Unimodal Maps as Boundary-Restrictions of Two-Dimensional Full-Folding Maps

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Abstract

It is shown that every unimodal map is realized as a restriction of a simple map defined on the unit disc to a part of its boundary. Our two-dimensional map is called a full-folding map, which is defined generally on a compact metric space. It is a generalization of the full tent map in that it has two homeomorphic inverse maps and thus every non-critical point has two inverse images.

1 Introduction

Dynamics of unimodal maps is one of the most fruitful research area in which combinatorial tools such as the kneading sequence play an important role. In contrast to the simple definition, a unimodal maps has a rather complicated behavior, and it is shown to be roughly controlled by the kneading sequence. Kneading theory is developped in 70's by Milnor and Thurston, Guckenheimer, and others [7, 10], and since then it has been an active research area and a lot of excellent books about this topic are written[5, 9, 4, 6].

In this paper, we provide a simple insight into the study of unimodal maps. We show that every unimodal map $(h, \mathbb{I})$ can be extended to a two-dimensional map $(f, \mathbb{D})$ with simple dynamical properties defined on the unit disc $\mathbb{D}$ so that h is the restriction of f to a part of the boundary of $\mathbb{D}$. The dynamical property we consider is a generalizations of that of the full tent map, and a map with this property is introduced more generally on a compact metric space and is called a full-folding map.

Among unimodal maps, the full tent map is a special one with a simple chaotic dynamics. It homeomorphically maps both $[0, 1/2]$ and $[1/2, 1]$ to the unit interval $\mathbb{I}$, and thus it has two inverse maps g_0 from $\mathbb{I}$ to $[0, 1/2]$ and g_1 from $\mathbb{I}$ to $[1/2, 1]$. Therefore, through n-times repetitive application, the unit interval is divided into 2^n homeomorphic copies of $\mathbb{I}$ which are coded by $\{0, 1\}$-sequences, and as the limit, each infinite $\{0, 1\}$-sequence designates a point on $\mathbb{I}$. On the other hand, each point has at most two such $\{0, 1\}$-sequence codes which differ only at one digit, and by replacing this digit with $\perp$ (in this paper, we use the character "$\perp$" meaning undefinedness in place of 'C' or '$*$'), we obtain the itinerary of the point. In this way, a full tent map can be investigated through the two inverse maps, which is in contrast to a unimodal map in general where one or both of the inverse images of a point may disappear.

We generalize this property of a tent map to compact metric spaces and define (exact) full-folding maps, and study full-folding maps on the unit disc $\mathbb{D}$. As we will show, every unimodal map on $\mathbb{I}$ can be considered as a restriction of a full-folding map on $\mathbb{D}$ to a part of the boundary. In particular, the core of a unimodal map can be considered as a restriction of a particular kind of full-folding maps. Thus, it is expected that combinatorial properties of unimodal maps can be studied through the investigation of two-dimensional full-folding maps.

Note that this research has its origin in computer science. As mentioned above, we can consider the itinerary of the full tent map as a continuous code of the unit interval in $\{0, 1, \perp\}^\omega$, and the author has been working on computation over the reals with itineraries as codes [12, 13, 11]. He came to the idea of a full-folding map in his attempt to generalize this representation to other spaces.

In the next section, we introduce the notion of a full-folding map. Then, we study full-folding maps on the two-dimensional unit disc in Section 3, and show that unimodal maps appear as the marginal behavior of it in Section 4. In Section 5, we show that, in some cases, itineraries of two particular points roughly determine the behavior of a full-folding map.

Preliminaries and Notations:

If $f : X \to X$ is a map on X, we sometime write (f, X) for f. Let $\mathbb{I} = [0, 1]$ be the unit interval. We call a map $(h, \mathbb{I})$ a unimodal map if h is strictly increasing on $[0, c]$ and strictly decreasing on $[c, 1]$ for some $0 < c < 1$. Note that we only consider strict unimodal maps. The point c is called the critical point of h. If h is a unimodal map such that $h^2(c) < c < h(c)$ and $h^2(c) \leq h^3(c)$, then we call the interval $[h^2(c), h(c)]$ the core of h. Note that the core is mapped to itself. The (full) tent map $(\mathbf{t}, \mathbb{I})$ is the map $\mathbf{t}(x) = 2x$

for $x \leq 1/2$ and $\mathbf{t}(x) = 2 - 2x$ for $x \geq 1/2$. For compact metric spaces X and Y, two continuous maps (f, X) and (g, Y) are *conjugate* if there is a homeomorphism $h : X \to Y$ such that $g \circ h = h \circ f$.

In Section 2, we define a full-folding map on a compact metric space X. In this section, we consider closure, interior, and boundary in X, and we denote by $\overline{U}$ the closure of a subset U in X. Recall that a subset U of X is *regular open* if U is the interior of its closure. In Section 3 and later, we focus on the case X is a space homeomorphic to the unit disc $\mathbb{D}$. For $U \subset \mathbb{D}$, we denote by $\bar{U}$ the closure of U in $\mathbb{R}^2$, by $\operatorname{int} U$ the interior of U in $\mathbb{R}^2$, and by ∂U the boundary of U in $\mathbb{R}^2$. For example, $\operatorname{int} \mathbb{D}$ is the open unit disc and $\partial \mathbb{D} = S^1$. Note that the closure of U in $\mathbb{R}^2$ and the closure of U in $\mathbb{D}$ are identical and there is no confusion of notation.

We say that a map (g, X) is a similitude if there exists $s > 0$ such that $d(x, y) = sd(g(x), g(y))$ for every $x, y \in X$. A curve is a continuous image of $[0, 1]$, and a curve is said to be simple if its map is injective. If C is a curve in $\mathbb{D}$ and a and b are points on C, we call the segment of C from a to b an interval and denote it by $[a, b]$ (or (a, b) if the endpoints are excluded) if it is not ambiguous.

For a character set Σ, we denote by Σ^* the set of finite sequences of Σ and by Σ^ω the set of infinite sequences of Σ, with the index starting with 0. We denote by ϵ the empty sequence. For $p \in \Sigma^\omega \cup \Sigma^*$, we denote by $|p|$ the length of p, which is infinity if $p \in \Sigma^\omega$. For $n < |p|$, we denote by $p(n)$ the n-th character of p and $p_{<n}$ the prefix of p of length n, that is, the sequence $p(0)p(1) \dots p(n-1)$. We denote by σ the left shift operation on Σ^ω and Σ^*, with $\sigma(\epsilon) = \epsilon$. We use letters i, j, k, l, m, n to denote non-negative integers.

2 Full-Folding Maps

Though we are mainly interested in the case X is the unit disc $\mathbb{D}$, we define the notion of a full-folding map more generally on a compact metric space and show properties we need in the following sections.

