Coherence Spaces for **Resource-Sensitive** Computation in Analysis

Kei Matsumoto

(RIMS, Kyoto University)

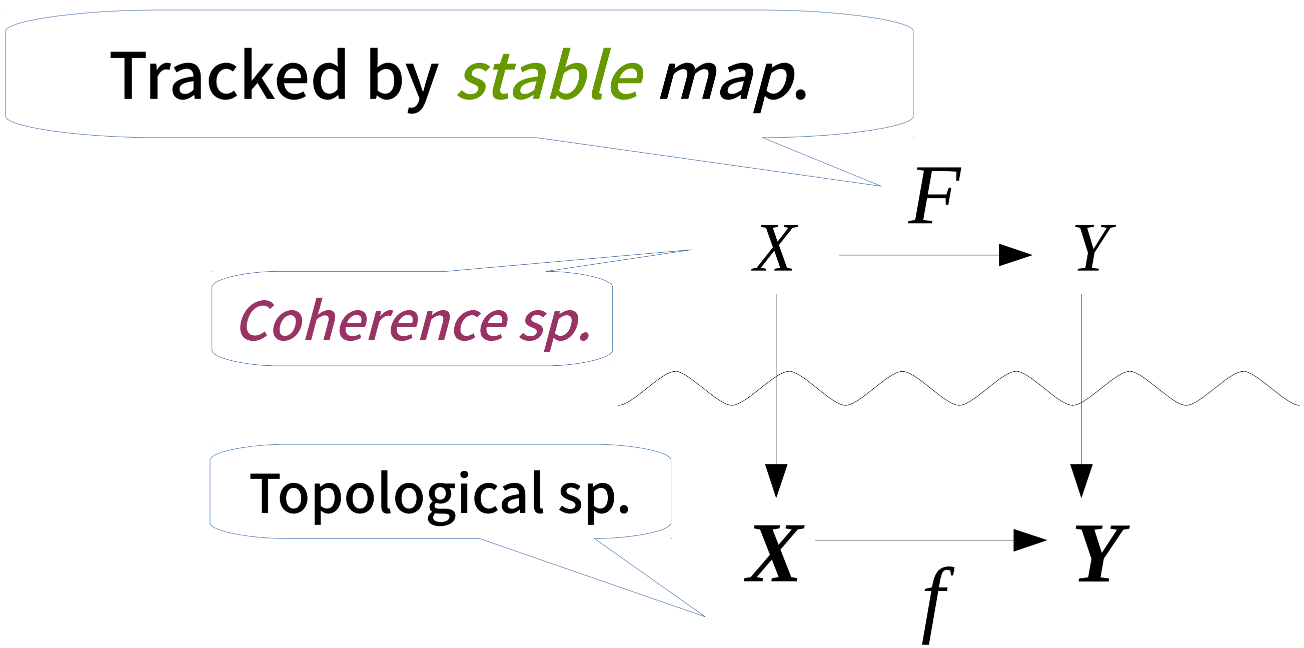
Background

- Computable analysis studies computation over topological spaces, by giving *representations.*
  - *Type two theory of Effectivity*
  - *Domain representations*
- Their approaches are to track computation by *continuous maps* over "symbolic" spaces.

Baire sp., Scott domains, ...

The principle: Computable $\Rightarrow$ *Continuous*

Our Proposal



- Our principle: Computable ⇒ *Stable* [Berry '78]

Using instead of Scott-domains *coherence spaces* [Girard '86].

- BTW, two morphisms **coexists** in coherence spaces: stable & linear maps.
- A new question then arrises:

What are Linear Computations in Topology?

Girard's Linear Logic

EDITOR-IN-CHIEF:
M. NIVAT
ASSOCIATE EDITOR:
M.S. PATERSON
EDITORIAL BOARD:
G. AUSIELLO
J.W. DE BAKKER
J. BEĆVAŘ
C. BÖHM
R.V. BOOK
E. ENGELER
A.P. ERSHOV
H. GENRICH
M.A. HARRISON
HU SHIH-HUA
S. IGARASHI
T. ITO
R.M. KARP
LIU BI WU
Z. MANNA
A.R. MEYER
R. MILNER
G. MIRKOWSKA
D. PARK
D. PERRIN
M. RABIN
A. SALOMAA
A. SCHONHAGE
E. SHAMIR
R. SIROMONEY
M. TAKAHASHI
J.D. ULLMAN
S. WINOGRAD
H. YAMADA
F. YAO
P. YOUNG
ADVISORY EDITOR:
M. SINTZOFF
liaison with Science of Computer Programming

Volume 50
Number 1 1987
TCSDI 50(1) 1–102 (1987)

ISSN 0304-3975

theoretical computer science

the journal of the EATCS

FUNDAMENTAL STUDIES

CONTENTS

J.-Y. GIRARD, Linear logic 1

I. Introduction and abstract 2
II. Linear logic explained to a proof-theorist 4
III. Linear logic explained to a (theoretical) computer scientist 7
IV. Pons asinorum: from usual implication to linear implication 14
1. The phase semantics 17
2. Proof nets 28
3. The coherent semantics 48
4. Normalization in PN2 60
5. Some useful translations 78
6. Work in progress: slices 93
V. Two years of linear logic; selection from the garbage collector 97
Acknowledgment 101
References 102

Volume 50 will be published at irregular intervals in 1987 alongside the issues of Volumes 49, 51, 52, 53, 54, 55, which will be published on a monthly basis.

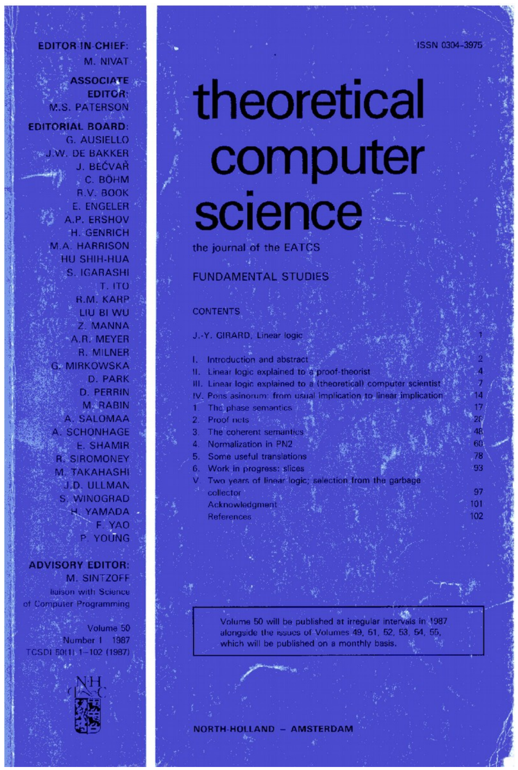
NORTH-HOLLAND – AMSTERDAM

One of the most influential papers in 80's in both logic and computer science.

Attractive Ideas:

- Restructuring both Classical & Intuitionistic Logic
- Proof Nets
- Resource-Conciousness

Girard's Linear Logic



One of the most influential papers in 80's in both logic and computer science.

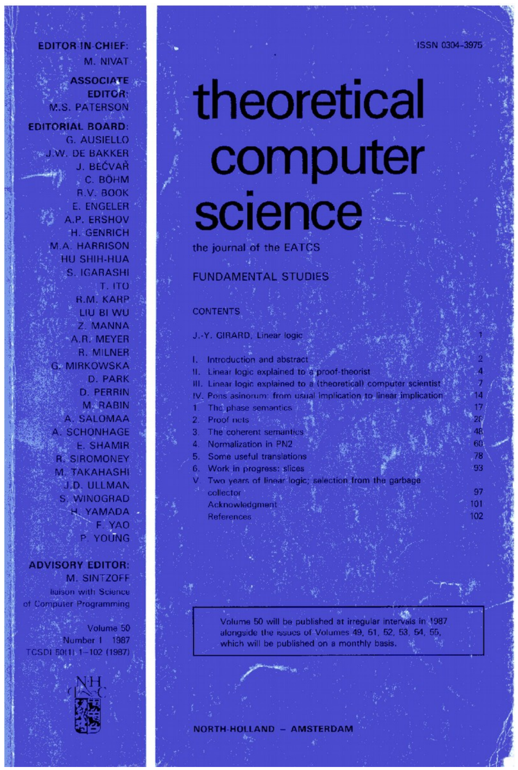
$$
\begin{array}{rcl}
X^T & = & X \\
(A \to B)^T & = & !?A^T \multimap ?B^T \\
(A \wedge B)^T & = & ?A^T \mathbin{\&} ?B^T \\
T^T & = & \top \\
(A \vee B)^T & = & ?A^T \parr ?B^T \\
F^T & = & \bot \\
(\neg A)^T & = & ?!(A^T)^\perp \\
(\forall \xi A)^T & = & \forall \xi ?A^T \\
(\exists \xi A)^T & = & \exists \xi !?A^T
\end{array}
$$

$$
\begin{array}{rcl}
X^n & = & X \\
(A \to B)^n & = & !A^n \multimap B^n \\
(A \wedge B)^n & = & A^n \mathbin{\&} B^n \\
T^n & = & \top \\
(A \vee B)^n & = & !A^n \oplus !B^n \\
F^n & = & 0 \\
(\forall \xi A)^n & = & \forall \xi A^n \\
(\exists \xi A)^n & = & \exists \xi !A^n
\end{array}
$$

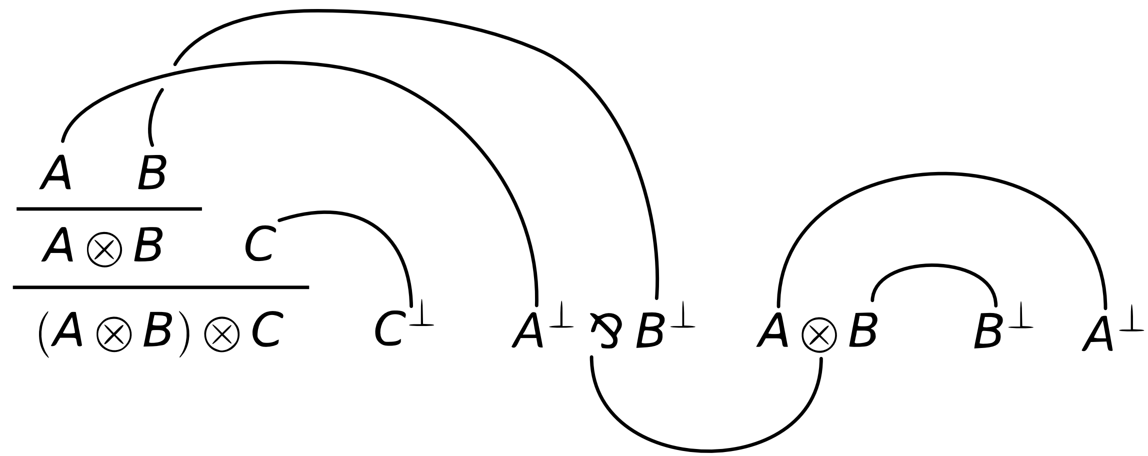
Attractive Ideas:

- Restructuring both Classical & Intuitionistic Logic
- Proof Nets
- Resource-Conciousness

Girard's Linear Logic



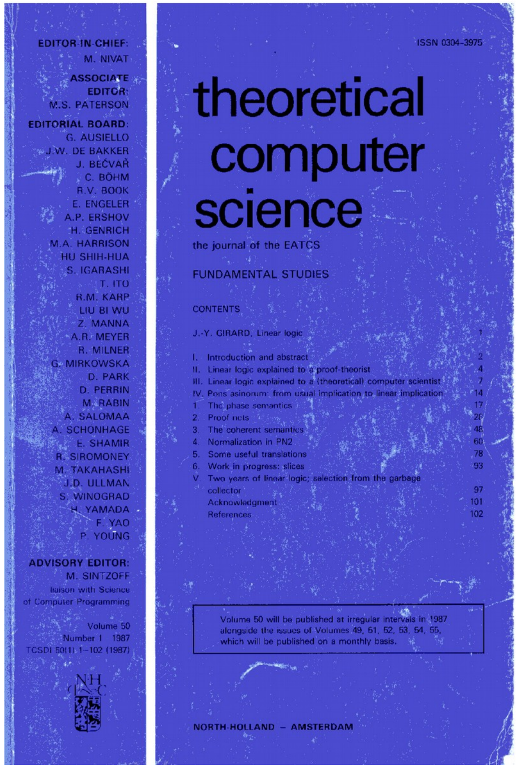
One of the most influential papers in 80's in both logic and computer science.



