EFFICIENCY  
PROPERTIES OF THE INDUCED  
 $\{0, 1, \perp\}^\omega$ -REPRESENTATIONS**

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ABSTRACT. A dyadic subbase induces a representation of a second-countable regular space as a subspace of the space of infinite sequences of  $\{0, 1, \perp\}$ , which is known as Plotkin's  $\mathbb{T}^\omega$ . We study four properties of dyadic subbases – full-representing, canonically representing, independent, and minimal – which express efficiency properties of the induced representations.

1. INTRODUCTION

When a subbase (with an enumeration)  $S = (S_0, S_1, \dots)$  of a  $T_0$  space  $X$  is given, we can represent  $X$  as a subspace of  $\mathcal{P}_\omega$ , the powerset of  $\mathbb{N}$ , through the map  $\varphi'_S : X \rightarrow \mathcal{P}_\omega$  defined as  $\varphi'_S(x) = \{n \mid S_n \ni x\}$ , or equivalently, we can represent each point of the space as an infinite sequence of 1 and  $\perp$ . When  $X$  is a second-countable regular space and the subbase  $S = (S_0^0, S_0^1, S_1^0, S_1^1, \dots)$  is dyadic in the sense that  $S_n^j$  are regular open and  $S_n^0$  and  $S_n^1$  are exteriors of each other, we can represent  $X$  also as a subspace of Plotkin's  $\mathbb{T}^\omega$ , which is the set of infinite sequences of  $\mathbb{T} = \{0, 1, \perp\}$  [6]. Here, the representation function  $\varphi_S : X \rightarrow \mathbb{T}^\omega$  is defined as  $\varphi_S(x)[n]$  is 0, 1, and  $\perp$  iff  $x$  is in  $S_n^0$ ,  $S_n^1$ , and on the boundary of

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$S_n^0$  (= the boundary of  $S_n^1$ ), respectively. When  $X$  is realized as a subspace of  $\mathbb{T}^\omega$ , we can study topological properties of  $X$  through the domain structure of  $\mathbb{T}^\omega$ , and we can define computation on  $X$  through string manipulation on  $\mathbb{T}^\omega$  [7, 8]. Pairs of regular opens which are exteriors of each other are the maximal points of the domain studied in [2].

We give two examples of dyadic subbases and the induced representations of the closed unit interval  $\mathbb{I} = [0, 1]$ .

**Example 1.1** (Dedekind subbase). *Fix a numbering  $q_i$  of rational numbers in  $(0, 1)$ . Define the dyadic subbase  $D = (D_0^0, D_0^1, D_1^0, D_1^1, \dots)$  as  $D_n^0 = [0, q_n)$  and  $D_n^1 = (q_n, 1]$ . The induced representation  $\varphi_D : [0, 1] \rightarrow \mathbb{T}^\omega$  is  $\varphi_D(x)[n] = 0, \perp$ , and  $1$  iff  $x < q_n$ ,  $x = q_n$ , and  $x > q_n$ , respectively.*

**Example 1.2** (Gray subbase [7]). *We first define the representation function  $\varphi_G$ . Let the tent function  $t : \mathbb{I} \rightarrow \mathbb{I}$  be*

$$t(x) = \begin{cases} 2x & (0 \leq x \leq 1/2) \\ 2(1-x) & (1/2 < x \leq 1) \end{cases}$$

and the function  $P : \mathbb{I} \rightarrow \mathbb{T}$

$$P(x) = \begin{cases} 0 & (x < 1/2) \\ \perp & (x = 1/2) \\ 1 & (x > 1/2) \end{cases} .$$

Gray embedding  $\varphi_G$  is a function from  $\mathbb{I}$  to  $\mathbb{T}_{\perp,1}^\omega$  defined as  $\varphi_G(x)[n] = P(t^n(x))$  ( $n = 0, 1, \dots$ ). Figure 1 shows this embedding. Here, a horizontal line means that the corresponding bit has value 1, and the edge of each line corresponds to the value  $\perp$ . It is just the itinerary of the tent function, which is essential for symbolic dynamical systems [5]. It is also just the expansion of  $[0, 1]$  with binary reflected Gray-code, which is a binary coding of natural numbers other than the ordinary one [4].

Define  $G_n^0 = \{x \mid \varphi_G(x)[n] = 0\}$  and  $G_n^1 = \{x \mid \varphi_G(x)[n] = 1\}$ .  $G = (G_0^0, G_0^1, G_1^0, G_1^1, \dots)$  forms a dyadic subbase of  $[0, 1]$ .

When we compare these two representations, one can see that  $\varphi_G$  is, in some sense, more efficient than  $\varphi_D$ , and  $\varphi_D$  is more redundant than  $\varphi_G$ . To characterize this efficiency of  $\varphi_G$ , we define four properties of dyadic subbases, that is, full-representing, canonically representing, independent, and minimal. Full-representing roughly

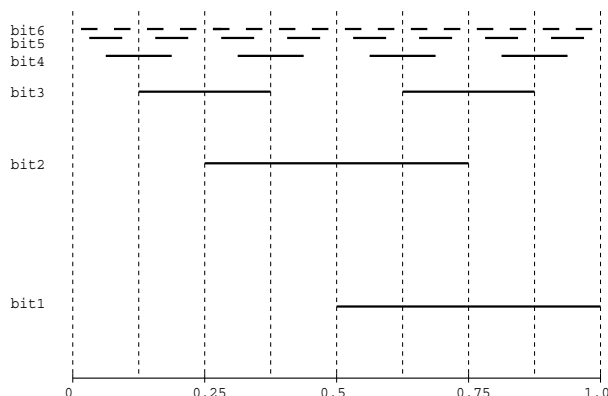


FIGURE 1. Gray code Embedding of real numbers.

means that every  $\Sigma^\omega$  sequence appears as a representation of some point when we fill the bottoms with 0 and 1, and canonically representing means that every digit of the induced representation is indispensable to identify a point. Independent subbase is analogous to independent family of subsets, and minimal subbase is simply defined as a minimal dyadic subbase.

