Weak concrete mathematical incompleteness, phase transitions and reverse mathematics

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Overview

1. Introduction:
   - Reverse mathematics
   - Weak concrete mathematical incompleteness
   - Phase transitions
2. Mixing
   - Example: Dickson's lemma
   - Paris–Harrington and adjacent Ramsey

Introduction: Reverse Mathematics

Reverse mathematics is a program founded by Harvey Friedman and developed by, among others, Stephen Simpson. The program is motivated by the foundational question:

What are appropriate axioms for mathematics?

One of the main themes of reverse mathematics[2] is that a large number of theorems from the mathematics literature are either provable in RCA_0 or equivalent to one of only four logical principles: WKL_0, ACA_0, ATR_0 and Π^1_1-CA.

In this talk, unless specified otherwise, the base theory will always be RCA_0.

[2] Subsystems of Second Order Arithmetic, Stephen G. Simpson

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Note that this falls outside of the 'Big Five', hence the principles can be considered to be part of the *Reverse Mathematics Zoo*.

Introduction: weak concrete mathematical incompleteness

Introduction: weak CMI

Thanks to Gödel's incompleteness theorems we know that for every 'reasonable' theory T of arithmetic there exist statements in the language of T which are independent of T.

Kurt Gödel (1906-1978)

Introduction: weak CMI

We will call such statements incompleteness phenomena or unprovable statements. The unprovable statements in this talk will be Π_2 (concrete) and independent of fragments of PA (weak).

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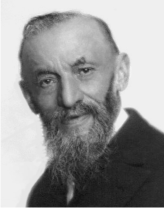
We are interested in *natural* unprovability, in the sense that our statements should closely resemble theorems from the mathematics literature.

Introduction: PA



Giuseppe Peano (1858-1932)

Peano Arithmetic is a first order theory which consists of defining axioms for

$$0, 1, +, \times, <$$

and the scheme of arithmetic induction:

$$[\varphi(0) \wedge \forall x(\varphi(x) \rightarrow \varphi(x+1)] \rightarrow \forall x \varphi(x),$$

Introduction: fragments of PA

Formulas of the form:

$$\exists x_1 \forall x_2 \ldots Q x_n \varphi$$

are called Σ_n-formulas.

When we restrict the scheme of induction to Σ_n formulas, we call the theory:

$$\mathrm{I}\Sigma_n.$$

Introduction: Why weak CMI?

$\mathrm{I}\Sigma_1$ has the same strength of primitive recursive arithmetic, as such it is considered to be important in a partial realisation of Hilbert's program.

$\mathrm{I}\Sigma_2$ has the strength of 'multiply recursive arithmetic'.

PA is a canonical first order theory of arithmetic. It is mutually interpretable with $\mathrm{ZFC}-$infinity$+\neg$infinity.

Introduction: Why weak CMI?

Already for $\mathrm{I}\Sigma_1$, examples of concrete incompleteness are unlikely to occur during conventional mathematical practice. This was expressed by Harvey Friedman's Grand Conjecture[3]:

> *Every theorem published in the Annals of Mathematics whose statement involves only finite mathematical objects (...) can be proved in EFA.*

It took until the late 70's before natural examples for PA showed themselves, and they remain few in number.

[3] FOM: grand conjectures, Fri Apr 16 15:18:28 EDT 1999

Introduction: Why weak CMI when interested in RM?

The proof theoretic ordinals of fragments of PA are all below ε_0.

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The proof theoretic ordinals of fragments of PA are all below ε_0.

It may be possible to convert weak CMI results into principles equivalent to the well-foundedness of the corresponding ordinal!

Introduction: Phase transitions in unprovability

Phase transitions

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$\psi_{x \mapsto c}$ is provable, but ψ_{id} is independent.

Question:

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$\psi_{x \mapsto c}$ is provable, but ψ_{id} is independent.

Question:

Where between constant functions and identity does ψ_f change from provable to independent?

This ends the first part of the talk.

Mixing: Dickson's lemma

Dickson's lemma

We order d-tuples of natural numbers coordinatewise:

$$(a_1, \ldots a_d) \leq (b_1, \ldots, b_d) :\Leftrightarrow a_1 \leq b_1 \wedge \cdots \wedge a_d \leq b_d.$$

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Any $\mathbb{N}^d$, ordered coordinatewise, is a well partial order.

Dickson's lemma is known to be equivalent to the well-foundedness of ω^ω (Simpson).