Attractive Ideas:

- Restructuring both Classical & Intuitionistic Logic
- Proof Nets
- Resource-Conciousness

Girard's Linear Logic



Volume 50
Number 1 1987
TCSDI 50(1) 1–102 (1987)

ISSN 0304-3975

theoretical computer science

the journal of the EATCS

FUNDAMENTAL STUDIES

CONTENTS

Volume 50 will be published at irregular intervals in 1987 alongside the issues of Volumes 49, 51, 52, 53, 54, 55, which will be published on a monthly basis.

NORTH-HOLLAND – AMSTERDAM

One of the most influential papers in 80's in both logic and computer science.

Attractive Ideas:

- Restructuring both Classical & Intuitionistic Logic
- Proof Nets
- Resource-Conciousness

Resource-Sensitivity

Imagine:

> there's no ***resource-conciousness***
>
> It isn't easy to do
>
> Nothing to ***comsume*** or ***lose*** for
>
> And no ***modalities*** too
>
> Imagine all the people
>
> Living life in ***Intuitionistic Logic*** ...

Resource-Sensitivity

Ex. In Intuitionistic Logic,

$$(A \longrightarrow B) \wedge (A \longrightarrow C) \longrightarrow (A \longrightarrow B \wedge C)$$ is true.

Substitute:

- A:= “to pay ￥400”
- B:= “to get a pack of cigarettes”
- C:= “to get a cup of cake”

a person in the Intuitionistic Logic world

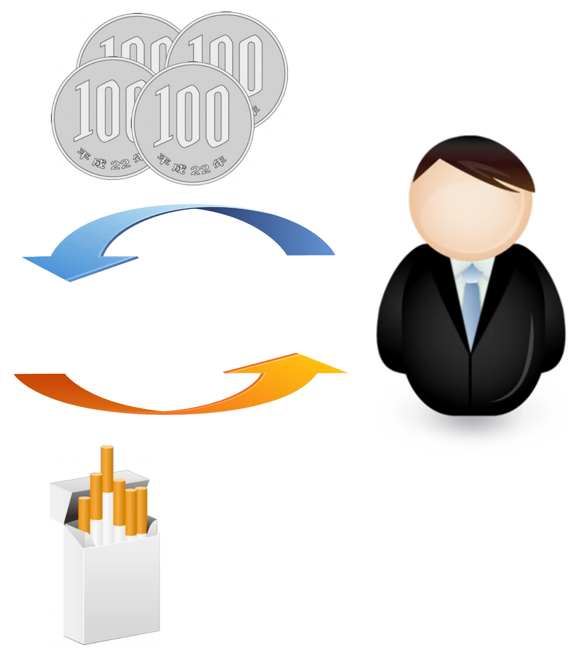
Resource-Sensitivity

Ex. In Intuitionistic Logic,

$$\boxed{(A \longrightarrow B)} \wedge (A \longrightarrow C) \longrightarrow (A \longrightarrow B \wedge C)$$ is true.

Substitute:

- A:= “to pay ￥400”
- B:= “to get a pack of cigarettes”
- C:= “to get a cup of cake”



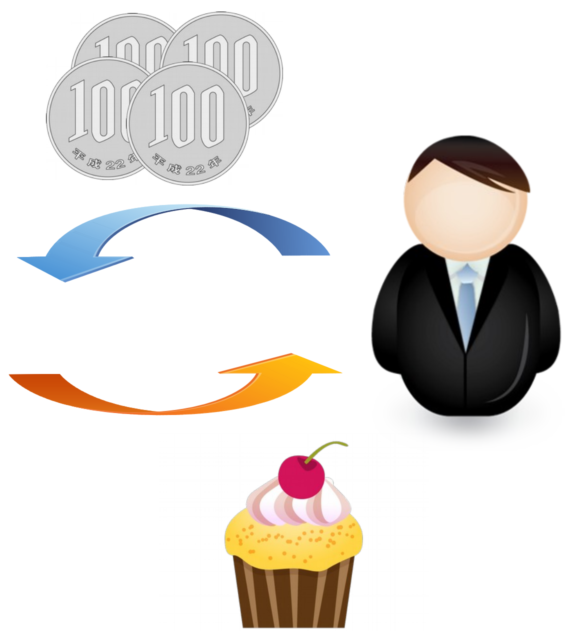
Resource-Sensitivity

Ex. In Intuitionistic Logic,

$$(A \longrightarrow B) \wedge \boxed{(A \longrightarrow C)} \longrightarrow (A \longrightarrow B \wedge C)$$ is true.

Substitute:

- A:= “to pay ￥400”
- B:= “to get a pack of cigarettes”
- C:= “to get a cup of cake”



Resource-Sensitivity

Ex. In Intuitionistic Logic,

$$(A \longrightarrow B) \wedge (A \longrightarrow C) \longrightarrow \boxed{(A \longrightarrow B \wedge C)} \text{ is true.}$$

Substitute:

- A:= “to pay ￥400”
- B:= “to get a pack of cigarettes”
- C:= “to get a cup of cake”

Paradox

Resource-Sensitivity

Ex. In Intuitionistic Logic,

$(A \longrightarrow B) \wedge (A \longrightarrow C) \longrightarrow (A \longrightarrow B \wedge C)$ is true.

A is used twice.

Substitute:

- A:= “to pay ￥400”
- B:= “to get a pack of cigarettes”
- C:= “to get a cup of cake”

Lack of conciousness to comsume assumptions !

Resource-Sensitivity

Ex. In Linear Logic,

$$(A \multimap B) \otimes (A \multimap C) \multimap (A \multimap B \otimes C)$$ is **false.**

New conjunction/ implication

Because: In LL, we must use the assumption exactly once in the proof.

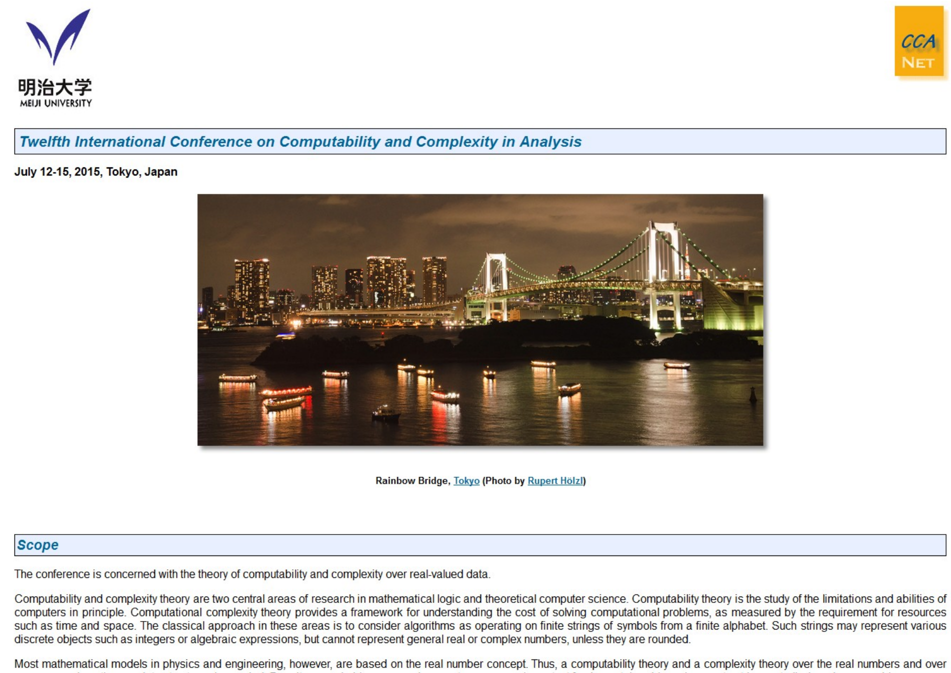
Coherence Spaces are proposed as a denotational semantics which reflects this property.

Via the Curry-Howard isomorphism,
they are also a model of resource-sensitive computations of linear function programs.

Main Result from CCA'15

Representations based on coherence spaces have an interesting feature:

for every ***real funcitons***, we have shown that


明治大学
MEIJI UNIVERSITY

CCA NET

Twelfth International Conference on Computability and Complexity in Analysis

July 12-15, 2015, Tokyo, Japan

Rainbow Bridge, Tokyo (Photo by Rupert Hölzl)

Scope

The conference is concerned with the theory of computability and complexity over real-valued data.

Computability and complexity theory are two central areas of research in mathematical logic and theoretical computer science. Computability theory is the study of the limitations and abilities of computers in principle. Computational complexity theory provides a framework for understanding the cost of solving computational problems, as measured by the requirement for resources such as time and space. The classical approach in these areas is to consider algorithms as operating on finite strings of symbols from a finite alphabet. Such strings may represent various discrete objects such as integers or algebraic expressions, but cannot represent general real or complex numbers, unless they are rounded.

Most mathematical models in physics and engineering, however, are based on the real number concept. Thus, a computability theory and a complexity theory over the real numbers and over

- stably realizable ⇔ continuous
- linearly realizable ⇔ uniformly continuous.

Let us emphasize that these correspondences hold for real functions.

Next step: generalize them to a wider class.

I. Review: Coherent Spaces

II. Coherence as Uniformity

III. Linear Admissibility

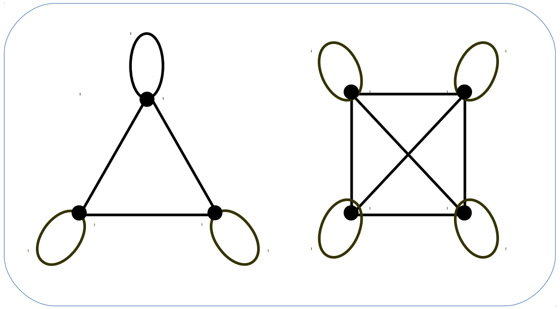
IV. Concluding Comments

Coherence Spaces

Def. A *coherence space* $\boldsymbol{X} = (X, \frown\!\!\!\!\smile)$ is a reflexive graph:

- a countable set of *tokens* X with
- a symmetric reflexive. binary rel. $\frown\!\!\!\!\smile$ on X

Write $x \frown y$ iff $x \frown\!\!\!\!\smile y$ and $x \neq y$ (*strict coherence*)
A *clique* is a set of tokens which are pairwise coherent.
An *anticlique* is a set of tokens in which every pair is not coherent.