After defining (proper) dyadic subbases in Section 3, we define the above four properties and show that the strength of them decrease in this order in Section 4. In Section 5, we study the case  $X$  is compact and show that the first three properties are equivalent. In Section 6, we give examples of such subbases, and mention some open problems.

## 2. NOTATIONS

We will write  $\text{Int } O, \text{Ext } O, \text{Bd } O, \overline{O}$  for the interior, exterior, boundary, and closure of  $O$ , respectively. An open set  $O$  is *regular* when  $\text{Int } \overline{O} = O$ , and in this case, we have  $\text{Ext } \text{Ext } O = O$ .

Throughout this paper,  $\Sigma$  is the alphabet  $\{0, 1\}$ . We write  $\mathbb{T}$  for  $\Sigma \cup \{\perp\}$ . We write  $\Sigma^\omega$  and  $\mathbb{T}^\omega$  for the set of infinite sequences of  $\Sigma$  (i.e. the Cantor set) and  $\mathbb{T}$ , respectively. Each element of  $\mathbb{T}^\omega$  is called a *bottomed sequence*. We call each copy of 0 and 1 which appears in  $d \in \mathbb{T}^\omega$  a *digit* of  $d$ . A *finite (an infinite) element* of  $\mathbb{T}^\omega$  is an element with finite (infinite) number of digits. When we express

a finite element of  $\mathbb{T}^\omega$ , we omit the  $\perp^\omega$  which comes at the end of the sequence. For example, we write  $\perp 110\perp 1$  for  $\perp 110\perp 1\perp^\omega \in \mathbb{T}^\omega$ . We write  $K(\mathbb{T}^\omega)$  and  $L(\mathbb{T}^\omega)$  for the set of finite and infinite elements of  $\mathbb{T}^\omega$ , respectively. We use variables  $p, q$ , and  $r$  for elements of  $\mathbb{T}^\omega$ ,  $d$  and  $e$  for elements of  $K(\mathbb{T}^\omega)$ ,  $c$  for elements of  $\mathbb{T}$ , and  $a$  for elements of  $\Sigma$  (i.e. digits). *not* is a function to invert a digit, i.e.,  $\text{not}(0) = 1$  and  $\text{not}(1) = 0$ .

We consider the order  $1 > \perp$  and  $0 > \perp$  on  $\mathbb{T}$ , and its product order on  $\mathbb{T}^\omega$ . That is,  $p \geq q$  iff  $p[n] = q[n]$  holds when  $q[n]$  is a digit.  $(\mathbb{T}^\omega, \leq)$  is an  $\omega$ -algebraic domain, and  $\Sigma^\omega$  is the set of maximal elements of  $(\mathbb{T}^\omega, \leq)$ . This paper is related to, but does not explicitly use domain theory [1]. We write  $\uparrow p$  for  $\{r \mid r \geq p\}$ . When  $p$  and  $q$  have a common upper bound (i.e.,  $r \geq p$  and  $r \geq q$  for some  $r$ ), we say that  $p$  and  $q$  are *compatible*, and write  $p \uparrow q$ . One can see that  $p \uparrow q$  iff  $p[n] = q[n]$  holds when both  $p[n]$  and  $q[n]$  are digits.

We call the number of digits in  $p$  the *length* of  $p$ . When  $p \in \mathbb{T}^\omega$ , we write  $p|_n$  for the finite element of length  $n$  which has the first  $n$  digits of  $p$ .  $p|_n$  satisfies  $p|_n \leq p$ . We write  $p[n := c] \in \mathbb{T}^\omega$  for  $p$  with the  $n$ -th component replaced by  $c \in \mathbb{T}$ .

We consider the topology  $\{\emptyset, \{0\}, \{1\}, \{0, 1\}, \mathbb{T}\}$  on  $\mathbb{T}$  and its product topology on  $\mathbb{T}^\omega$ , which coincides with the Scott topology of the domain  $(\mathbb{T}^\omega, \leq)$ .

### 3. DYADIC SUBBASE

**Definition 3.1.** Let  $X$  be a Hausdorff space. We call a countable subbase  $S = (S_0^0, S_0^1, S_1^0, S_1^1 \dots)$  of  $X$  with a pairing and an enumeration of the pairs a *dyadic subbase* when  $S_n^j$  ( $n = 0, 1, 2, \dots, j = 0, 1$ ) are regular open and  $S_n^0$  and  $S_n^1$  are exteriors of each other.

A space is called *semiregular* when it is Hausdorff and it has a base which consists of regular open sets. When a Hausdorff space  $X$  has a dyadic subbase,  $X$  is a second-countable semiregular space because the intersection of two regular open sets is also regular open. On the other hand, we have

**Proposition 3.2.** *Every second-countable semiregular space has a dyadic subbase.*

*Proof.* Let  $\mathcal{B}$  be a countable base of a second-countable semiregular space  $X$ . Let  $\mathcal{P} = \{(B_1, B_2) \mid B_1, B_2 \in \mathcal{B} \text{ and there is a regular open set } U \text{ such that } B_1 \subseteq U \subseteq B_2\}$ . For each  $P = (B_1, B_2) \in \mathcal{P}$ , we choose a regular open set  $U_P$  such that  $B_1 \subseteq U_P \subseteq B_2$ . Then,  $\mathcal{U} = \{U_P \mid P \in \mathcal{P}\}$  is a countable regular base. By extending it with the exteriors of each element, we have a dyadic subbase of  $X$ .  $\square$

Since a regular space is semiregular, every second-countable regular space has a dyadic subbase. Though we are mainly interested in regular spaces, we use regularity only in Theorem 5.2 and we only assume that  $X$  is Hausdorff when we say that  $X$  has a dyadic subbase. See the open problems on regularity in Section 6.