Definition 2.1 *Let X be a compact metric space. A continuous map (f, X) is a* full-folding map *if, for a regular open subset X_0 of X and $X_1 = X \setminus \overline{X_0}$, $f|_{\overline{X_i}} : \overline{X_i} \to X$ $(i < 2)$ are homeomorphisms.*

Suppose that $f : X \to X$ is a full-folding map. X_1 is also a regular open set. We denote by C the boundary of X_0 in X, which is also the boundary of X_1 in X. We call C the set of critical points of f. We denote by $g_i : X \to \overline{X_i}$ the inverse of $f|_{\overline{X_i}}$ for $i < 2$. If we need to specify the map f, we add the suffix f and write X_{f0}, X_{f1}, C_f for X_0, X_1, C, and so on. The map f is one-to-one on C and two-to-one on $X_0 \cup X_1$. If g_0 and g_1 are similitudes, then we call f a *similarity full-folding map*.

We define the itinerary function φ (or φ_f) from X to $\{0, 1, \perp\}^\omega$ as

$$\varphi(x)(k) = \begin{cases} 0 & (f^k(x) \in X_0), \\ 1 & (f^k(x) \in X_1), \\ \perp & (f^k(x) \in C). \end{cases}$$

We have

$$\varphi(f(x)) = \sigma(\varphi(x)) \tag{2.1}$$

and, for $i < 2$,

$$\varphi(g_i(x)) = \begin{cases} i\varphi(x) & (x \notin f(C)), \\ \perp\varphi(x) & (x \in f(C)). \end{cases} \tag{2.2}$$

For $n < \omega$ and $j < 2$, let

$$R_{n,j} = f^{-n}(X_j) = \{x : f^n(x) \in X_j\}.$$

$R_{n,j}$ is the set of points whose itineraries have the value j at the n-th coordinate. Note that $R_{0,0} = X_0$ and $R_{0,1} = X_1$. For $p \in \{0, 1\}^* \cup \{0, 1\}^\omega$, let

$$R(p) = \bigcap_{k<|p|} R_{k,p(k)}.$$

$R(p)$ is the set of points with the itinerary p if $p \in \{0, 1\}^\omega$, and the set of points with the itinerary starting with p if $p \in \{0, 1\}^*$.

Let $v : X \to X$ be the homeomorphism defined as follows.

$$v(x) = \begin{cases} g_1(f(x)) & (x \in X_0), \\ g_0(f(x)) & (x \in X_1), \\ x & (x \in C). \end{cases}$$

We define $\mathrm{not} : \{0, 1, \perp\} \to \{0, 1, \perp\}$ as $\mathrm{not}(0) = 1$, $\mathrm{not}(1) = 0$ and $\mathrm{not}(\perp) = \perp$, and $\mathrm{nh} : \{0, 1, \perp\}^\omega \to \{0, 1, \perp\}^\omega$ (or $\mathrm{nh} : \{0, 1, \perp\}^n \to \{0, 1, \perp\}^n$ for $n < \omega$) as $\mathrm{nh}(\epsilon) = \epsilon$ and $\mathrm{nh}(ip) = \mathrm{not}(i)p$ for $i \in \{0, 1\}$ and $p \in \{0, 1\}^* \cup \{0, 1\}^\omega$.

Lemma 2.2 *(1)* $v^2(x) = x$.

(2) $\varphi(v(x)) = \mathrm{nh}(\varphi(x))$.

(3) $v(R_{0,i}) = R_{0,\mathrm{not}(i)}$ *and* $v(R_{n,i}) = R_{n,i}$ *for* $0 < n < \omega$ *and* $i < 2$.

(4) $v(R(p)) = R(\mathrm{nh}(p))$ *for* $p \in \{0, 1\}^* \cup \{0, 1\}^\omega$.

(5) *For a point* $x \in C$ *and an open set* $V \ni x$*, there is an open set* U *such that* $x \in U \subset V$ *and* $v(U) = U$.

Proof: (1), (2), (3), (4): immediate from the definition.
(5): Take $U = V \cap v(V)$. ∎

Proposition 2.3 *Let* $n < \omega$, $p \in \{0,1\}^n$, *and* $i < 2$.

(1) $R(ip) = g_i(R(p)) \cap X_i$.

(2) $\overline{R(p)} = \bigcap_{k<n} \overline{R_{k,p(k)}} = \{x : \varphi(x)(k) \in \{p(k), \perp\}$ *for* $k < n\}$.

(3) $\overline{R(ip)} = g_i(\overline{R(p)})$.

(4) $\overline{R(p)}$ *is homeomorphic to* X *with* $g_p = g_{p(0)} \circ g_{p(1)} \circ \ldots \circ g_{p(n-1)}$ *the homeomorphism from* X *to* $\overline{R(p)}$.

Proof: (1) By Equation (2.2).

(2) Suppose that $\overline{R(p)} \subsetneq \bigcap_{k<n} \overline{R_{k,p(k)}}$ for some $n < \omega$ and $p \in \{0,1\}^n$. We choose such a p with minimal length, and suppose that $x \in \bigcap_{k<n} \overline{R_{k,p(k)}}$ and $x \notin \overline{R(p)}$. If $x \notin C$, we have $x \in X_{p(0)}$ and through the homeomorphism $f|_{\overline{X_{p(0)}}}$, $f(x) \in \bigcap_{k<n-1} \overline{R_{k,p(k+1)}}$ and $f(x) \notin \overline{R(\sigma(p))}$, and thus $\sigma(p)$ satisfies the condition and contradicts to the minimality of the length of p. If $x \in C$, we consider an open neighbourhood U of x in Lemma 2.2(5). For each index $k > 0$, we have $R_{k,p(k)} \cap U \neq \emptyset$. Since $R_{k,p(k)} \cap U$ is open, $R_{k,p(k)} \cap U \not\subset C$ and therefore $R_{k,p(k)} \cap U \cap X_i \neq \emptyset$ for $i = 0$ or 1. However, they are homeomorphic by Lemma 2.2(3) and both are non-empty. Therefore, $R_{k,p(k)} \cap U \cap X_{p(0)} \neq \emptyset$. Thus, in $\overline{X_{p(0)}}$, $x \in \bigcap_{0<k<n} \overline{R_{k,p(k)} \cap X_{p(0)}} \subseteq \bigcap_{0<k<n} \overline{R_{k,p(k)} \cap \overline{X_{p(0)}}}$. On the other hand, $x \notin \overline{R(p)} = \overline{\bigcap_{0<k<n} R_{k,p(k)} \cap X_{p(0)}} = \overline{\bigcap_{0<k<n}(R_{k,p(k)} \cap \overline{X_{p(0)}})}$. Therefore, by applying the homeomorphism $f|_{\overline{X_{p(0)}}}$, $f(x) \in \bigcap_{k<n-1} \overline{R_{k,p(k+1)}}$ and $f(x) \notin \overline{\bigcap_{k<n-1} R_{k,p(k+1)}} = \overline{R(\sigma(p))}$. Therefore, $\sigma(p)$ also satisfies the condition and again we have contradiction.