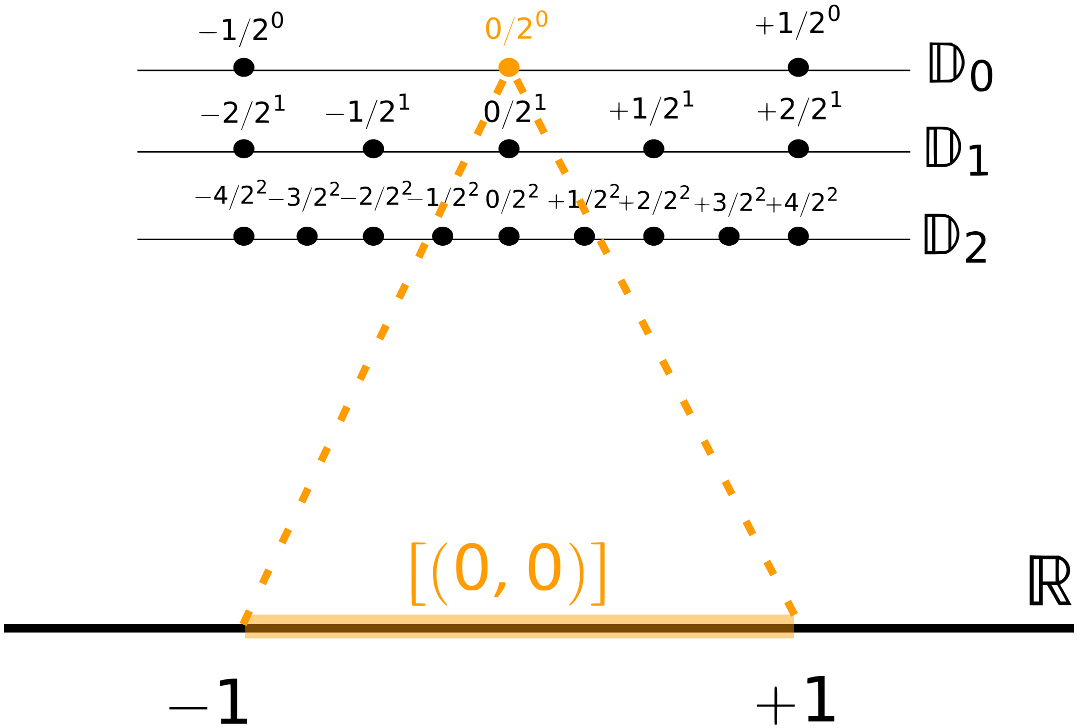
- $\boldsymbol{X}$:the set of all cliuqes.
- $\boldsymbol{X}_{fin}$:the set of all finite cliques.
- $\boldsymbol{X}_{max}$:the set of all maximal cliques.

Example: Cauchy Sequences

Let $\mathbb{D} = \mathbb{Z} \times \mathbb{N}$. Each member of $\mathbb{D}$ is identified with the *dyadic rational* as $(m, n) \in \mathbb{D} \sim m/2^n$.

For each $x := (m, n) \in \mathbb{D}$, define

- $den(x) := n$
- $\mathbb{D}_n := \{x \in \mathbb{D} : den(x) = n\}$
- $[x] := [(m-1)/2^n; (m+1)/2^n]$



Ex. Define a coherence space $\mathbf{R} := (\mathbb{D}, \frown)$ for *dyadic Cauchy sequences* as:

$$x \frown y \iff den(x) \neq den(y) \text{ and } [x] \cap [y] \neq \emptyset$$
$$\iff den(x) \neq den(y) \text{ and } |x - y| \leqslant 2^{-den(x)} + 2^{-den(y)}$$

Maximal cliques ≈ (*rapidly converging*) Cauchy sequences

$|x_n - x_m| \leqslant 2^{-n} + 2^{-m}$ for every $n, m \in \mathbb{N}$

Realization of Real Numbers

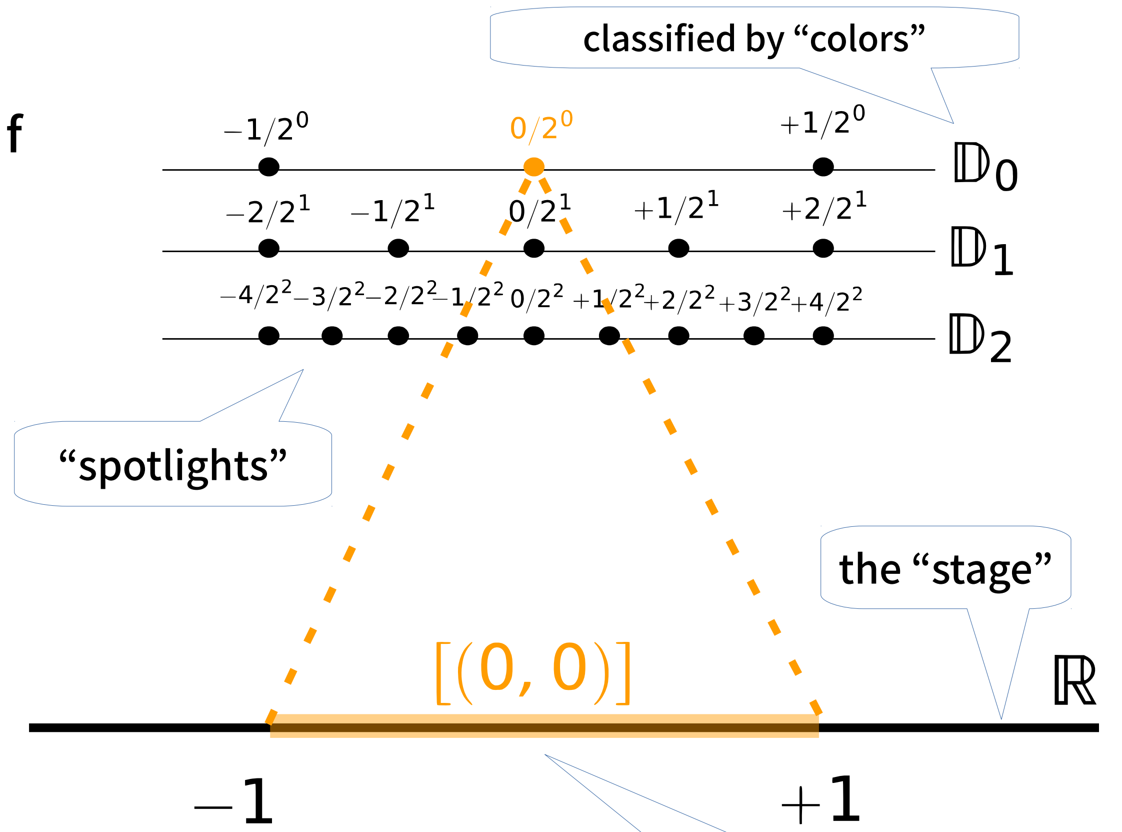
Example: Cauchy Sequences

Let $\mathbb{D} = \mathbb{Z} \times \mathbb{N}$. Each member of $\mathbb{D}$ is identified with the *dyadic rational* as $(m, n) \in \mathbb{D} \sim m/2^n$.

For each $x := (m, n) \in \mathbb{D}$, define

- $den(x) := n$
- $\mathbb{D}_n := \{x \in \mathbb{D} : den(x) = n\}$
- $[x] := [(m-1)/2^n ; (m+1)/2^n]$

Ex. Define a coherence space $\mathbf{R} := (\mathbb{D}, \frown)$ for *dyadic Cauchy sequences* as:



$$x \frown y \iff den(x) \neq den(y) \text{ and } [x] \cap [y] \neq \emptyset$$
$$\iff den(x) \neq den(y) \text{ and } |x - y| \leqslant 2^{-den(x)} + 2^{-den(y)}$$

Maximal cliques ≈ (*rapidly converging*) Cauchy sequences

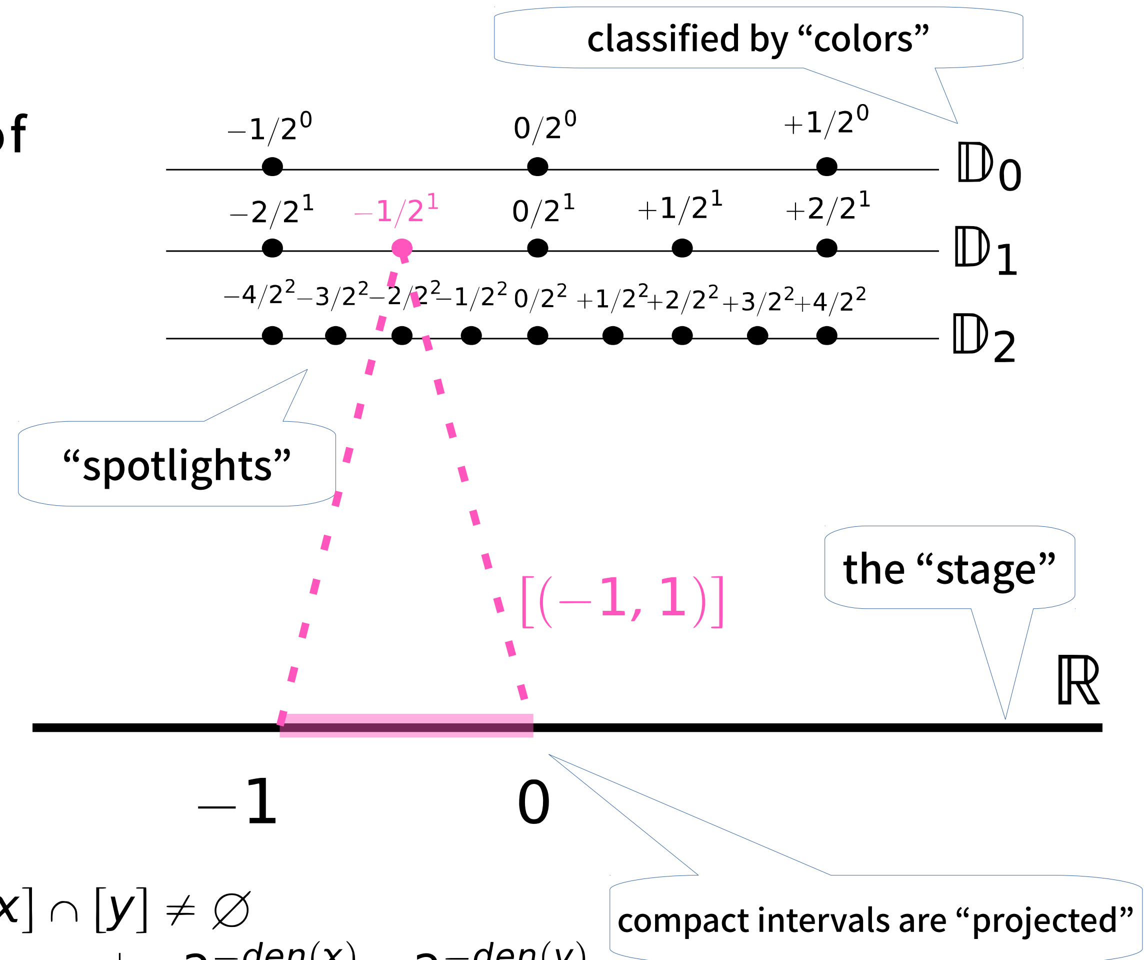
$|x_n - x_m| \leqslant 2^{-n} + 2^{-m}$ for every $n, m \in \mathbb{N}$

Example: Cauchy Sequences

Let $\mathbb{D} = \mathbb{Z} \times \mathbb{N}$. Each member of $\mathbb{D}$ is identified with the *dyadic rational* as $(m, n) \in \mathbb{D} \sim m/2^n$.