**Proposition 3.3.** *When  $S$  is a dyadic subbase of  $X$ , we have a topological embedding  $\varphi_S : X \rightarrow \mathbb{T}^\omega$  defined as*

$$\varphi_S(x)[n] = \begin{cases} 0 & (x \in S_n^0) \\ \perp & (x \in \text{Bd } S_n^0 = \text{Bd } S_n^1) \\ 1 & (x \in S_n^1) \end{cases} .$$

*Proof.*  $\varphi_S$  is one-to-one because  $X$  is a Hausdorff space. It is an embedding because  $\varphi_S$  induces a correspondence between the open base generated by  $S$  and the family  $\{\uparrow e \cap \text{Image}(\varphi_S) \mid e \in K(\mathbb{T}^\omega)\}$ , which is an open-base of  $\text{Image}(\varphi_S)$ .  $\square$

**Definition 3.4.** We define  $\psi_S, \bar{\psi}_S : \mathbb{T}^\omega \rightarrow \mathcal{P}(X)$  as follows

$$\begin{aligned} \psi_S(p) &= \bigcap S_n^{p[n]}, \\ \bar{\psi}_S(p) &= \bigcap \overline{S_n^{p[n]}}. \end{aligned}$$

Here, we define  $S_n^\perp = X$ .

Both  $\psi_S$  and  $\bar{\psi}_S$  are anti-monotonic ( $\psi_S(p) \supseteq \psi_S(q)$  and  $\bar{\psi}_S(p) \supseteq \bar{\psi}_S(q)$  if  $p \leq q$ ) and  $\psi_S(p) \subseteq \bar{\psi}_S(p)$ . We can consider each infinite sequence  $p$  in  $\mathbb{T}^\omega$  as a specification of points in  $X$  in two ways. The first one is to consider each digit  $a$  on the  $n$ -th cell of  $p$  as giving the specification that the point is in  $S_n^a$ . The other one is to consider it as giving the specification that the point is in  $\overline{S_n^a}$ .  $\psi_S(p)$  and  $\bar{\psi}_S(p)$  are just the sets of points satisfying the specification  $p$  in these two ways, respectively.  $\{\psi_S(d) \mid d \in K(\mathbb{T}^\omega)\}$  is the base generated by  $S$ .

The sets  $\psi_S(p)$  and  $\overline{\psi_S(p)}$  are also expressed using the order relation on  $\mathbb{T}^\omega$  as follows.

**Proposition 3.5.** 1)  $\psi_S(p) = \varphi_S^{-1}(\uparrow p)$ .  
2)  $\overline{\psi_S(p)} = \varphi_S^{-1}(\{q \mid p \uparrow q\})$ .

*Proof.* 1) Obvious.

2)  $p \uparrow q$  iff  $p[n] \neq \perp$  implies  $(q[n] = p[n] \text{ or } q[n] = \perp)$ . Therefore,  $p \uparrow \varphi_S(x)$  iff  $x \in S_n^{p[n]}$  for every  $n$ .  $\square$

In the following, we place one more condition on dyadic subbases which connects these two interpretations of  $p \in \mathbb{T}^\omega$ .

**Definition 3.6.** A dyadic subbase is *proper* if  $\overline{\psi_S(d)} = \overline{\psi_S(d)}$  for  $d \in K(\mathbb{T}^\omega)$ .

Both the Dedekind subbase and the Gray subbase defined in the introduction are proper.

**Example 3.7.** An example of a non-proper dyadic subbase of  $\mathbb{I}$ . Replace the first two components  $[0, 1/2)$  and  $(1/2, 1]$  of the Gray subbase with the four elements  $[0, 1/2) \cup (5/8, 3/4)$ ,  $[0, 1/2) \cup (3/4, 7/8)$ ,  $(1/2, 1] \cup (1/8, 1/4)$ ,  $(1/2, 1] \cup (1/4, 3/8)$  and their exteriors. As another example, simply duplicate a non-clopen component  $(S_k^0, S_k^1)$  of a proper dyadic subbase.

**Proposition 3.8.** Let  $S$  be a proper dyadic subbase. When  $x \neq y$ ,  $x$  and  $y$  are separated by  $S_n^0$  and  $S_n^1$  for some  $n$ .

*Proof.* Suppose that  $x$  and  $y$  are separated by open base elements  $\psi_S(d)$  and  $\psi_S(e)$ . Then,  $x \in \psi_S(d)$  and  $y \notin \overline{\psi_S(d)} = \overline{\psi_S(d)}$ . Therefore, for some  $n$ ,  $x \in S_n^{d[n]}$  and  $y \notin S_n^{d[n]}$ , and thus  $y \in S_n^{not(d[n])}$ .  $\square$

In Example 3.7, the two points  $1/4$  and  $3/4$  cannot be separated by subbase elements. Thus, it gives a counter-example to Proposition 3.8 when  $S$  is not proper.