(3) $g_i(\overline{R(p)}) = g_i(\bigcap_{k<n} \overline{R_{k,p(k)}}) = \bigcap_{k<n} \overline{R_{k+1,p(k)} \cap \overline{X_i}} = \bigcap_{k<n} \overline{R_{k+1,p(k)} \cap X_i} = \overline{R(ip)}$ by (2) and Equation (2.2).

(4) Immediate from (3). ∎

Corollary 2.4 *For every* $p \in \{0,1\}^\omega$*, there is an* $x \in X$ *such that* p *is obtained by filling* $\perp$ *appearing in* $\varphi(x)$ *with 0 or 1.*

Proof: The sequence $\overline{R(p_{<n})}(n = 0, 1, 2, \ldots)$ of non-empty closed subsets is shrinking and thus their intersection is non-empty. We take $x \in \cap_{n<\omega}\overline{R(p_{<n})}$. By applying Proposition 2.3(2) to each $\overline{R(p_{<n})}$, we have $x \in \bigcap_{k<\omega} \overline{R_{k,p(k)}}$. Therefore, $\varphi(x)(k) \in \{p(k), \perp\}$ for $k < \omega$. ∎

If the itinerary function φ_f is injective, we say that f is an *exact full-folding map*. A similarity full-folding map is exact. Note that, if f is an exact full-folding map, $R = \{R_{n,i} : n < \omega, i < 2\}$ forms a full-representing subbase studied in [11, 13]. If f is an exact full-folding map, the element x in Corollary 2.4 is uniquely determined by $p \in \{0, 1\}^\omega$. We denote by ρ this map which assigns x to p. It is shown that ρ is continuous in Theorem 5.2 of [13].

Definition 2.5 *Let X be a compact metric space. A continuous map $f : X \to X$ is a* folding map *if, for a regular open subset X_0 and $X_1 = X \setminus \overline{X_0}$, $f|_{\overline{X_i}} : \overline{X_i} \to X$ is a homeomorphism into X for $i < 2$ and for each $b \in C = \overline{X_0} \setminus X_0$ and each open neighbourhood $U \ni f(b)$, there exists $a \in U$ such that* $\mathrm{card}(f^{-1}(a)) = 2$. *Here* $\mathrm{card}(A)$ *is the cardinality of a set A.*

A full-folding map is a folding map. As we did for a full-folding map, for a folding map f, we call C the set of critical points of f, define the itinerary function φ_f of f, and define an exact folding map.

Example 2.6 *The shift map σ is an exact full-folding map on the Cantor Set $\{0, 1\}^\omega$.*

Example 2.7 *Consider the case $X = \mathbb{I}$. A map on $\mathbb{I}$ is full-folding if and only if it is a unimodal map such that $f(0) = f(1) = 0$ and $f(c) = 1$ for the critical point c, or its inverted map $g(x) = 1 - f(x)$. The tent map $(\mathbf{t}, \mathbb{I})$ is an exact full-folding map, and all the exact full-folding maps on $\mathbb{I}$ are conjugate to it. The tent map and its inverted map are the only similarity full-folding maps on $\mathbb{I}$. $(f, \mathbb{I})$ is a folding map if and only if f is a unimodal map or its inverted map. Figure 1 shows regions $R_{n,i}$ and $R(p)$ of the tent map.*

3 Full-Folding Maps on the Two-Dimensional Unit Disc

We study the case X is homeomorphic to the unit disc $\mathbb{D}$. Suppose that $(f, \mathbb{D})$ is a full-folding map. Since $\overline{X_0}$ and $\overline{X_1}$ share the set C of critical

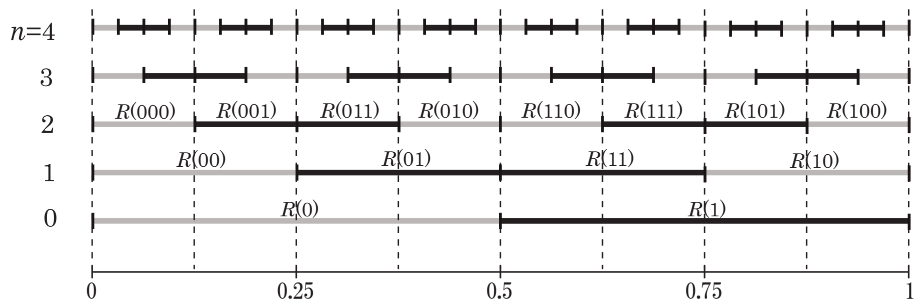

Figure 1: $R_{n,i}$ and $R(p)$ of the tent map. On each line, the gray part is $R_{n,0}$ and the black part is $R_{n,1}$.

points of f, one of g_0 or g_1 is orientation preserving and the other one is orientation reversing. We choose X_0 so that g_0 is orientation preserving and thus X_0, X_1, g_0, g_1 are uniquely determined by f. In the following, recall that ∂A is the boundary of A in $\mathbb{R}^2 \supset \mathbb{D}$.

Lemma 3.1 *C is a simple curve in $\mathbb{D}$ with different endpoints on $\partial\mathbb{D}$.*

Proof: Since f homeomorphically maps $\overline{X_0}$ to $\mathbb{D}$, C, which is a closed subset of $\partial\overline{X_0}$, is mapped to a closed subset of $\partial\mathbb{D}$. Since $\partial\mathbb{D}$ is compact, $f(C)$ consists of finite number of connected components. Suppose that one of the connected components of $f(C)$ is a one-point set $\{x\}$. Then, there is an open neighbourhood U of x such that $U \cap f(C) = \{x\}$. Therefore, $f^{-1}(U)$ is an open neighbourhood of $f^{-1}(x)$ such that $f^{-1}(U) \cap C = \{f^{-1}(x)\}$. It means that $f^{-1}(U) \setminus \{f^{-1}(x)\}$ consists of two disjoint open sets $f^{-1}(U) \cap X_0$ and $f^{-1}(U) \cap X_1$, and we have contradiction. Therefore, each connected component of $f(C)$ is an arc in $\partial\mathbb{D}$, and each connected component of C is a curve in $\mathbb{D}$ with endpoints on $\partial\mathbb{D}$. Thus, each connected component divides $\mathbb{D}$ into two regions, and if C has more than one components, then X_0 or X_1 comes to be disconnected. Therefore, C is connected and is a curve with endpoints on $\partial\mathbb{D}$. If both of the endpoints of C are the same, then C comes to be a closed curve and $\overline{X_0}$ or $\overline{X_1}$ has a hole and is not homeomorphic to $\mathbb{D}$. Therefore, C is a simple curve. ∎

We first study similarity full-folding maps on X, where X is a space homeomorphic to $\mathbb{D}$. Since $\overline{X_0}$ and $\overline{X_1}$ share the set C, the shrinking rates