For each $x := (m, n) \in \mathbb{D}$, define

- $den(x) := n$
- $\mathbb{D}_n := \{x \in \mathbb{D} : den(x) = n\}$
- $[x] := [(m-1)/2^n ; (m+1)/2^n]$



Ex. Define a coherence space $\mathbf{R} := (\mathbb{D}, \frown)$ for *dyadic Cauchy sequences* as:

$$x \frown y \iff den(x) \neq den(y) \text{ and } [x] \cap [y] \neq \emptyset$$
$$\iff den(x) \neq den(y) \text{ and } |x - y| \leqslant 2^{-den(x)} + 2^{-den(y)}$$

Maximal cliques ≈ (*rapidly converging*) Cauchy sequences

$|x_n - x_m| \leqslant 2^{-n} + 2^{-m}$ for every $n, m \in \mathbb{N}$

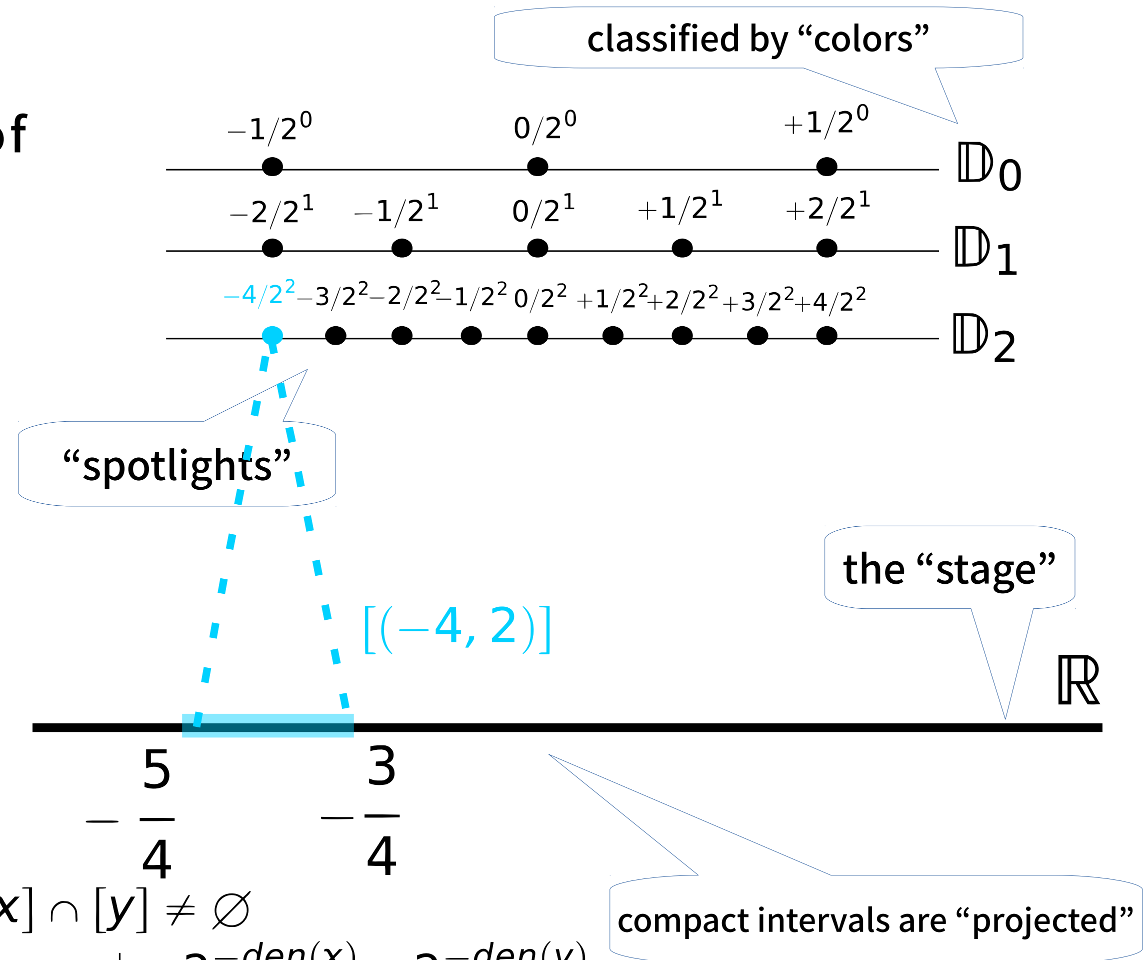
Realization of Real Numbers

Example: Cauchy Sequences

Let $\mathbb{D} = \mathbb{Z} \times \mathbb{N}$. Each member of $\mathbb{D}$ is identified with the *dyadic rational* as $(m, n) \in \mathbb{D} \sim m/2^n$.

For each $x := (m, n) \in \mathbb{D}$, define

- $den(x) := n$
- $\mathbb{D}_n := \{x \in \mathbb{D} : den(x) = n\}$
- $[x] := [(m-1)/2^n; (m+1)/2^n]$



Ex. Define a coherence space $\mathbf{R} := (\mathbb{D}, \frown)$ for *dyadic Cauchy sequences* as:

$$x \frown y \iff den(x) \neq den(y) \text{ and } [x] \cap [y] \neq \emptyset$$
$$\iff den(x) \neq den(y) \text{ and } |x - y| \leqslant 2^{-den(x)} + 2^{-den(y)}$$

Maximal cliques ≈ (*rapidly converging*) Cauchy sequences

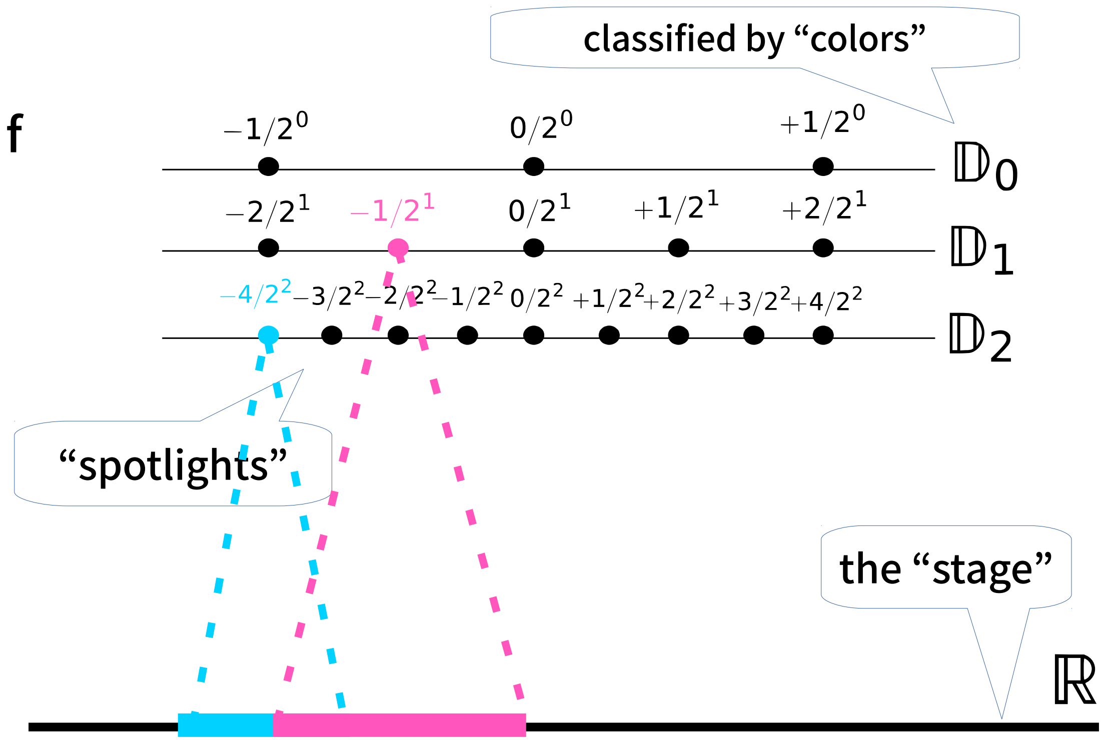
$|x_n - x_m| \leqslant 2^{-n} + 2^{-m}$ for every $n, m \in \mathbb{N}$

Example: Cauchy Sequences

Let $\mathbb{D} = \mathbb{Z} \times \mathbb{N}$. Each member of $\mathbb{D}$ is identified with the *dyadic rational* as $(m, n) \in \mathbb{D} \sim m/2^n$.

For each $x := (m, n) \in \mathbb{D}$, define

- $den(x) := n$
- $\mathbb{D}_n := \{x \in \mathbb{D} : den(x) = n\}$
- $[x] := [(m-1)/2^n ; (m+1)/2^n]$



Ex. Define a coherence space $\mathbf{R} := (\mathbb{D}, \frown)$ for *dyadic Cauchy sequences* as:

$$x \frown y \iff den(x) \neq den(y) \text{ and } [x] \cap [y] \neq \emptyset$$
$$\iff den(x) \neq den(y) \text{ and } |x - y| \leqslant 2^{-den(x)} + 2^{-den(y)}$$

Lights of different colors are overlapped

Maximal cliques ≈ (*rapidly converging*) Cauchy sequences

$|x_n - x_m| \leqslant 2^{-n} + 2^{-m}$ for every $n, m \in \mathbb{N}$

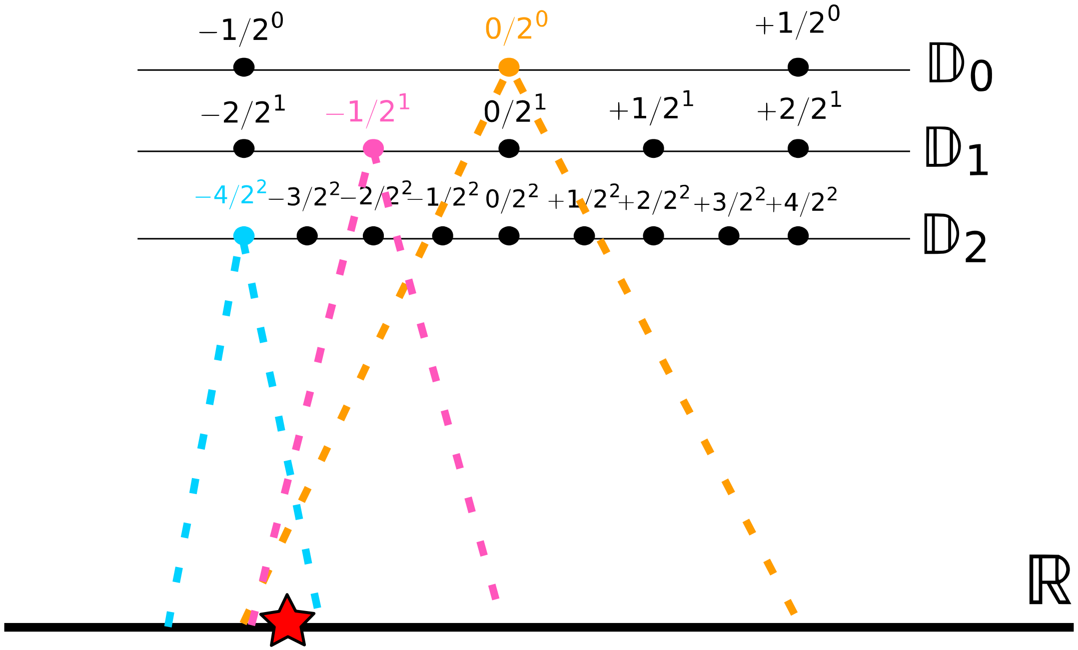
Realization of Real Numbers

Example: Cauchy Sequences

Let $\mathbb{D} = \mathbb{Z} \times \mathbb{N}$. Each member of $\mathbb{D}$ is identified with the *dyadic rational* as $(m, n) \in \mathbb{D} \sim m/2^n$.