**Proposition 3.9.** When  $S$  is a proper dyadic subbase, for  $p, q \in \text{Image}(\varphi_S)$ ,  $p \uparrow q$  implies  $p = q$ .

*Proof.* Suppose that  $p = \varphi_S(x)$  and  $q = \varphi_S(y)$  satisfy  $p \uparrow q$ . If  $p \neq q$ ,  $x$  and  $y$  are separated by  $S_n^0$  and  $S_n^1$  for some  $n$ . Thus,  $p[n]$  and  $q[n]$  are different digits. This contradicts  $p \uparrow q$ .  $\square$

**Proposition 3.10.** (1) When  $S$  is a dyadic subbase,  $\psi_S \circ \varphi_S(x) = \{x\}$ , and  $\psi_S(q) = \emptyset$  for all  $q \not\geq \varphi_S(x)$ .

(2) When  $S$  is a proper dyadic subbase,  $\overline{\psi}_S \circ \varphi_S(x) = \{x\}$ . Moreover,  $\overline{\psi}_S(q) = \{x\}$  for all  $q \geq \varphi_S(x)$ .

*Proof.* (1) Obvious. Note that  $q \geq \varphi_S(x)$  means that there is an index  $n$  of  $q$  such that  $q[n] = 0$  or  $1$  and  $\varphi_S(x)[n] = \perp$ .

(2) Suppose that  $y \in \overline{\psi}_S \circ \varphi_S(x)$ . It means that  $y$  is in the closure of every element  $S_n^j$  such that  $x \in S_n^j$ . Since  $S_n^j \cap S_m^i = S_n^j \cap S_m^i$ ,  $y$  is in the closure of every base element containing  $x$ . Since  $X$  is Hausdorff, it means  $y = x$ . Since  $\overline{\psi}_S$  is anti-monotonic,  $\overline{\psi}_S(q) \subseteq \{x\}$  when  $q \geq \varphi_S(x)$ .  $\square$

#### 4. FOUR PROPERTIES OF DYADIC SUBBASES

We consider four conditions each of which distinguishes the Gray subbase from the Dedekind subbase.

From Proposition 3.10, every bottomed sequence  $q \geq \varphi_S(x)$  specifies  $\{x\}$  with respect to  $\overline{\psi}_S$  when  $S$  is a proper dyadic subbase. First, we consider the condition that the converse is also true.

**Definition 4.1.** A proper dyadic subbase  $S$  is called a *canonically representing subbase* if  $\overline{\psi}_S(q) = \{x\}$  implies  $q \geq \varphi_S(x)$ .

It is obvious that if  $q$  is not compatible with  $\varphi_S(x)$ , then  $x$  is not in  $\overline{\psi}_S(q)$ . Therefore, this condition is equivalent to saying that when  $q \not\geq \varphi_S(x)$ ,  $\overline{\psi}_S(q)$  includes at least one point other than  $x$ . Since  $\overline{\psi}_S$  is anti-monotonic, we only need to consider the case  $q$  is maximal, i.e.,  $q = \varphi_S(x)[n := \perp]$  for an index  $n$  such that  $\varphi_S(x)[n]$  is a digit. Thus,  $S$  is canonically representing means that the representation function  $\varphi_S$  has no redundancy in that if we erase any digit of  $\varphi_S(x)$ , then there is another point satisfying the specification with respect to  $\overline{\psi}_S$  and the sequence no longer identifies the point  $x$ .

Next, we consider the following condition.

**Definition 4.2.** A proper dyadic subbase  $S$  is called a *full-representing subbase* if for each  $p \in \Sigma^\omega$ , there is an element  $z \in X$  such that  $\varphi_S(z) \leq p$ .

Note that such an element  $z$  is unique by Proposition 3.9. In Type-2 Theory of Effectivity[9], a surjective partial function from

$\Sigma^\omega$  to  $X$  is called a representation. When a dyadic subbase  $S$  is given, we can define such a representation  $\rho_S : \Sigma^\omega \rightarrow X$  as  $\rho_S(p) = x$  iff  $x \in \overline{S_n^{p[n]}}$  for all  $n$ . The condition that  $S$  is full-representing means that  $\rho_S$  is a total function from  $\Sigma^\omega$  to  $X$ . Thus, in this case,  $X$  is a dyadic space (the continuous image of the Cantor cube  $D^m$  for  $m \geq \aleph_0$  [3, 3.12.12]). Note that not all dyadic spaces have full-representing subbases. See Corollary 6.2.

Thirdly, we define independent subbase. A countable family  $A_0, A_1, \dots$  of subsets of  $X$  is called an independent family iff  $\bigcap A_i^{d(i)} \neq \emptyset$  for all  $d \in K(\mathbb{T}^\omega)$ , when we define  $A^0 = A$ ,  $A^1 = X \setminus A$ , and  $A^\perp = X$ . By analogy, we define as follows.

**Definition 4.3.** A proper dyadic subbase  $S$  is called an *independent subbase* if  $\psi_S(d) \neq \emptyset$  for all  $d \in K(\mathbb{T}^\omega)$ .

It is equivalent to saying that  $\text{Image}(\varphi_S)$  is dense in  $\mathbb{T}^\omega$ . Finally, we define as follows.

**Definition 4.4.** A proper dyadic subbase is *minimal* if any proper subset is not a dyadic subbase.

**Proposition 4.5.** (1) *The subbase  $G$  is canonically representing, full-representing, independent, and minimal.*

(2) *The subbase  $D$  is not canonically representing, nor full-representing, nor independent, nor minimal.*

We study relations among these four properties.