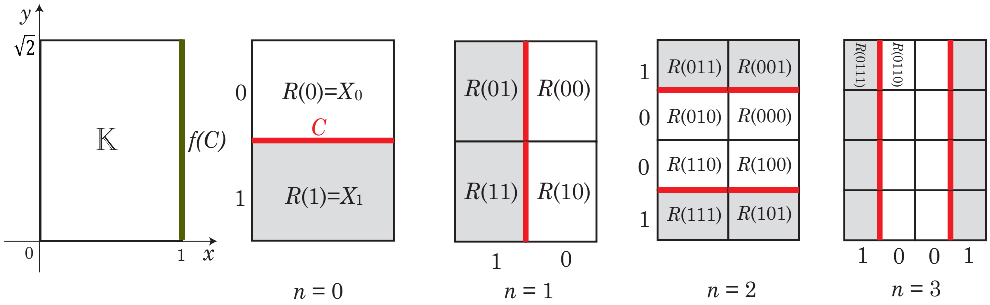

Figure 2: $R_{n,i}$ and $R(p)$ of the rectangular similarity full-folding map. In each figure, the white region is $R_{n,0}$ and the gray region is $R_{n,1}$.

of g_i $(i < 2)$ are the same. Since X is the union of two similar copies of X, this rate is $1/\sqrt{2}$. Since the congruence map $g_1 \circ f$ from $\overline{X_0}$ to $\overline{X_1}$ is the identity on the curve C, C is a line segment and $g_1 \circ f$ is the map flipping over C. Therefore, $f(C)$ is a line segment on ∂X with the length $\sqrt{2}$ of that of C. Through this kind of observation, one can see that there are only two similarity full-folding maps.

One is what we call the rectangular similarity full-folding map $(\mathbf{q}, \mathbb{K})$. Here, the space $\mathbb{K}$ is a rectangle with the height $\sqrt{2}$ of the width. We place $\mathbb{K}$ as in Figure 2 by flipping it if necessary so that the orientation preserving copy $\overline{X_0}$ is the upper half of $\mathbb{K}$ and $f(C)$ is the right-hand side edge of $\overline{X_0}$. $\mathbf{q}$ is the composition of two maps. First, $\mathbb{K}$ is mapped to X_0 by flipping X_1 over the line segment C. Then, X_0 is enlarged with the ratio of $1 : \sqrt{2}$ and rotated to the left by 90 degrees so that it coincides with $\mathbb{K}$. As in the figure, we set the width of $\mathbb{K}$ as 1 and locate the lower-left corner of $\mathbb{K}$ at the origin of the axis of coordinates. Then, we have

$$\mathbf{q}((x,y)) = \begin{cases} (\sqrt{2}(\sqrt{2}-y), \sqrt{2}x) & (\ y \geq 1/\sqrt{2}), \\ (\sqrt{2}y, \sqrt{2}x) & (\ y \leq 1/\sqrt{2}). \end{cases}$$

Let $\hat{\varphi}_{\mathbf{t}}$ be the inverted itinerary of the tent map $\mathbf{t}$. that is,

$$\hat{\varphi}_{\mathbf{t}}(x)(n) = \begin{cases} 1 - \varphi_{\mathbf{t}}(x)(n) & (\ \varphi_{\mathbf{t}}(x)(n) \neq \bot), \\ \bot & (\ \varphi_{\mathbf{t}}(x)(n) = \bot). \end{cases}$$

The itinerary $\varphi_{\mathbf{q}}((x,y))$ of (x,y) is the interleaving of $\hat{\varphi}_{\mathbf{t}}(x)$ and $\hat{\varphi}_{\mathbf{t}}(y)$ through an appropriate scaling. That is,

$$\begin{aligned} \varphi_{\mathbf{q}}((x,y))(2n) &= \hat{\varphi}_{\mathbf{t}}(y/\sqrt{2})(n), \\ \varphi_{\mathbf{q}}((x,y))(2n+1) &= \hat{\varphi}_{\mathbf{t}}(x)(n). \end{aligned}$$

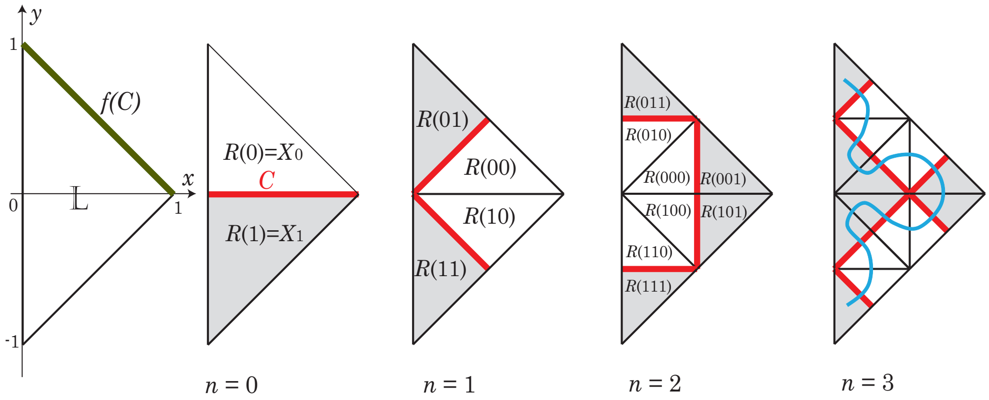

Figure 3: $R_{n,i}$ and $R(p)$ of the triangular similarity full-folding map. In each figure, the white region is $R_{n,0}$ and the gray region is $R_{n,1}$.

The other similarity full-folding map is $(\mathbf{p}, \mathbb{L})$ in Figure 3, which we call the triangular similarity full-folding map. This time, $\mathbb{L}$ is an isosceles right triangle. We place $\mathbb{L}$ on the coordinate plane as in Figure 3 so that the orientation preserving copy $\overline{X_0}$ comes in the first quadrant. Then, we have

$$\mathbf{p}((x, y)) = (1 - (x + |y|), x - |y|).$$

We consider the map $\beta : \mathbb{I} \to \mathbb{L}$ defined as

$$\beta(x) = \rho_{\mathbf{p}}(\mathrm{fill}_0(\hat{\varphi}_{\mathbf{t}}(x))).$$

Here, $\mathrm{fill}_i : \{0, 1, \perp\}^\omega \to \{0, 1\}^\omega$ is the function to replace $\perp$ in the sequence with i ($i < 2$), and $\rho_{\mathbf{p}}$ is defined below Corollary 2.4. Though $\mathrm{fill}_0 \circ \hat{\varphi}_{\mathbf{t}}$ is not continuous, $\rho_{\mathbf{p}}(\mathrm{fill}_0(\hat{\varphi}_{\mathbf{t}}(x)))$ and $\rho_{\mathbf{p}}(\mathrm{fill}_1(\hat{\varphi}_{\mathbf{t}}(x)))$ coincide and β comes to be a continuous surjective map. β is a space-filling curve sometimes called the Peano Curve.