For each $x := (m, n) \in \mathbb{D}$, define

- $den(x) := n$
- $\mathbb{D}_n := \{x \in \mathbb{D} : den(x) = n\}$
- $[x] := [(m-1)/2^n; (m+1)/2^n]$



Ex. Define a coherence space $\mathbf{R} := (\mathbb{D}, \frown)$ for *dyadic Cauchy sequences* as:

$$x \frown y \iff den(x) \neq den(y) \text{ and } [x] \cap [y] \neq \varnothing$$
$$\iff den(x) \neq den(y) \text{ and } |x - y| \leqslant 2^{-den(x)} + 2^{-den(y)}$$

Maximal cliques ≈ (*rapidly converging*) Cauchy sequences

$|x_n - x_m| \leqslant 2^{-n} + 2^{-m}$ for every $n, m \in \mathbb{N}$

Realization of Real Numbers

Coherence as Topology

The set $\boldsymbol{X}$ of cliques is ordered by $\subseteq$, endowed with the Scott topology generated by the upper sets of finite cliques:

$$\uparrow a := \left\{ b \in \boldsymbol{X} : b \supseteq a \right\} \qquad (a \in \boldsymbol{X}_{fin})$$

- Coherence spaces are very simplified domains
- Compact Elements $=$ finite cliques

Finite Cliques induce Topology

Stable & Linear Maps

Def. A function $F : \boldsymbol{X} \longrightarrow \boldsymbol{Y}$ is *stable* if it is $\subseteq$ -monotone and satisfies

$$\forall a \in \boldsymbol{X},\ \forall y \in Y,$$

$$F(a) \ni y \quad \Longrightarrow \quad \exists!\ a_0 \subseteq_{\mathrm{fin}} a.\ \text{minimal s.t. } F(a_0) \ni y$$

Collection of resources

Model of computations in which the amount of resources to be used is *uniquely determined*.

Def. A function $F : \boldsymbol{X} \longrightarrow \boldsymbol{Y}$ is *linear* if it is $\subseteq$ -monotone and satisfies

$$\forall a \in \boldsymbol{X},\ \forall y \in Y,$$

$$F(a) \ni y \quad \Longrightarrow \quad \exists x \in a.\ \text{unique s.t. } F(\{x\}) \ni y$$

Model of computations in which resources are used *exactly once*.

Stable & Linear Maps are Resource-Sensitive.

Girard's Formula

Model of Intuit. Logic

Two closed structures of coherence spaces:

- The category **Stbl** of coh. spaces and stable maps is *cartesian closed*
  - $\boldsymbol{X} \Rightarrow \boldsymbol{Y}$: the coherence space for stable maps.
- The category **Lin** of coh. spaces and linear maps is *monoidal closed*
  - $\boldsymbol{X} \multimap \boldsymbol{Y}$: the coherence space for linear maps.

They are combined by introducing the "of course" modality:

$!\boldsymbol{X} = (\boldsymbol{X}_{fin}, \frown)$ naturally defined by $a \frown b \iff a \cup b \in \boldsymbol{X}$.

Th. $\boldsymbol{X} \Rightarrow \boldsymbol{Y} = !\boldsymbol{X} \multimap \boldsymbol{Y}$

Linear Decompostion of Cartesian closed Structure.

I. Review: Coherent Spaces

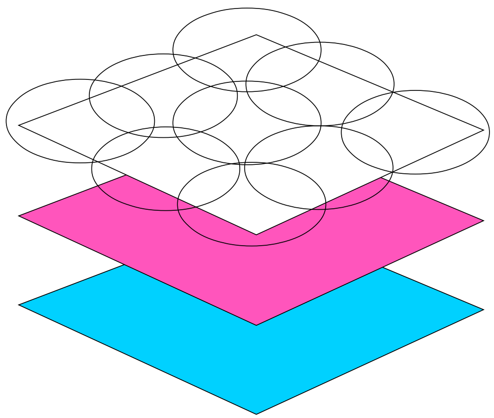
II. Coherence as Uniformity

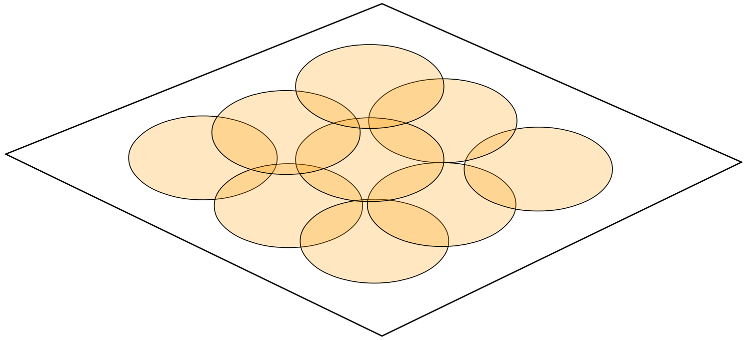
III. Linear Admissibility

IV. Concluding Comments

Review: Uniform Space

- A uniform space is a set with a uniformity: a collection of coverings of the set.
  - Each uniform cover is considered to be consisting of balls of the "same size"
  - They are partially ordered by the refinement relation and form a filter.
- They also induce a topology in the "vertical" way.
  - The balls surrounding each point form a neighborhood filter, which generates the uniform topology.
- Every uniformizable space has the finest uniformity: the fine uniform space.

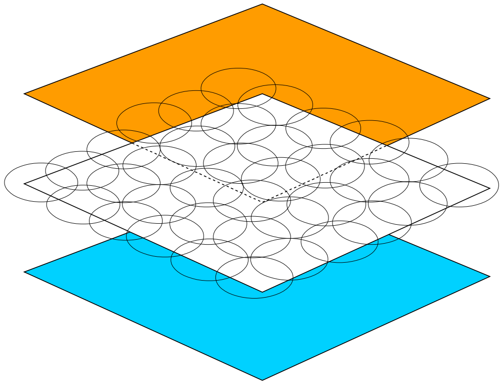


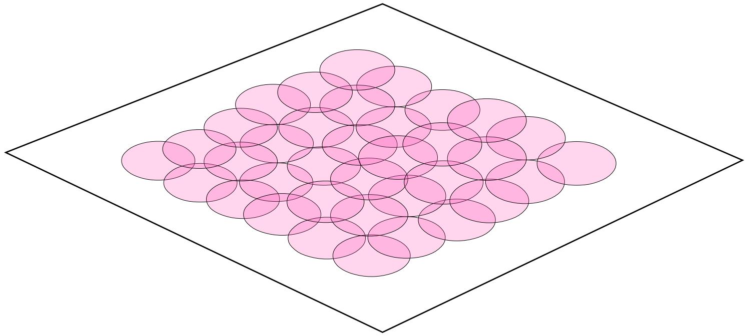


Uniform Covers are given *horizontally*

Review: Uniform Space

- A uniform space is a set with a uniformity: a collection of coverings of the set.
  - Each uniform cover is considered to be consisting of balls of the "same size"
  - They are partially ordered by the refinement relation and form a filter.
- They also induce a topology in the "vertical" way.
  - The balls surrounding each point form a neighborhood filter, which generates the uniform topology.
- Every uniformizable space has the finest uniformity: the fine uniform space.

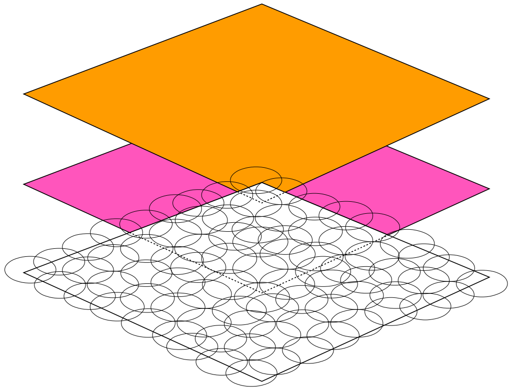


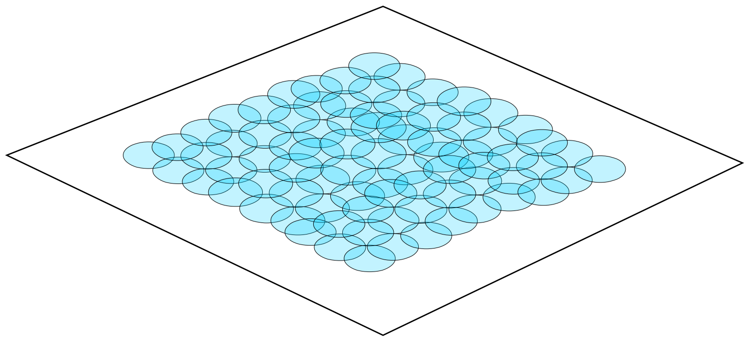


Uniform Covers are given *horizontally*

Review: Uniform Space

- A uniform space is a set with a uniformity: a collection of coverings of the set.
  - Each uniform cover is considered to be consisting of balls of the "same size"
  - They are partially ordered by the refinement relation and form a filter.
- They also induce a topology in the "vertical" way.
  - The balls surrounding each point form a neighborhood filter, which generates the uniform topology.
- Every uniformizable space has the finest uniformity: the fine uniform space.

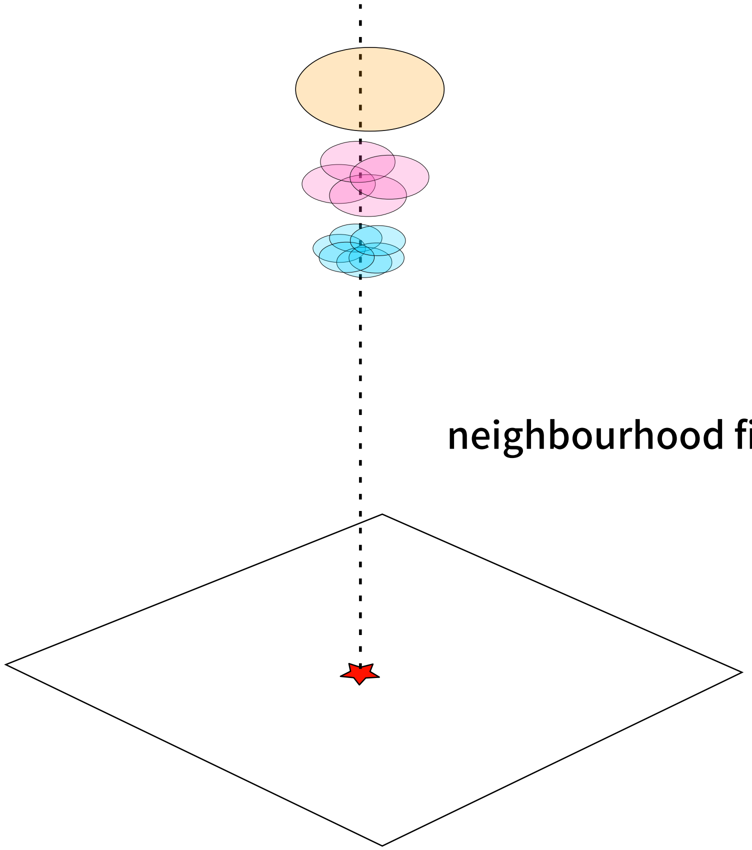




Uniform Covers are given *horizontally*

Review: Uniform Space

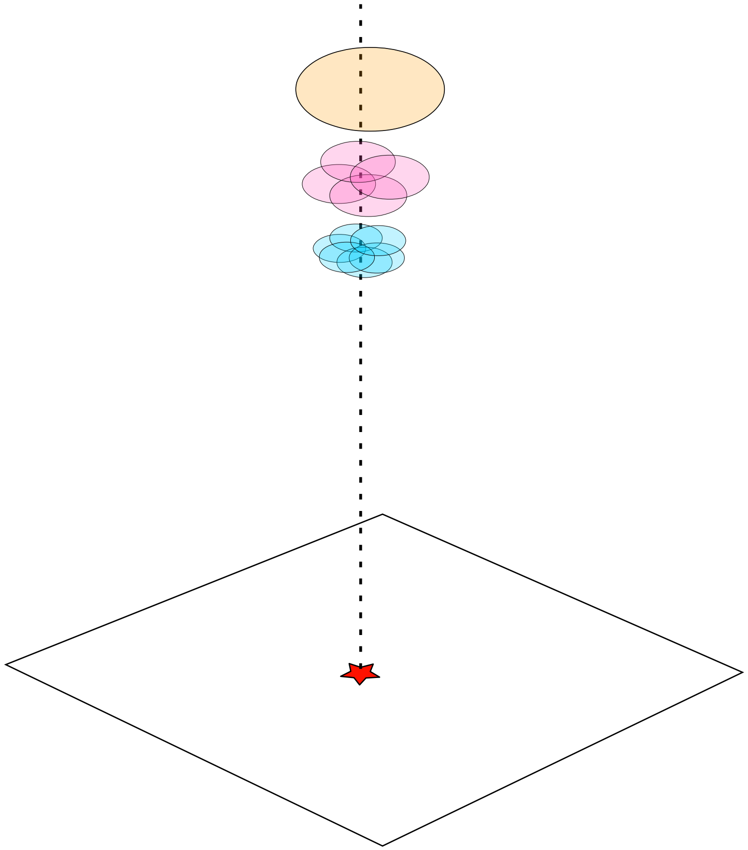
- A uniform space is a set with a uniformity: a collection of coverings of the set.
  - Each uniform cover is considered to be consisting of balls of the "same size"
  - They are partially ordered by the refinement relation and form a filter.
- They also induce a topology in the "vertical" way.
  - The balls surrounding each point form a neighborhood filter, which generates the uniform topology.
- Every uniformizable space has the finest uniformity: the fine uniform space.