**Proposition 4.6.** *A full-representing subbase is canonically representing.*

*Proof.* Suppose that  $S$  is a full-representing subbase of  $X$ . Let  $x$  be any point of  $X$ ,  $q = \varphi_S(x)$ , and  $q[n] \neq \perp$ . We will show that  $\overline{\psi_S(q[n := \perp])} \neq \{x\}$ . Without loss of generality, we assume  $q[n] = 0$ . Let  $q' = q[n := 1]$  and  $p$  be any element in  $\Sigma^\omega$  such that  $q' \leq p$ . From the assumption, there is an element  $y$  such that  $\varphi_S(y) \leq p$ . We have  $x \neq y$  because  $x \in S_n^0$  and  $y \in \overline{S_n^1}$ . We have  $y \in \overline{\psi_S(q[n := \perp])}$  because  $q[n := \perp] < p \geq \varphi_S(y)$  and Proposition 3.5(2).  $\square$

**Proposition 4.7.** *A canonically representing subbase of a non-empty space is independent.*



*Proof.* Suppose that  $P = \{d \in K(\mathbb{T}^\omega) \mid \psi_S(d) = \emptyset\}$  is not empty and  $d$  is an element of  $P$  which includes minimal number of digits. By reindexing the subbase, we can assume that  $d = 0^n$  ( $n \geq 1$ ). When  $n = 1$ , we have  $S_0^0 = \emptyset$  and  $S_0^1 = X$ . Therefore, for every element  $x \in X$ ,  $\varphi_S(x)[0] = 1$  and  $q = \varphi_S(x)[0 := \perp]$  satisfies  $\overline{\psi}_S(q) = \{x\}$ . Thus,  $S$  is not canonically representing.

When  $n > 1$ , consider the sets  $\psi_S(0)(= S_0^0)$ ,  $\psi_S(1)(= S_0^1)$ , and  $\psi_S(\perp 0^{n-1})$ . By assumption, they are not empty and  $\psi_S(0) \cap \psi_S(\perp 0^{n-1}) = \psi_S(0^n) = \emptyset$ . Then, since  $\psi_S(\perp 0^{n-1})$  is an open set which does not intersect with  $\psi_S(0)$ , it is in the exterior of  $\psi_S(0)$ . That is, we have  $\psi_S(\perp 0^{n-1}) \subseteq \psi_S(1)$ . Since  $\psi_S(\perp 0^{n-1})$  is not empty, take a point  $x \in \psi_S(\perp 0^{n-1})$ . We have  $\varphi_S(x)[0] = 1$  because  $x \in \psi_S(1)$ . Let  $p = \varphi_S(x)[0 := \perp] \in \mathbb{T}^\omega$ . Suppose that  $y \in \overline{\psi}_S(p)$ . Then,  $y$  belongs to  $\bigcap_{1 \leq i \leq n-1} \overline{S_i^0}$ , which is equal to  $\bigcap_{1 \leq i \leq n-1} \overline{S_i^0} = \overline{\psi_S(\perp 0^{n-1})}$  because  $S$  is proper. Since  $\psi_S(\perp 0^{n-1}) \subseteq \psi_S(1)$ ,  $y \in \overline{\psi}_S(1)$  and thus  $y \in \overline{\psi}_S(\varphi_S(x))$ . Therefore, we have  $y = x$ . It means that  $\overline{\psi}_S(p) = \{x\}$ , and contradicts the fact that  $S$  is canonically representing.  $\square$

**Proposition 4.8.** *When  $S$  is a proper dyadic subbase, the following conditions are equivalent.*

- 1)  $S$  is an independent subbase.
- 2) If  $d, e \in K(\mathbb{T}^\omega)$  are compatible, then  $\psi_S(d) \cap \psi_S(e) \neq \emptyset$ .
- 3) If  $d, e \in K(\mathbb{T}^\omega)$  satisfy  $\psi_S(d) \subseteq \psi_S(e)$ , then  $d \geq e$ .
- 4) If  $d, e \in K(\mathbb{T}^\omega)$  satisfy  $\psi_S(d) = \psi_S(e)$ , then  $d = e$ .

*Proof.* (1)  $\leftrightarrow$  (2) is obvious from the definition.

(2)  $\rightarrow$  (3) Suppose that  $\psi_S(d) \subseteq \psi_S(e)$ . We need to show that, if  $e[n]$  is a digit  $a$ , then  $d[n] = a$ . If  $d[n] = \text{not}(a)$ , then  $\psi_S(d) \cap \psi_S(e) = \emptyset$  and thus contradicts. Suppose that  $d[n] = \perp$ . Let  $d' = d[n := \text{not}(a)]$ . We have  $d' > d$  and thus  $\psi_S(d') \subseteq \psi_S(d) \subseteq \psi_S(e)$ . On the other hand,  $d' \not\geq e$  and thus  $\psi_S(d') \cap \psi_S(e) = \emptyset$ . Since  $\psi_S(d') = \psi_S(d') \cap \psi_S(d')$  is not empty, it contradicts.

(3)  $\rightarrow$  (4) Obvious.

(4)  $\rightarrow$  (1) If  $\psi_S(d) = \emptyset$ , then  $\psi_S(e) = \emptyset$  for all  $e > d$ .  $\square$

When  $S$  is an independent subbase,  $\text{Image}(\varphi_S) \subseteq L(\mathbb{T}^\omega)$  because  $\psi_S(d)$  cannot be a one-point set for  $d \in K(\mathbb{T}^\omega)$ .