Now, we move on to the study of a full-folding map f on $\mathbb{D}$. In the following, we fix an orientation of $\mathbb{D}$. Let a_1 and b_1 be the two endpoints of C. We select a_1 so that the orientation of C induced by the orientation of $\overline{X_0}$ is from b_1 to a_1, and define $a = f(a_1)$ and $b = f(b_1)$. The four points a, b, a_1, b_1 are on $\partial\mathbb{D}$. One can see that there are 16 possibilities for the order of four points a, b, a_1, b_1 on a circle under the condition that a_1 and b_1 are different and a and b are different. Thus we can divide full-folding maps of $\mathbb{D}$ into 16 categories. We list eleven of them in Figure 4 with the names

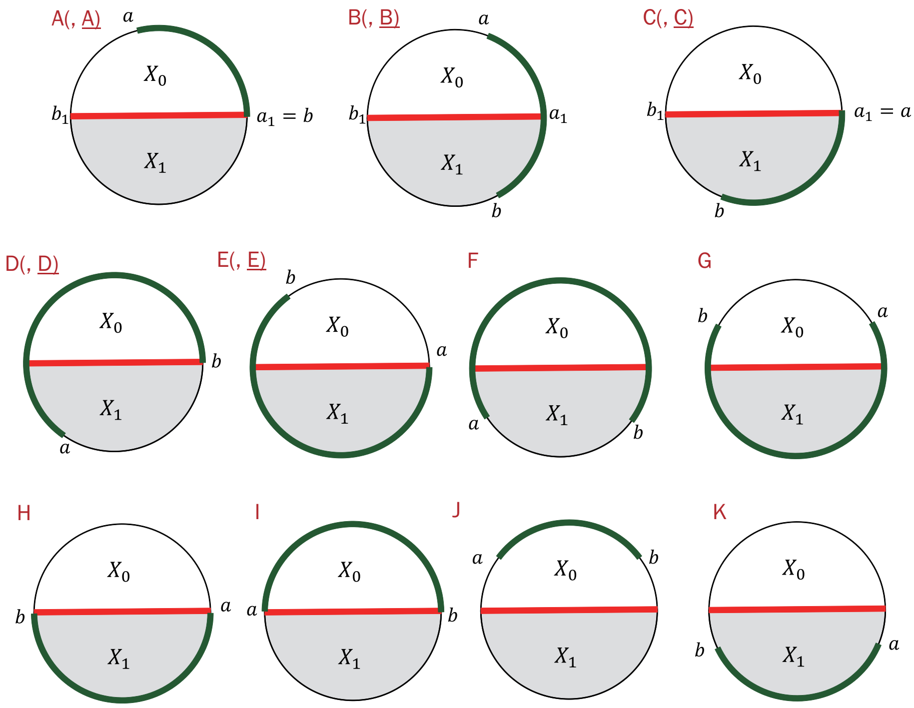

Figure 4: 16 categories of full-folding maps. $\underline{\mathrm{A}}$,..., $\underline{\mathrm{E}}$. are obtained by reversing the figure horizontally and replacing a_i and b_i for $i = 0$ and 1.

A to K. The rests are obtained from A, B, C, D, and E by reversing the orientation of $\mathbb{D}$, that is, flipping the figure horizontally, swapping a and b, and swapping a_1 and b_1. We name them $\underline{\mathrm{A}}$, $\underline{\mathrm{B}}$, $\underline{\mathrm{C}}$, $\underline{\mathrm{D}}$, and $\underline{\mathrm{E}}$, respectively. Note that Category F to K are stable under this operation. Thus, we have 11 categories if we do not consider the orientation of $\mathbb{D}$. In order to distinguish, we call them CATEGORY A to K. The 16 Categories are preserved by orientation-preserving conjugacy, that is, for a full-folding map f and an orientation-preserving homeomorphism e, f and $e^{-1} \circ f \circ e$ belong to the same Category. The 11 CATEGORIES are preserved by conjugacy. One can see that a triangular similarity full-folding map belongs to CATEGORY A and a rectangular similarity full-folding map belongs to CATEGORY B.

Figure 5 shows inverse images of the set C of critical points for a full-folding map in Category K. Let a_2 and b_2 be the images of a_1 and b_1 by g_0, respectively, and $\hat{a_2}$ and $\hat{b_2}$ be those by g_1. As the second figure shows, there are many possibilities for the order of $a, b, \hat{a_2}$, and $\hat{b_2}$ on $\partial X_1 \setminus C$. In

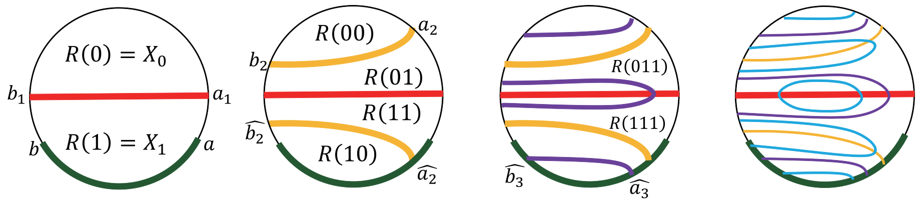

Figure 5: Inverse images of C for a full-folding map in Category K.

this figure, $\hat{a_2}$ is on $f(C)$ and therefore $g_0(\hat{a_2}) = g_1(\hat{a_2})$ is on C and thus the regions $R(111)$ and $R(011)$ share a part of their boundaries. See Figure 7 for different arrangements of the regions induced by different orders of the four points. In the third figure of Figure 5, we have choices of the order of b, $\hat{a_3}$, and $\hat{b_3}$ where $\hat{a_3} = g_1(a_2)$ and $\hat{b_3} = g_1(b_2)$, and a different choice induces a different arrangement of the regions in the next level. It is easy to see inductively that, with any of the choices, $R(1^n) \cap \partial\mathbb{D}$ consists of two connected components which exist on the intervals $(a_1, \hat{a_2})$ and $(b_1, \hat{b_2})$ of $\partial\mathbb{D}$ ($n \geq 2$) because $g_1(R(1^n)) = R(1^{n+1})$. Therefore, the diameter of $\overline{R(1^n)}$ is greater than the distance between the two intervals $[a_1, \hat{a_2}]$ and $[b_1, \hat{b_2}]$ and thus full-folding maps in Category K are not exact.

In the same way, one can show that only CATEGORY A and B contain exact full-folding maps. The non-existence for CATEGORY C and J are proved in the same way as for K. For the rests, the non-existence can be shown via the non-existence of a core of the corresponding unimodal map given in Theorem 4.1.

4 Unimodal Maps as Full-Folding Maps Restricted to the Boundaries

Suppose that $(f, \mathbb{D})$ is a full-folding map. One can see that f maps $\partial\mathbb{D}$ into itself and that any neighbourhood of a (and b) intersects with $\partial\mathbb{D} \setminus f(C)$. Therefore, $f|_{\partial\mathbb{D}}$ is a folding map on $\partial\mathbb{D} = S^1$ with the set of critical points $\{a_1, b_1\}$. We cut $\partial\mathbb{D}$ at a_1 and identify $\partial\mathbb{D}$ with $\mathbb{I} = [0, 1]$ so that the orientation of $\partial\mathbb{D}$ coincides with that of $\mathbb{I}$ and define the induced map $(h, \mathbb{I})$ of $(f, \mathbb{D})$. Here, if $f(x) = a_1$, we define $h(x) = 0$ if $x \in \{0, 1\}$ and $h(x) = 1$ otherwise. Figure 6 shows graphs of induced maps of full-folding maps in Category A, B, and K.