Neighbors are given *vertically*

Review: Uniform Space

- A uniform space is a set with a uniformity: a collection of coverings of the set.
  - Each uniform cover is considered to be consisting of balls of the "same size"
  - They are partially ordered by the refinement relation and form a filter.
- They also induce a topology in the "vertical" way.
  - The balls surrounding each point form a neighborhood filter, which generates the uniform topology.
- Every uniformizable space has the finest uniformity: the fine uniform space.



Th. Let $\mathbb{X}_{\text{fine}}$ be the fine space compatible with a topological space $\mathbb{X}$.

$$f : \mathbb{X} \to \mathbb{Y} \text{ is continuous} \iff f : \mathbb{X}_{\text{fine}} \to \mathbb{Y} \text{ is uniformly continuous}$$

We've seen that situation!

Girard's Formula

Model of Intuit. Logic

Two closed structures of coherence spaces:

- The category **Stbl** of coh. spaces and stable maps is *cartesian closed*
  - $\boldsymbol{X} \Rightarrow \boldsymbol{Y}$: the coherence space for stable maps.
- The category **Lin** of coh. spaces and linear maps is *monoidal closed*
  - $\boldsymbol{X} \multimap \boldsymbol{Y}$: the coherence space for linear maps.

They are combined by introducing the "of course" modality:

$!\boldsymbol{X} = (\boldsymbol{X}_{fin}, \frown)$ naturally defined by $a \frown b \iff a \cup b \in \boldsymbol{X}$.

Th. $$\boldsymbol{X} \Rightarrow \boldsymbol{Y} = !\boldsymbol{X} \multimap \boldsymbol{Y}$$

Linear Decompostion of Cartesian closed Structure.

Coherence as Uniformity

Recall: $\uparrow x := \{a \in \boldsymbol{X} : a \ni x\}$ for each token $x \in X$

Incoherence implies disjointness: $\neg(x \frown y) \implies \uparrow x \cap \uparrow y = \emptyset$

A *partition* of $\boldsymbol{X}_{max}$ is an anticlique which induces the disjoint covering.

Condition: a coherence space $\boldsymbol{X} = (X, \frown)$ is *disjointly coverable* if every token can be extended to a partition:

$$\forall x \in X.\ \exists \mathfrak{a} \in \boldsymbol{X}^{\perp}.\quad x \in \mathfrak{a} \quad \text{and} \quad \textstyle\sum_{y \in \mathfrak{a}} \uparrow y \supseteq \boldsymbol{X}_{max}$$

Coh. Sp. for anticliques

Prop. $\underset{a}{\boldsymbol{X}_{max}} \simeq \underset{!a := \{a_0 : a_0 \subseteq_{fin} a\}}{!\boldsymbol{X}_{max}}$ (homeomorphic.)

Th. 1) Partitions of $\boldsymbol{X}_{max}$ form a subbasis for a uniformity.
2) Partitions of $!\boldsymbol{X}_{max}$ form a basis for a uniformity.
3) Both uniformities induce the Scott topology as the uniform topologies.
4) The induced uniformity on $!\boldsymbol{X}_{max}$ is fine.

Anticliques induce Uniformity

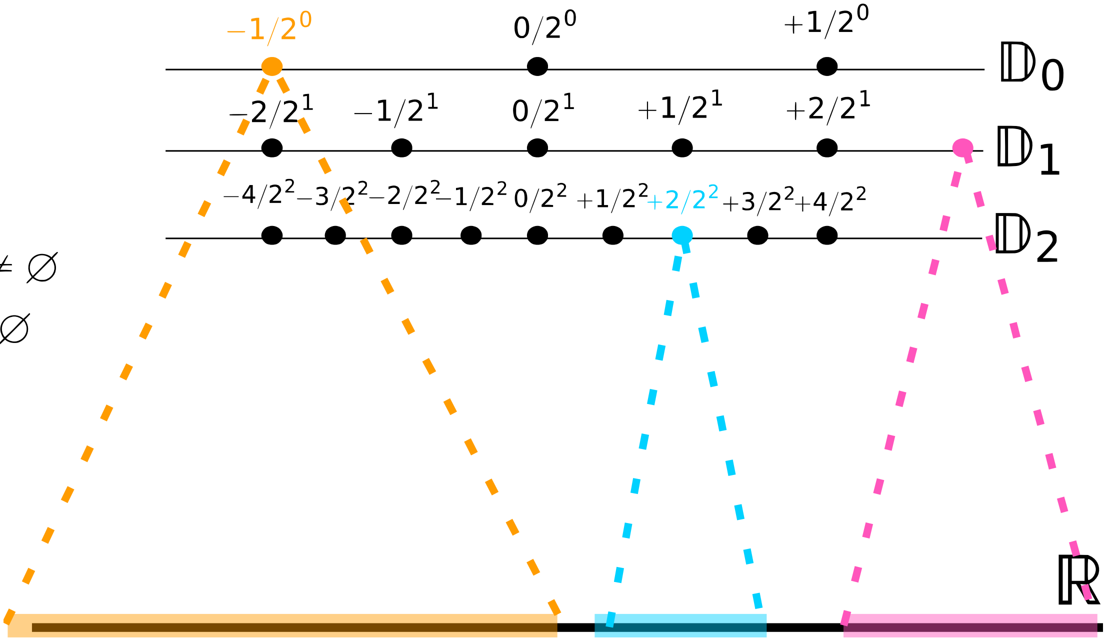
Cauchy Sequences Again

Ex. Define a coherence space $\mathbf{R} := (\mathbb{D}, \frown)$ for *dyadic Cauchy sequences* as:

$$x \frown y \iff den(x) \neq den(y) \text{ and } [x] \cap [y] \neq \varnothing$$

$$x \asymp y \iff den(x) = den(y) \text{ or } [x] \cap [y] = \varnothing$$

(incoherence)



Each $\mathbb{D}_n$ is a partition of $\mathbf{R}_{max}$, because a maximal clique must contain "all colors". There are no other partitions consisting of "several colors".

- In any partition, spotlights of *different* colors must project *disjoint* sets by the second condition of the incoherence.
- This is impossible essentially due to Sierpiński's theorem.
- The partitions then generate the uniformity compatible with the real line, the representation is a *uniformly open* map.

Linear ⇒ Uniform Continuous

Recall that

Th. $\boldsymbol{X} \Rightarrow \boldsymbol{Y} \quad = \quad !\boldsymbol{X} \multimap \boldsymbol{Y}$

Th. $f : \mathbb{X} \to \mathbb{Y}$ is continuous $\Longleftrightarrow f : \mathbb{X}_{\mathrm{fine}} \to \mathbb{Y}$ is uniformly continuous

We then combine these results.
Assume that F preserves maximality of cliques.

Th. Every linear map $F : \boldsymbol{X} \longrightarrow \boldsymbol{Y}$ is uniformly continuous with respect to the uniformities induced by partitions.

Cor. Every stable map $F : \boldsymbol{X} \longrightarrow \boldsymbol{Y}$ is continuous.

(although it is a reinvention of the wheel...)

I. Review: Coherent Spaces

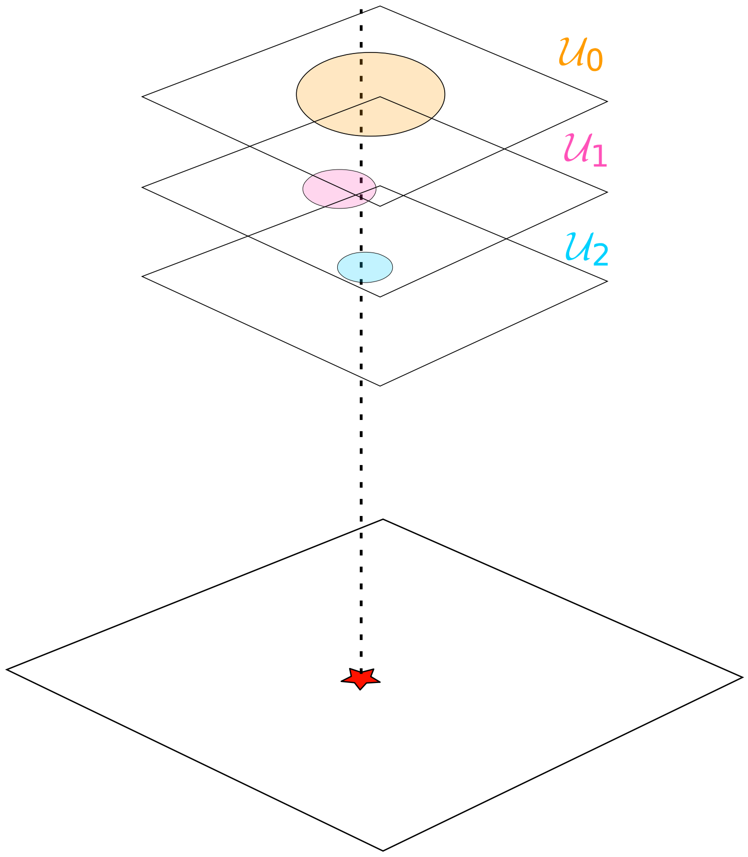
II. Coherence as Uniformity

III. Linear Admissibility

IV. Concluding Comments

Standard Representations

Let $\mathbb{X}$ be a Hausdorff uniform space with a countable basis $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ consisting of countable coverings.



Fact. Such a space is known to be *separable metrizable*.

Def. The *standard representation* of $\mathbb{X}$ is given by the coherence space $\mathbf{B} = (B, \frown)$ defined by $B = \{(n, U) : n \in \mathbb{N}, U \in \mathcal{U}_n\}$ and

$$(n, U) \frown (m, V) \iff n \neq m \text{ and } U \cap V \neq \varnothing$$

Each Maximal clique of $\mathbf{B}$ specifies *at most* one point.