**Proposition 4.9.** *An independent subbase is minimal.*

*Proof.* Suppose that an independent subbase  $S = (S_0^0, S_0^1, S_1^0, S_1^1, \dots)$  is not minimal and  $S' = (S_1^0, S_1^1, \dots)$  is also a subbase. Then,  $S_0^0 \supset \psi_S(d)$  for some  $d \in K(\mathbb{T}^\omega)$  such that  $d[0] = \perp$ . Thus, for  $d' = d[0 := 1]$ ,  $\psi_S(d') = \emptyset$ .  $\square$

So far, we have proved full-representing  $\Rightarrow$  canonically representing  $\Rightarrow$  independent  $\Rightarrow$  minimal, when the space is not empty. These implications are strict as the following examples show.

**Example 4.10.** *An example of a canonically representing subbase which is not full-representing. Consider the subset  $\mathcal{J} = \{x \in \mathbb{I} \mid x = k/2^m \text{ for } k, m \in \mathbb{N}\}$  and the subspace  $\mathcal{P} = \mathbb{I} \setminus \mathcal{J}$  with the relative topology of  $\mathbb{I}$ .  $\mathcal{P}$  is homeomorphic to the Baire space. Let  $S$  be the subbase on  $\mathcal{P}$  which is relative to the Gray subbase on  $\mathbb{I}$ . We have  $\mathcal{P} = \{x \in \mathbb{I} \mid \varphi_G(x) \text{ contains infinite number of } 1\}$ , and  $\varphi_S(x) \in \Sigma^\omega$  for  $x \in \mathcal{P}$ . Obviously,  $S$  is not full-representing. Suppose that  $x \in \mathcal{P}$  and  $\varphi_S(x)[n] = a$ . Let  $q = \varphi_S(x)[n := \perp]$ .  $\overline{\psi}_G(q)$  is always a two-point set and both of the elements contain infinite number of 1. Therefore,  $\overline{\psi}_S(q) = \overline{\psi}_G(q)$  and thus  $S$  is canonically representing.*

**Example 4.11.** *An example of an independent subbase which is not canonically representing. Consider  $[0, 1)$  with the relative subbase  $S$  of the Gray subbase. Then, since  $\psi_S(10^\omega)$  is empty,  $\psi_S(\perp 0^\omega) = \{0\}$ , whereas  $\varphi_S(0) = 0^\omega$ .*

**Example 4.12.** *An example of a minimal subbase which is not independent. Consider the space  $[0, 1] \cup [2, 3]$  and the subbase  $S$  whose 0th component  $S_0^0$  is  $[0, 1]$ , (thus  $S_0^1 = [2, 3]$ ) and the other components are composed from the Dedekind subbase  $D$  on  $[0, 1]$  and the Gray subbase  $G$  on  $[2, 3]$ , defined as  $S_{n+1}^0 = D_n^0 \cup G_n^0$  and  $S_{n+1}^1 = D_n^1 \cup G_n^1$ .*

## 5. THE COMPACT CASE

We have stronger results when  $X$  is compact.

**Theorem 5.1.** *When  $X$  is compact and  $S$  is a proper dyadic subbase of  $X$ , the followings are equivalent.*

- 1)  $S$  is full-representing.
- 2)  $S$  is canonically representing.
- 3)  $S$  is independent.

*Proof.* (1)  $\rightarrow$  (2): Proposition 4.6.

(2)  $\rightarrow$  (3): Proposition 4.7.

(3)  $\rightarrow$  (1): Let  $p \in \Sigma^\omega$ . Consider the infinite sequence  $\overline{\psi}_S(p|_1) \supset \overline{\psi}_S(p|_2) \supset \dots$  of non-empty closed sets. Since  $X$  is compact, their intersection is not empty. Let  $z$  be in their intersection. Then, we have  $\varphi_S(z)[n] \leq p[n]$  for each  $n$  and thus  $\varphi_S(z) \leq p$ . Therefore,  $z \in \overline{\psi}_S(p)$ . The uniqueness of such an  $z$  is by Proposition 3.9.  $\square$

Note that the spaces in Example 4.10 and 4.11 are not compact but the space in Example 4.12 is compact. Therefore, independent subbase and minimal subbase are different even for the compact case.

**Theorem 5.2.** *Suppose that  $S$  is a full-representing subbase of a space  $X$  with the representation function  $\rho_S : \Sigma^\omega \rightarrow X$ .*

- 1)  $X$  is compact.
- 2)  $X$  is regular.
- 3)  $\rho_S$  is continuous.

*Proof.* (1)  $\rightarrow$  (2) : Every compact Hausdorff space is regular.

(2)  $\rightarrow$  (3) : Let  $p \in \Sigma^\omega$ ,  $x = \rho_S(p)$ , and  $\psi_S(d) \ni x$ . Since  $\{\psi_S(d) \mid d \in K(\mathbb{T}^\omega)\}$  is a base of  $X$ , when  $x \in \psi_S(d)$ , there is  $e \geq d$  such that  $x \in \psi_S(e) \subset \overline{\psi}_S(e) \subset \psi_S(d)$ . Then, for all  $q \in \uparrow e \cap \Sigma^\omega$ , we have  $\varphi_S(\rho_S(q)) \uparrow e$  because  $\varphi_S(\rho_S(q)) \leq q \geq e$ . Therefore,  $\rho_S(q) \in \overline{\psi}_S(e)$  by Proposition 3.5, which is equal to  $\overline{\psi}_S(e)$  because  $S$  is proper. Therefore,  $\rho_S(q) \in \psi_S(d)$ .

(3)  $\rightarrow$  (1) : Because  $\Sigma^\omega$  is compact and  $X$  is the image of  $\rho_S$ .  $\square$

This theorem shows that when we only consider regular spaces, the existence of a full-representing subbase implies compactness. On the other hand, it is still open whether there exists a non-regular space with a full-representing subbase. See Section 6.