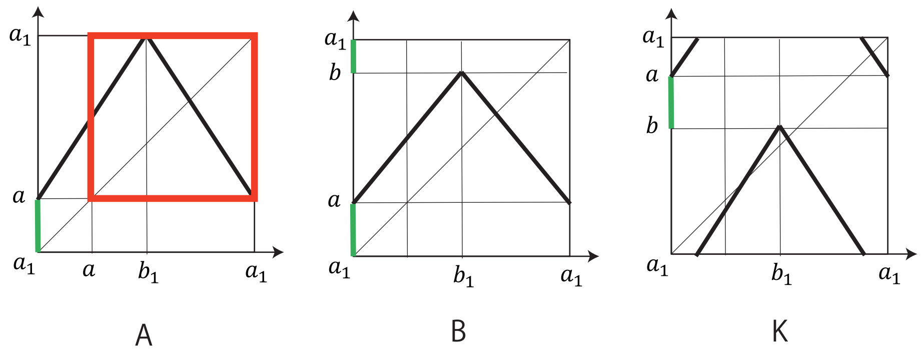

Figure 6: Graphs of induced maps of typical full-folding maps in Category A, B, and K. For A, the product of the core of the induced unimodal map is drawn with a thick-lined box.

Theorem 4.1 *Suppose that* $(f, \mathbb{D})$ *is a full-folding map and* $(h, \mathbb{I})$ *is the induced map defined above.*

(1) h *is a unimodal map if and only if* $a_1 \in f(C)$, *that is,* f *is in Category A to I,* $\underline{\mathrm{D}}$, *or* $\underline{\mathrm{E}}$. *The critical point of* h *is* b_1. *In other cases,* h *becomes a discontinuous map.*

(2) If h *is a unimodal map with a core, then* $b_1 \notin f(C)$, *that is,* f *belongs to Category* A, B, *or* C.

(3) If f *is a full-folding map in Category* A, *then* h *is a unimodal map with a core.*

Proof: Immediate from the definition. ∎

Remark 4.2 *If we cut* $\partial\mathbb{D}$ *at* b_1 *and identify* $\partial\mathbb{D}$ *with* $[0, 1]$ *in the opposite direction, then we have similar results, and, for example, we obtain a unimodal map if and only if* $b_1 \in f(C)$.

We also have the converse of this theorem. That is, for a unimodal map $(h, \mathbb{I})$, we can find a full-folding map $(f, \mathbb{D})$ such that $(h, \mathbb{I})$ is obtained as the restriction of $(f, \mathbb{D})$ to a part of the boundary of $\mathbb{D}$.

Theorem 4.3 *(1) Suppose that* $(h, \mathbb{I})$ *is a unimodal map with the critical point* c *such that* $h(0) = h(1)$. *Suppose also that* $(h, \mathbb{I})$ *is not a full-folding map, that is,* $h(0) \neq 0$ *or* $h(c) \neq 1$. *Then, there is a full-folding map* $(f, \mathbb{D})$ *which induces* $(h, \mathbb{I})$.

(2) *Suppose that* $(h, \mathbb{I})$ *is a unimodal map. There is a full-folding map* $(f, \mathbb{D})$ *and an interval* $[a, b]$ *on* $\partial\mathbb{D}$ *such that* $h = f|_{[a,b]}$ *through an identification of the interval* $[a, b]$ *with* $[0, 1]$.

(3) *Suppose that a unimodal map* $(h, \mathbb{I})$ *has a core. Suppose also that* $h^3(c) \neq h^2(c)$. *Then, there is a full-folding map* $f : \mathbb{D} \to \mathbb{D}$ *in Category* A *which extends the core of* h. *That is,* $h|_{[h^2(c),h(c)]} = f|_{[a,a_1]}$ *through the identification of* $[a, a_1] \subset \partial\mathbb{D}$ *with* $[h^2(c), h(c)]$.

Proof: (1) We consider h as a map (h, S^1) on $S^1 = \partial\mathbb{D}$ by identifying 0 and 1. Let D be the interval $[h(c), h(0)]$ on S^1 which contains $0 = 1$. We connect the two points c and 0 by a line segment C in $\mathbb{D}$. Let u be any homeomorphism from C to D such that $u(c) = h(c)$ and $u(0) = h(0)$. Now, C divides $\mathbb{D}$ into two regular open sets X_0 and X_1 of $\mathbb{D}$ such that $\partial X_0 = [0, c] \cup C$ and $\partial X_1 = C \cup [c, 1]$. Here, $[0, c]$ and $[c, 1]$ are intervals on S^1. Thus, we obtain a homeomorphism f_0' from ∂X_0 to $\partial\mathbb{D}$ defined as $f_0'(x) = h(x)$ for $x \in [0, c]$ and $f_0'(x) = u(x)$ for $x \in C$. Similarly, we define a homeomorphism f_1' from ∂X_1 to $\partial\mathbb{D}$. One can extend f_i' to a homeomorphism f_i from $\overline{X_i}$ to $\mathbb{D}$ for $i < 2$. Such a homeomorphism obviously exists. In the following, we give one construction of f_0. Take a point o in $\operatorname{int} X_0$ and o' in $\operatorname{int}\mathbb{D}$. Since $\mathbb{D}$ and X_0 are both convex, one can connect o and $x \in \partial X_0$ by a line segment $[o, x]$ and o' and $f_0'(x) \in \partial\mathbb{D}$ by a line segment $[o', f_0'(x)]$. Thus, we define the map f_0 so that $[o, x]$ is mapped to $[o', f_0'(x)]$ linearly. Because f_0 and f_1 coincide on C, we obtain a map $(f, \mathbb{D})$, which is full-folding.

(2) Suppose that a unimodal map $(h, \mathbb{I})$ is given. One can extend $\mathbb{I}$ to $[a', b'] \supset \mathbb{I}$ so that h is a unimodal map which satisfies the conditions of (1) through the identification of $[a', b']$ with $[0, 1]$. We apply (1) to form $(f, \mathbb{D})$.

(3) For simplicity, we redefine $[h^2(c), h(c)]$ as $\mathbb{I}$ and $(h|_{[h^2(c),h(c)]}, [h^2(c), h(c)])$ as $(h, \mathbb{I})$. We have $h(c) = 1, h(1) = 0$, and $h(0) \neq 0$. For some $a < 0$, we extend h to a unimodal map on $[a, 1]$ so that $h(a) = 0$. Then, we apply the method in (1) to obtain a full-folding map in Category A with the desired property. ∎

In this way, one can view a unimodal map as a restriction of a full-folding map on the unit disc to a part of its boundary. One important property of a full-folding map $(f, \mathbb{D})$ is that both of the inverse images $g_0(x)$ and $g_1(x)$ are guaranteed to exist. Therefore, if a unimodal map $(h, \mathbb{I})$ does not have the inverses of a point $x \in \mathbb{I}$, they do exist in the extended space $\mathbb{D}$ for the extended full-folding map. Thus, the author suggests one can investigate in unimodal maps through full-folding maps, in particular through full-folding maps in Category A since interesting combinatorial behavior of a unimodal map appear in the core. Possible areas of application would be the kneading theory (for example, [2, 5, 8]), and the inverse limit space theory ([1, 3]).