Linear Admissibility

Standard representation does not depend on the choice of basis $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$:

This generalizes to the notion of admissibility.

$$
\begin{tikzcd}
\mathbf{B}_1 \arrow[rr, bend left] \arrow[dr] & & \mathbf{B}_2 \arrow[ll, bend left] \arrow[dl] \\
& \mathbb{X} &
\end{tikzcd}
$$

An Idea. A representation $\boldsymbol{Y} \xrightarrow{\rho} \mathbb{X}$ is *linear admissible* if

1. for every uniform cover $\mathcal{U}$ of $\mathbb{X}$ there exists a uniform cover of $\boldsymbol{Y}_{max}$ which refines $\mathcal{U}$, and
2. for every subspace $\mathbb{X}_0$ and its representation $\boldsymbol{X} \xrightarrow{\gamma} \mathbb{X}_0$ satisfying (1) the inclusion map is tracked by some linear map F .

uniform cont.

$$
\begin{tikzcd}
\boldsymbol{X} \arrow[r, "F"] \arrow[d, "\gamma"'] & \boldsymbol{Y} \arrow[d, "\rho"] \\
\mathbb{X}_0 \arrow[r, "i"] & \mathbb{X}
\end{tikzcd}
$$

A naive idea is to mimic Admissibility in TTE, but it doesn't work!!

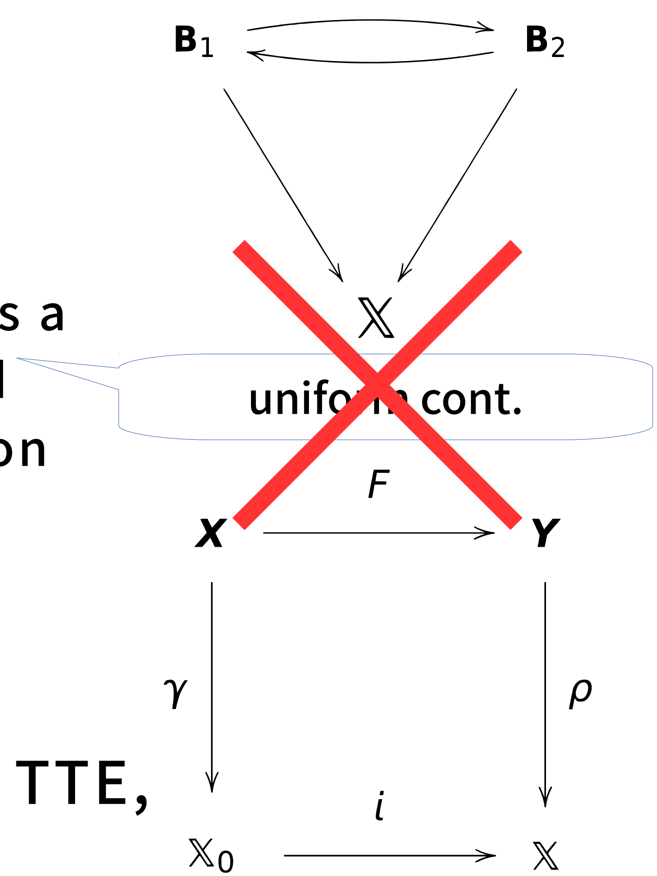
Linear Admissibility

Standard representation does not depend on the choice of basis $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$:

This generalizes to the notion of admissibility.

~~**An Idea:** A representation $\boldsymbol{Y} \xrightarrow{\rho} \mathbb{X}$ is *linear admissible* if~~

1. ~~for every uniform cover $\mathcal{U}$ of $\mathbb{X}$ there exists a uniform cover of $\boldsymbol{Y}_{map}$ which refines $\mathcal{U}$, and~~
2. ~~for every subspace $\mathbb{X}_0$ and its representation $\boldsymbol{X} \xrightarrow{\gamma} \mathbb{X}_0$ satisfying (1) the inclusion map is tracked by some linear map F.~~



A naive idea is to mimic Admissibility in TTE, but it doesn't work!!

Linear Admissibility

Standard representation does not depend on the choice of basis $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$:

This generalizes to the notion of admissibility.

$$
\begin{tikzcd}
\mathbf{B}_1 \arrow[rr, bend left] \arrow[dr] & & \mathbf{B}_2 \arrow[ll, bend left] \arrow[dl] \\
& \mathbb{X} &
\end{tikzcd}
$$

Def. A representation $\boldsymbol{Y} \xrightarrow{\rho} \mathbb{X}$ is *linear admissible* if

1. for every uniform cover $\mathcal{U}$ of $\mathbb{X}$ there exists a *partition* of $\boldsymbol{Y}_{max}$ which refines $\mathcal{U}$, and
2. for every subspace $\mathbb{X}_0$ and its representation $\boldsymbol{X} \xrightarrow{\gamma} \mathbb{X}_0$ satisfying (1) the inclusion map is tracked by some linear map F .

strongly uniform cont.

$$
\begin{tikzcd}
\boldsymbol{X} \arrow[r, "F"] \arrow[d, "\gamma"'] & \boldsymbol{Y} \arrow[d, "\rho"] \\
\mathbb{X}_0 \arrow[r, "i"] & \mathbb{X}
\end{tikzcd}
$$

Th. (1) Every uniformly continuous maps is then linearly realizable w.r.t. linear admissible representations.
(2) Every standard representation is linear admissible.

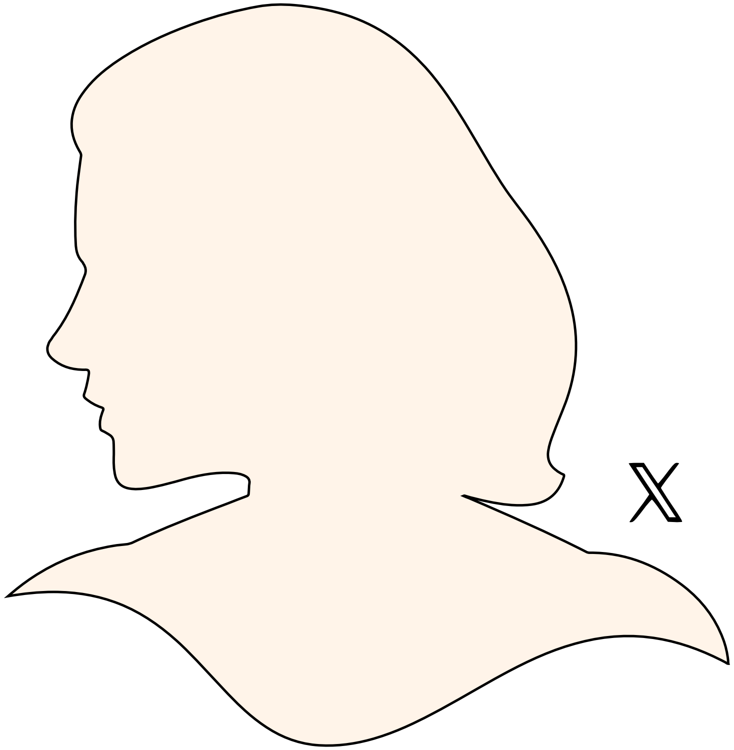
Chain-Connectedness

Def. A uniform space $\mathbb{X} = (X, \mu)$ is *chain-connected* if

$$\forall x, y \in X.\ \forall \mathcal{U} \in \mu.\ \exists U_1, \ldots, U_n \in \mathcal{U}.$$
$$\text{s.t. } x \in U_1,\ y \in U_n \text{ and}$$
$$U_i \cap U_{i+1} \neq \varnothing \text{ for every } i < n.$$

A typical example: $\mathbb{Q}$ (though it is totally disconnected)

In particular, every two members of a uniform cover is "chainly connected"



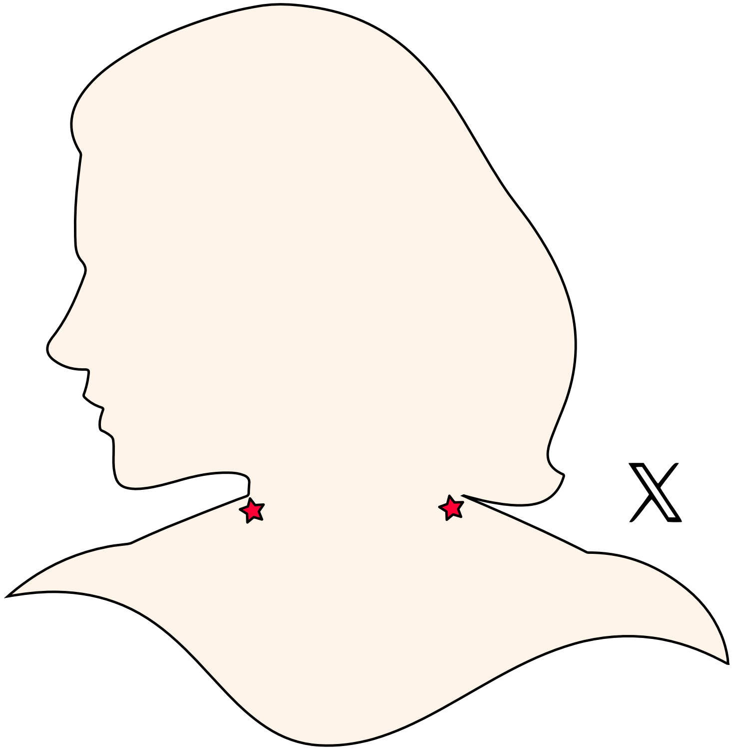
Chain-Connectedness

Def. A uniform space $\mathbb{X} = (X, \mu)$ is *chain-connected* if

$$\forall x, y \in X.\ \forall \mathcal{U} \in \mu.\ \exists U_1, \ldots, U_n \in \mathcal{U}.$$
$$\text{s.t. } x \in U_1,\ y \in U_n \text{ and}$$
$$U_i \cap U_{i+1} \neq \varnothing \text{ for every } i < n.$$

A typical example: $\mathbb{Q}$ (though it is totally disconnected)

In particular, every two members of a uniform cover is "chainly connected"



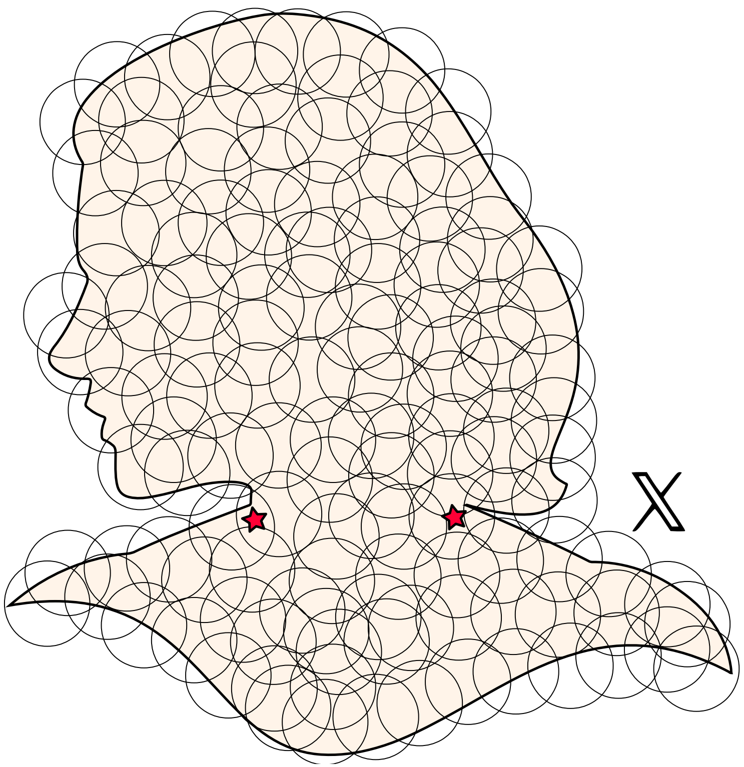
Chain-Connectedness

Def. A uniform space $\mathbb{X} = (X, \mu)$ is *chain-connected* if

$$\forall x, y \in X.\ \forall \mathcal{U} \in \mu.\ \exists U_1, \ldots, U_n \in \mathcal{U}.$$
$$\text{s.t. } x \in U_1,\ y \in U_n \text{ and}$$
$$U_i \cap U_{i+1} \neq \varnothing \text{ for every } i < n.$$