**Proposition 5.3.** 1) *When  $X$  is a compact space with a dyadic subbase  $S$ ,  $\varphi_S^{-1}(\Sigma^\omega \cap \text{Image}(\varphi_S))$  is a dense  $G_\delta$  set in  $X$ .*

2) *When  $X$  is a compact space with an independent subbase  $S$ ,  $\Sigma^\omega \cap \text{Image}(\varphi_S)$  is a dense  $G_\delta$  set in  $\Sigma^\omega$ .*

*Proof.* 1) This set is equal to  $\bigcap_n (S_n^0 \cup S_n^1)$ . Since  $S_n^0 \cup S_n^1$  is a dense open subset of  $X$ , it holds because of the Baire category theorem.

2) Recall that  $\text{Image}(\varphi_S)$  is dense in  $\mathbb{T}^\omega$  when  $S$  is an independent subbase, and  $\Sigma^\omega \cap \text{Image}(\varphi_S)$  is dense in  $\text{Image}(\varphi_S)$  by (1). Therefore,  $\Sigma^\omega \cap \text{Image}(\varphi_S)$  is also dense in  $\mathbb{T}^\omega$ , and also dense in  $\Sigma^\omega$ .  $\square$

This proposition shows that the set of points whose representation by  $\varphi_S$  does not include a  $\perp$  is a dense  $G_\delta$  set.

Finally, we consider the possibility of another definition for a canonically representing subbase. The intuitive meaning of a canonically representing subbase is that if we erase a digit from the sequence  $\varphi_S(x)$ , it no longer identifies a point with respect to  $\overline{\psi}_S$ . If we use  $\psi_S$  instead of  $\overline{\psi}_S$ , one may as well define as follows. (See also Proposition 3.10.)

**Definition 5.4.** A dyadic subbase  $S$  is *strongly canonically representing* iff  $\psi_S(q) = \{x\}$  implies  $q = \varphi_S(x)$ .

However, we can prove that, when  $X$  is compact, only the Cantor space has such a subbase.

**Proposition 5.5.** *Suppose that  $S$  is a strongly canonically representing subbase of  $X$ . If  $p \in \text{Image}(\varphi_S)$  and  $q$  is obtained by inverting finite number of digits of  $p$ , then  $q \in \text{Image}(\varphi_S)$ .*

*Proof.* Suppose that  $p = \varphi_S(x)$  and  $p[n] = 0$ . Let  $r = p[n := \perp]$ . Then, since  $\psi_S(r)$  contains an element other than  $x$ , there is  $y \neq x$  such that  $\varphi_S(y) \geq r$ . In particular,  $\varphi_S(y) \geq r[n := 1] = p[n := 1]$  because  $\varphi_S(y) \not\geq r[n := 0] = p$ . In the same way, there is  $z$  such that  $\varphi_S(z) \geq \varphi_S(y)[n := 0]$ . From this construction, we have  $\varphi_S(z) \geq \varphi_S(x)$ . Therefore,  $z = x$ . Thus,  $y$  satisfies  $\varphi_S(y) = \varphi_S(x)[n := 1]$ . Repeating this process, we have the result.  $\square$

**Theorem 5.6.** *A compact space with a strongly canonically representing subbase is homeomorphic to the Cantor space.*

*Proof.* Suppose that  $p \in \text{Image}(\varphi_S)$  and  $p[0] = \perp$ . Let  $q \in \Sigma^\omega$  be any element of  $\Sigma^\omega$  and  $r < q$  be the element which satisfies  $r[n] = \perp$  iff  $p[n] = \perp$  for every  $n$ . In particular,  $r[0] = \perp$ . Since  $r|_m$  can be extended to an infinite sequence which is same as  $p$  except for finite number of coordinates, from Proposition 5.5, we have, for every  $m \geq 0$ , an element  $y_m$  in  $\psi_S(r|_m)$ .  $y_m$  satisfies  $y_m[0] = \perp$ . Let  $z$  be a cluster point of  $y_m$ . When  $r[n]$  is a digit  $a$ ,  $y_m \in S_n^a$  for

almost all  $m$ , and thus  $z \in \overline{S_n^a}$ . On the other hand, when  $r[n] = \perp$ ,  $y_m \in \text{Bd } S_n^0$  for every  $n$  and thus  $z \in \text{Bd } S_n^0$ . Therefore, we have  $\varphi_S(z)[n] \leq r[n]$  for every  $n$ , which means  $\varphi_S(z) \leq r$ . In particular,  $z[0] = \perp$ . Since  $q$  is arbitrary, it means that for every  $z \in X$ ,  $\varphi_S(z)[0] = \perp$ , which is impossible because  $S_0^0 \neq \emptyset$ . Thus,  $p[0]$  is a digit. For the same reason,  $p[n]$  is a digit for every  $n$ .  $\square$

## 6. EXAMPLES OF SPACES AND OPEN PROBLEMS

We show examples of compact spaces with independent (also canonically representing and full-representing by Theorem 5.1) subbases. First of all, since  $\text{Image}(\varphi_S)$  is dense in  $\mathbb{T}^\omega$ ,

**Proposition 6.1.** *If  $X$  has an independent subbase, then  $X$  has no isolated points (i.e.,  $X$  is dense in itself).*

**Corollary 6.2.** *Every countable compact Hausdorff space does not have an independent subbase.*

The characterization of dense-in-itself compact Hausdorff spaces with independent subbases is an open problem. We give some constructions of independent subbases.