5 Itineraries of Full-Folding Maps

Suppose that $(f, \mathbb{D})$ is a full-folding map in Category A, B, C, D, E, F, G, H, I, $\underline{\mathrm{D}}$, or $\underline{\mathrm{E}}$. By Theorem 4.1(1), the induced map $(h, \mathbb{I})$ of $(f, \mathbb{D})$ is a unimodal map. As in Section 4, we cut $\partial\mathbb{D}$ at a_1 and consider that $x \in \mathbb{I}$ specifies a point $\eta(x)$ on $\partial\mathbb{D}$, and compare the two itineraries $\varphi_h(x)$ and $\varphi_f(\eta(x))$ for $x \in \mathbb{I}$. Since $X_{h0} = \eta^{-1}(X_{f0} \cap \partial\mathbb{D}) \cup \{0\}$ and $X_{h1} = \eta^{-1}(X_{f1} \cap \partial\mathbb{D}) \cup \{1\}$, $\varphi_h(x)(n)$ and $\varphi_f(\eta(x))(n)$ differ only when $f^n(\eta(x)) = a_1$, and $\varphi_h(x)$ is obtained from $\varphi_f(\eta(x))$ by replacing some of the occurrences of $\perp$ with 0 or 1.

For every f, such replacements to 0 and 1 obviously occur as the first character of $\varphi_h(0)$ and $\varphi_h(1)$, respectively. If f is in Category B, C, E, F, G, H, $\underline{\mathrm{D}}$, or $\underline{\mathrm{E}}$ this is the only replacement. If f is in Category D or I, replacements occur only in itineraries of 0, 1, and $\eta^{-1}(b_1)$, which are easily calculated. If f is in Category A, the situation is a bit complicated.

Lemma 5.1 *Suppose that $(f, \mathbb{D})$ is in Category A and a_1 is not a periodic point of f. Let* $\mathrm{bottom}(x) = \{n : \varphi_f(x)(n) = \perp\}$ *for $x \in \mathbb{D}$.*

(1) $\mathrm{bottom}(a_1) = \{0\}$.

(2) For $x \in \partial\mathbb{D} \setminus \{a_1\}$, $\mathrm{bottom}(x)$ *is $\emptyset$ or $\{k, k+1\}$ for some k.*

(3) For $x \in \mathrm{int}\,\mathbb{D}$, $\mathrm{bottom}(x)$ *is $\emptyset$, one-point set, or $\{j, k, k+1\}$ for some $j < k$.*

Proof: (1) Obvious.

(2) Since $\partial\mathbb{D}$ is mapped to itself, $\varphi(x)(k) = \perp$ if and only if $f^k(x) \in \{a_1, b_1\}$. On the other hand, $f^k(x) = b_1$ if and only if $f^{k+1}(x) = a_1$ for $k \geq 0$. Since a_1 is not periodic, there is at most one k for which $f^k(x) = b_1$.

(3) Let $x \in \mathrm{int}\,\mathbb{D}$. If $f^j(x) \in C \setminus \{a_1, b_1\}$, then $f^k(x) \in \partial\mathbb{D}$ for every $k > j$. Therefore, $f^j(x) \in C \setminus \{a_1, b_1\}$ only for at most one j. On the other hand, if $f^k(x) \in \partial\mathbb{D}$, then for some $0 \leq j < k$, $f^j(x) \in C \setminus \{a_1, b_1\}$. ∎

Let $x \in \partial\mathbb{D} \setminus \{a_1\}$. If a_1 is not a periodic point of $(f, \mathbb{D})$ and $\varphi_f(x)$ contains a copy of $\perp$, then $\varphi_f(x)$ contains a subsequence $\perp\perp$ by Lemma 5.1(1), In this case, $\varphi_h(\eta^{-1}(x))$ is obtained from $\varphi_f(x)$ by replacing $\perp\perp$

	$\varphi(a)$	$\varphi(b)$
A	0...	$\perp\varphi(a)$
B	0...	1...
C	$\perp^{\omega}$	1...
D	1^{ω}	$\perp 1^{\omega}$
E	$\perp^{\omega}$	0^{ω}
F	1^{ω}	1^{ω}
G	0^{ω}	0^{ω}
H	$\perp^{\omega}$	$\perp^{\omega}$
I	$\perp^{\omega}$	$\perp^{\omega}$
J	0...	0...
K	1...	1...

	$\varphi(a)$	$\varphi(b)$
$\underline{\text{A}}$	$\perp\varphi(b)$	0...
$\underline{\text{B}}$	1...	0...
$\underline{\text{C}}$	1...	$\perp^{\omega}$
$\underline{\text{D}}$	$\perp 1^{\omega}$	1^{ω}
$\underline{\text{E}}$	0^{ω}	$\perp^{\omega}$

Table 1: Itineraries of a and b for the 16 categories of full-folding maps.

with $\perp 1$. If a_1 is periodic, then successive pairs of $\perp$ may appear repeatedly in $\varphi_f(x)$, and $\varphi_h(\eta^{-1}(x))$ is obtained from $\varphi_f(x)$ by making this replacement to all of them. Conversely, for $0 < x < 1$, $\varphi_f(\eta(x))$ is obtained from $\varphi_h(x)$ by replacing $\perp 1$ with $\perp\perp$.

It is known that, the itineraries of 0 and the critical point c roughly determine the behavior of a unimodal map $(h, \mathbb{I})$. We investigate similar results for a full-folding map $(f, \mathbb{D})$ and the two points a and b. Table 1 shows the forms of itineraries of a and b for the 16 categories of full-folding maps.

Lemma 5.2 *Suppose that $(f, \mathbb{D})$ is a full-folding map which belongs to Category other than H, I, J, or K. Then, $\varphi(a)$ and $\varphi(b)$ determine the category of f.*

Proof: Immediate from Table 1. ∎

We write a_f, b_f, and $R_f(p)$ for a, b, and $R(p)$ of a full-folding map $(f, \mathbb{D})$, respectively.