Dickson's lemma

A sequence $\bar{m}_0, \ldots \bar{m}_D$ of d-tuples of natural numbers is l-bounded if:

$$\max \bar{m}_i \leq l + i.$$

Definition (Miniaturised Dickson's Lemma)

For every d, l there exists D such that for every l-bounded sequence $\bar{m}_0, \ldots, \bar{m}_D$ of d-tuples there are $i < j \leq D$ with $\bar{m}_i \leq \bar{m}_j$.

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Miniaturised Dickson's Lemma is known to be independent of $\mathrm{I}\Sigma_1$ (Friedman?).

Dickson's lemma

A sequence $\bar{m}_0, \ldots \bar{m}_D$ of d-tuples of natural numbers is (f, l)-bounded if:

$$\max \bar{m}_i \leq l + f(i).$$

Definition (MDL_f)

For every d, l there exists D such that for every (f, l)-bounded sequence $\bar{m}_0, \ldots, \bar{m}_D$ of d-tuples there are $i < j \leq D$ with $\bar{m}_i \leq \bar{m}_j$.

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It is known that (Weiermann):

1. $\mathrm{I}\Sigma_1 \vdash \mathrm{MDL}_{\log}$, but
2. $\mathrm{I}\Sigma_1 \nvdash \mathrm{MDL}_{\sqrt[c]{}}$, for every c.

Dickson's lemma

Question:

What about the RM status of $\forall f.\mathrm{MDL}_f$?

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In general, given a Weiermann-style parametrised CMI-result ψ_f:

Question:

What is the RM status of $\forall f.\psi_f$?

Mixing: Paris–Harrington and adjacent Ramsey

Paris–Harrington

The following is independent of PA (Paris, Harrington 1977):

Definition (Paris–Harrington principle)

For all a, d, k there exists R such that every $C\colon [a, R]^d \to k$ has a large homogeneous set.

Additionally, if one fixes $d + 1$, the resulting variant becomes independent of $\mathrm{I}\Sigma_d$.

Paris–Harrington

A set X is called f-large if $|X| > f(\min X)$.

Definition (PH^d_f)

For all a, k there exists R such that every $C\colon [a, R]^d \to k$ has an f-large homogeneous set.

A variant of $\forall f.\mathrm{PH}^2_f$ is known to be equivalent to the well-foundedness of ω^ω (Kreuzer, Yokoyama).

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Theorem (P.)

$\forall f.\mathrm{PH}^d_f$ is equivalent to the well-foundedness of ω_d.

adjacent Ramsey

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Definition (adjacent Ramsey)

For all $C\colon \mathbb{N}^d \to \mathbb{N}^r$ there are $x_1 < \cdots < x_{d+1}$ with:

$$C(x_1, \dots, x_d) \leq C(x_2, \dots, x_{d+1}).$$

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Adjacent Ramsey is known to be equivalent to the well-foundedness of ε_0.

adjacent Ramsey

A colouring $C\colon \{0,\ldots,R\}^d \to \mathbb{N}^r$ is f-limited if

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A colouring $C\colon \{0,\ldots,R\}^d \to \mathbb{N}^r$ is f-limited if

$$\max C(x) \leq f(\max x).$$

Definition (FAR^d_f)

For every r there exists R such that for every f-limited function $C\colon \{0,\ldots,R\}^d \to \mathbb{N}^r$ there are $x_1 < \cdots < x_{d+1} \leq R$ with:

$$C(x_1,\ldots,x_d) \leq C(x_2,\ldots,x_{d+1}).$$

adjacent Ramsey

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It is already known that $\mathrm{PH}_{\mathrm{id}}^{d+1}$ is equivalent to $\mathrm{FAR}_{\mathrm{id}}^{d}$ (Friedman, P.).

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Friedman's proof of the upper bound for adjacent Ramsey is specific to ε_0 and does not work for fixed dimensions.

It is already known that $\mathrm{PH}_{\mathrm{id}}^{d+1}$ is equivalent to $\mathrm{FAR}_{\mathrm{id}}^{d}$ (Friedman, P.).

The proof needs only minor modifications to convert this result to PH_{f}^{d+1} is equivalent to FAR_{f}^{d}

adjacent Ramsey

Hence, we can use the status of $\forall f.\mathrm{FAR}_f$, with fixed dimension, to determine that:

Theorem (Friedman, P.)

Adjacent Ramsey with fixed dimension d is equivalent to the well-foundedness of ω_{d+1}.

Final remark

There is a rich interplay between proof theory for CMI and RM of well-foundedness of ordinals!

Thank you for listening.

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