**The Cantor space  $\Sigma^\omega$ :** The subbase  $S_n^0 = \uparrow \perp^n 0$  and  $S_n^1 = \uparrow \perp^n 1$ .

**The unit interval  $\mathbb{I}$ :** The Gray subbase  $G$  in Example 1.2.

**Products  $\mathbb{I}^n$ :** For the case of  $\mathbb{I}^2$ , let  $S_{2n}^j = G_n^j \times \mathbb{I}$  and  $S_{2n+1}^j = \mathbb{I} \times G_n^j$  ( $n = 0, 1, 2, \dots, j = 0, 1$ ). In the same way, we can form an independent subbase of  $X \times Y$  from those of  $X$  and  $Y$ .

**Hilbert cube  $\mathbb{I}^\omega$ :** We can apply the same kind of construction for infinite products of spaces with independent subbases.

**The circle  $S^1$ :** The endpoints 0 and 1 of  $\mathbb{I}$  have the representations  $\varphi_G(0) = 0^\omega$  and  $\varphi_G(1) = 10^\omega$ , respectively. Since they are different only at the first coordinate, we can give the representation  $\varphi_{G'}$  of  $S^1 = \mathbb{I}/(0 = 1)$  so that  $\varphi_{G'}([0 = 1]) = \perp 0^\omega$  and  $\varphi_{G'}(x) = \varphi_G(x)$  for other  $x$ . The corresponding subbase is independent.

**$n$ -dimensional surface  $S^n$ , torus  $T^2$ ,  $n$ -torus  $nT^2$  (orientable closed surface of genus  $n$ ):** Suppose that  $S$  is an independent subbase of  $X$  and  $Z$  is a nowhere-dense closed subset of  $X$ . Consider the space  $Y = \{0, 1\} \times X/\sim$  where  $\sim$  is defined as  $(0, z) \sim (1, z)$  for  $z \in Z$ . Then, we can form an independent subbase  $T$  of  $Z$  as  $T_0^0 = (\{0\} \times X) \setminus Z$ ,  $T_0^1 = (\{1\} \times X) \setminus Z$ ,  $T_{n+1}^0 = \{0, 1\} \times S_n^0$ ,

and  $T_{n+1}^1 = \{0, 1\} \times S_n^1$ . This technique applies to these cases, and we can form independent subbases for them.

We define the *dimension* of a dyadic subbase as the maximal number of  $\perp$  which appears in the representation of each point. For example  $G$  and  $D$  have dimension 1, and the dimension of each independent subbase we listed above is equal to the weak inductive dimension of the space. In [8], it is proved that every separable metric space of dimension  $n$  ( $n \leq \infty$ ) has a dyadic subbase with dimension  $n$ . The characterization of  $n$ -dimensional spaces with independent subbases of dimension  $n$  is an open problem. As another open problem, as we have mentioned in Section 5, we have not succeeded in constructing a non-regular space with an independent subbase.

Finally, we will briefly study about independent subbases of  $\mathbb{I}$ . The Gray subbase  $G$  is not the only independent subbase of  $\mathbb{I}$ ; there are uncountably many independent subbases of  $\mathbb{I}$ , even if we identify those which are conjugate through auto-homeomorphisms on  $\mathbb{I}$  and automorphisms on  $\mathbb{T}^\omega$ . Among them, the author has constructed, with professor Shuji Yamada, an independent subbase whose first coordinate has infinite number of points on its boundary. Note that the  $n$ -th coordinate of the Gray-subbase  $G$  has  $2^n$  points on its boundary. When  $e$  is a  $\Sigma$ -sequence of length  $n$ , let  $[l, r]$  be the closed interval  $\overline{\psi_G(e)}$  and let  $\tilde{e}$  denote the function from  $[0, 1/2^n]$  to  $\mathbb{T}^\omega$  defined as  $\tilde{e}(x)[m] = e[m]$  when  $m < n$  and  $\tilde{e}(x)[m] = \varphi_G(x + l)[m]$  when  $m \geq n$ . When  $f : [0, r_f] \rightarrow \mathbb{T}^\omega$  and  $g : [0, r_g] \rightarrow \mathbb{T}^\omega$  are functions, we define  $f ++ g : [0, r_f + r_g] \rightarrow \mathbb{T}^\omega$  such that  $f ++ g(x) = f(x)$  when  $x < r_f$ ,  $f ++ g(x) = g(x - r_f)$  when  $x > r_f$ , and  $f ++ g(r_f)[n] = f[n] \sqcap g[n]$ . Here,  $f[n] \sqcap g[n] = f[n]$  when  $f[n] = g[n]$ , and  $f[n] \sqcap g[n] = \perp$  when  $f[n] \neq g[n]$ . Then, we define  $a_n : [0, 1/2^{n+1}] \rightarrow \mathbb{T}^\omega$  as  $a_n = \widetilde{01^n0} ++ \widetilde{11^n0}$  when  $n$  is even and  $a_n = \widetilde{11^n0} ++ \widetilde{01^n0}$  when  $n$  is odd. Now, define a function  $\varphi'_P : [0, 1) \rightarrow \mathbb{T}^\omega$  as the infinite sum  $a_0 ++ a_1 ++ a_2 ++ \dots$ , and extend it to a function  $\varphi_P : [0, 1] \rightarrow \mathbb{T}^\omega$  so that  $\varphi_P(1) = \perp 1^\omega$ . The corresponding subbase  $P$  is independent. One can see that  $\text{Bd } P_0^0$  is an infinite set, which has an accumulation point on a boundary of  $[0, 1]$ .

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