Proposition 5.3 *Suppose that two full-folding maps $(f, \mathbb{D})$ and $(f', \mathbb{D})$ are not in Category H, I, J or K. Suppose also that they satisfy $\varphi_f(a_f) = \varphi_{f'}(a_{f'})$ and $\varphi_f(b_f) = \varphi_{f'}(b_{f'})$. Then, f and f' belong to the same category and, for each $n < \omega$, there is a homeomorphism α_n on $\mathbb{D}$ which maps a_f to $a_{f'}$, b_f to $b_{f'}$, and $R_f(p)$ to $R_{f'}(p)$ for every $p \in \{0, 1\}^n$.*

Proof: First, Lemma 5.2 shows that $(f, \mathbb{D})$ and $(f', \mathbb{D})$ belong to the same category. We write g_0, g'_0, R, R', a, a', b, b', φ, φ' for g_{f0}, $g_{f'0}$, R_f, $R_{f'}$, a_f, $a_{f'}$, b_f, $b_{f'}$, φ_f, $\varphi_{f'}$, and so on. We inductively define α_n for $n < \omega$ such that α_n is orientation-preserving and α_n maps a_f to $a_{f'}$, b_f to $b_{f'}$, and $R_f(p)$ to $R_{f'}(p)$ for every $p \in \{0,1\}^n$.

First, let α_0 be any orientation-preserving homeomorphism on $\mathbb{D}$ which maps a, b, a_1, and b_1 to a', b', a'_1, and b'_1, respectively. Note that f and f' are in the same category and therefore the orders of these four points are the same and thus such a homeomorphism exists. Suppose that a homeomorphism α_{n-1} is defined for $n > 0$. Since α_{n-1} is orientation-preserving and it maps a to a' and b to b', it maps $f(C)$ to $f'(C')$. As a candidate of α_n, we define

$$\delta_n(x) = \begin{cases} g'_0(\alpha_{n-1}(f(x))) & (x \in \overline{X_0}), \\ g'_1(\alpha_{n-1}(f(x))) & (x \in \overline{X_1}). \end{cases}$$

For $c \in C$, since $f(c) \in f(C)$, we have $\alpha_{n-1}(f(c)) \in f'(C')$ and therefore $g'_0(\alpha_{n-1}(f(c))) = g'_1(\alpha_{n-1}(f(c)))$. It means that δ_n is well-defined and δ_n becomes a homeomorphism.

δ_n maps $R(p)$ to $R'(p)$ for every $p \in \{0,1\}^n$. However, $\delta_n(a) \neq a'$ and $\delta_n(b) \neq b'$ in general, and therefore we need to make some modification to δ_n. Let $q = \varphi(a) = \varphi'(a')$, and $r = \varphi(b) = \varphi'(b')$.

Suppose that $q_{<n}$ contains a bottom and k is the first index at which $q(k) = \bot$. Then, a is on the boundary of two regions $\overline{R(q_{<k}0)}$ and $\overline{R(q_{<k}1)}$ and therefore $\delta_n(a)$ is on the boundary of $\overline{R'(q_{<k}0)}$ and $\overline{R'(q_{<k}1)}$. If $k = 0$, then $\overline{R'(q_{<k}0)} \cap \overline{R'(q_{<k}1)} \cap \partial\mathbb{D} = \{a'_1, b'_1\}$. If $a' = a'_1$ and $\delta_n(a) = b'_1$, then $a = a_1$ because f and f' belong to the same category, and thus $\delta_n(a_1) = b'_1$. Therefore, $a'_1 = b'_1$ and we have contradiction. Similarly, we have contradiction if $a' = b'_1$ and $\delta_n(a) = a'_1$, Therefore, $\delta_n(a) = a'$ for the case $k = 0$. If $k > 0$, then, since f is not in Category J or K, for every $p \in \{0,1\}^n$, $\overline{R(p)} \cap \partial\mathbb{D}$ is connected and thus is a curve in $\partial\mathbb{D}$. Therefore, $\overline{R'(q_{<k}0)} \cap \overline{R'(q_{<k}1)} \cap \partial\mathbb{D}$ is an one-point set. Since $\delta_n(a)$ and a' are in this set, we have $\delta_n(a) = a'$. Therefore, if $q_{<n}$ contains a bottom, we define β as the identity homeomorphism on $\mathbb{D}$. Now, consider the case that $q_{<n}$ do not contain a bottom. Then, the region $\overline{R(q_{<n})}$ is mapped to $\overline{R'(q_{<n})}$ by δ_n and $\delta_n(a)$ and a' are on the boundary of $\overline{R'(q_{<n})}$. We define β as the homeomorphism on $\mathbb{D}$ which is identity on $\mathbb{D} \setminus R'(q_{<n})$ and which maps $\delta_n(a)$ to a'. Such a homeomorphism exists by Schönflies theorem which says the inside of a Jordan closed curve is homeomorphic to $\mathbb{D}$. We define $\alpha'_n = \beta \circ \delta_n$. Then, it satisfies $\alpha'_n(a) = a'$

It is also the case for $\delta_n(b)$ and b', and there is a homeomorphism β' on

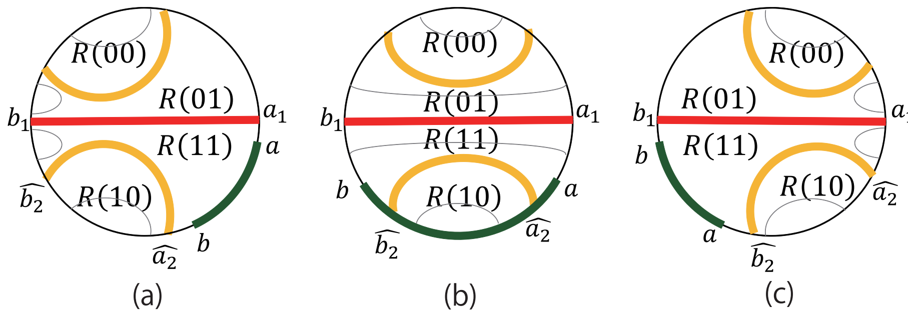

Figure 7: Three examples of full-folding maps in Category K such that $\varphi(a) = \varphi(b) = 1^\omega$. Note that, for (a) and (b), the order of a, b, $\hat{a_2}$, and $\hat{b_2}$ determine the whole sequences $\varphi(a)$ and $\varphi(b)$.

$\mathbb{D}$ which is identity on $\mathbb{D} \setminus R'(r_{<n})$ and which maps $\delta_n(b)$ to b'. Therefore, we define $\alpha_n = \beta' \circ \beta \circ \delta_n$ if $q_{<n} \neq r_{<n}$ or $q_{<n} = r_{<n}$ and it contains a bottom.

Finally, we consider the case $q_{<n} = r_{<n}$ and it does not contain a bottom, Since f is not in Category J or K, it happens only if f is in Category F or G by Table 1. Since δ_n is orientation-preserving, the order of $\delta_n(a)$ and $\delta_n(b)$ on $\partial\mathbb{D} \cap R'(q_{<n})$ is the same as that of a' and b'. Therefore, there is a homeomorphism β on $\mathbb{D}$ which is identity on $\mathbb{D} \setminus R'(q_{<n})$ and which maps $\delta_n(a)$ to a' and $\delta_n(b)$ to b'. Thus, we define $\alpha_n = \beta \circ \delta_n$. ∎

This proposition does not hold if we also consider full-folding maps in Category J and K. For example, Figure 7 shows three examples of full-folding maps in Category K such that $\varphi(a) = \varphi(b) = 1^\omega$ but a second level homeomorphism α_2 between each pair does not exist.

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