A typical example: $\mathbb{Q}$ (though it is totally disconnected)

In particular, every two members of a uniform cover is “chainly connected”



Chain-Connectedness

Def. A uniform space $\mathbb{X} = (X, \mu)$ is *chain-connected* if

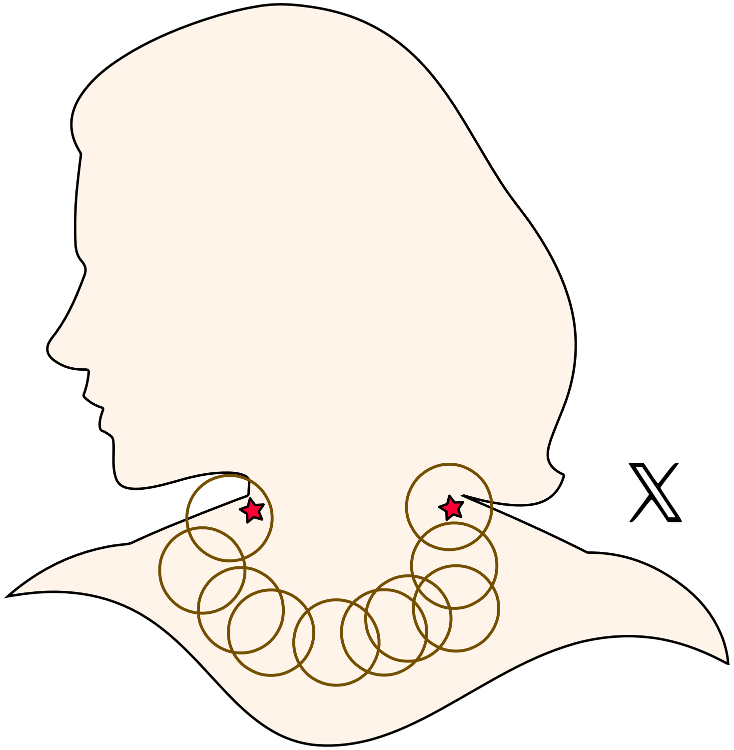
$\forall x, y \in X.\ \forall \mathcal{U} \in \mu.\ \exists U_1, \ldots, U_n \in \mathcal{U}.$

s.t. $x \in U_1$, $y \in U_n$ and

$U_i \cap U_{i+1} \neq \varnothing$ for every $i < n$.

A typical example: $\mathbb{Q}$ (though it is totally disconnected)

In particular, every two members of a uniform cover is "chainly connected"



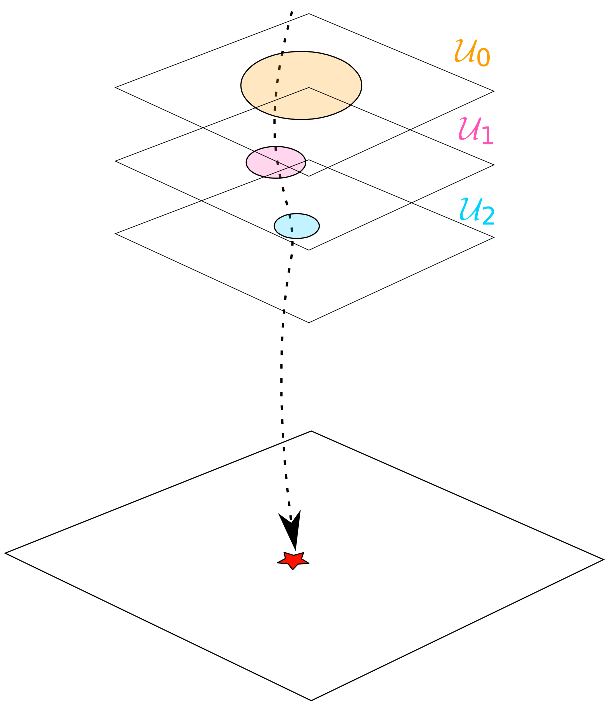
Principal Lemma

Suppose that $\mathbb{X}$ is a chain-connected separable metrizable space and that $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ is a uniform basis consiting of *open* coverings.

N.B. Every uniform space has a basis of open coverings.

Lem. The standard representation is a topologically and uniformly open map.

- Topological openness is immediate from the assumption.
- Uniform openness is essentially due to "one-coloredness" of partitions.



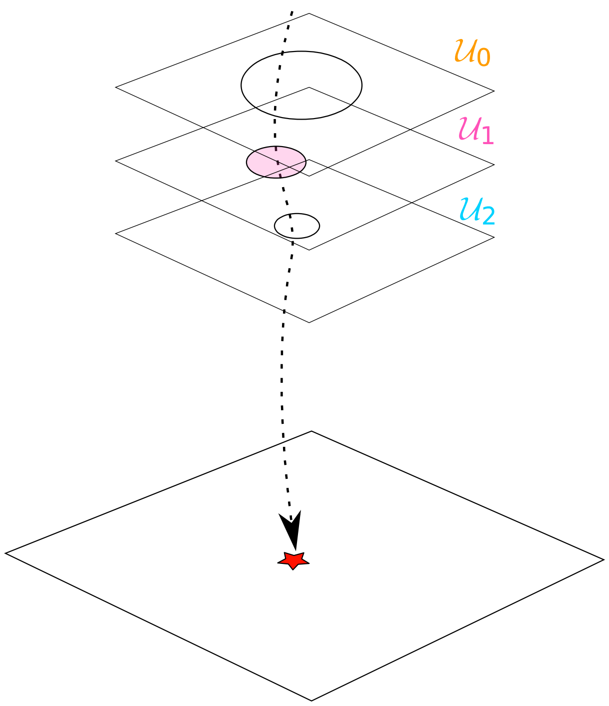
Principal Lemma

Suppose that $\mathbb{X}$ is a chain-connected separable metrizable space and that $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ is a uniform basis consiting of *open* coverings.

N.B. Every uniform space has a basis of open coverings.

Lem. The standard representation is a topologically and uniformly open map.

- Topological openness is immediate from the assumption.
- Uniform openness is essentially due to "one-coloredness" of partitions.



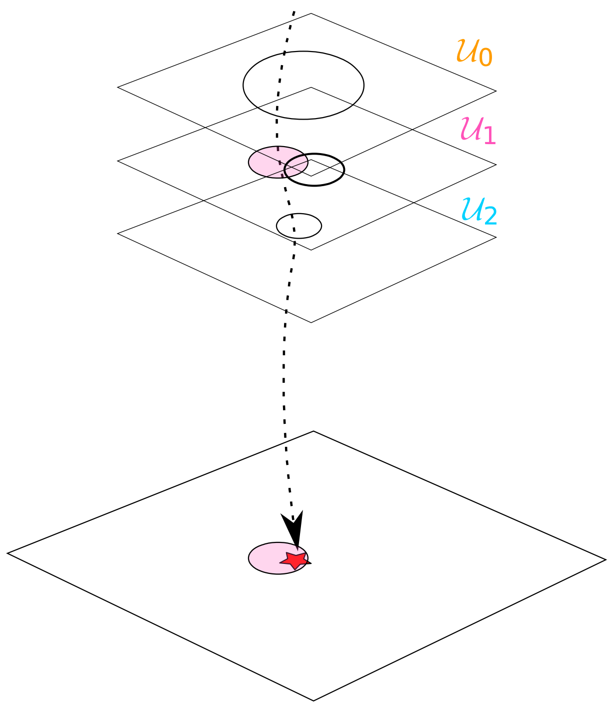
Principal Lemma

Suppose that $\mathbb{X}$ is a chain-connected separable metrizable space and that $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ is a uniform basis consiting of *open* coverings.

N.B. Every uniform space has a basis of open coverings.

Lem. The standard representation is a topologically and uniformly open map.

- Topological openness is immediate from the assumption.
- Uniform openness is essentially due to "one-coloredness" of partitions.



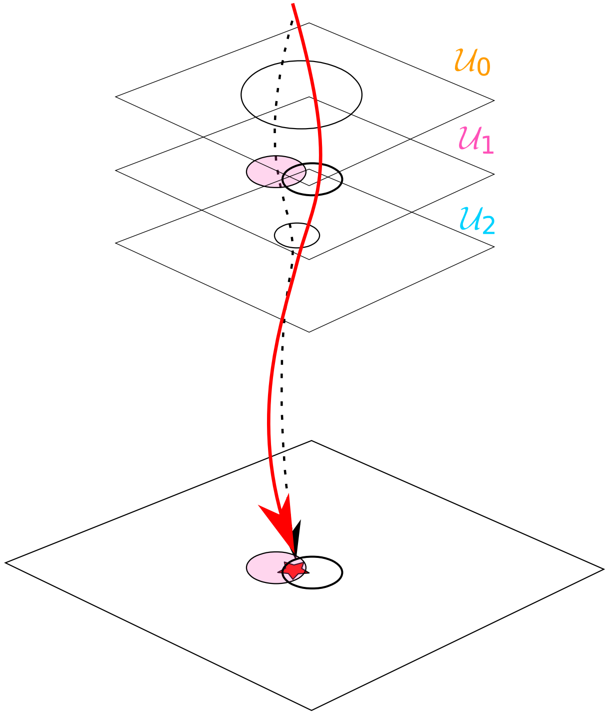
Principal Lemma

Suppose that $\mathbb{X}$ is a chain-connected separable metrizable space and that $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ is a uniform basis consiting of *open* coverings.

N.B. Every uniform space has a basis of open coverings.

Lem. The standard representation is a topologically and uniformly open map.

- Topological openness is immediate from the assumption.
- Uniform openness is essentially due to "one-coloredness" of partitions.

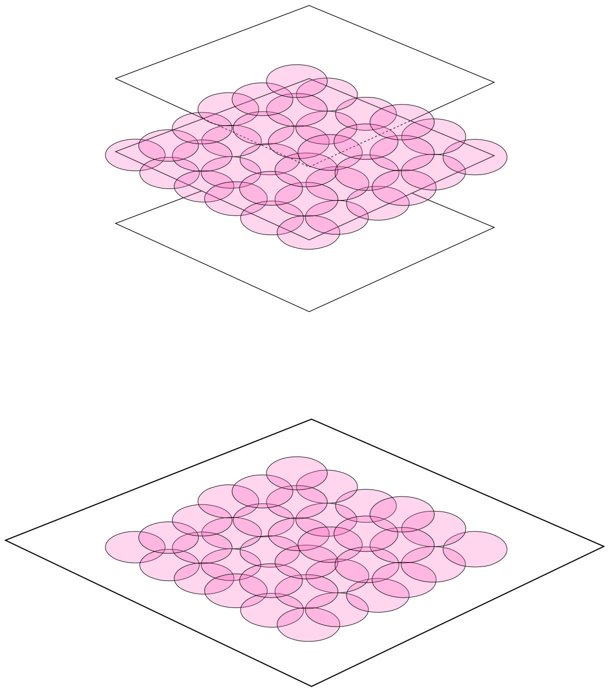


Principal Lemma

Suppose that $\mathbb{X}$ is a chain-connected separable metrizable space and that $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ is a uniform basis consiting of *open* coverings.

N.B. Every uniform space has a basis of open coverings.

Lem. The standard representation is a topologically and uniformly open map.



- Topological openness is immediate from the assumption.
- Uniform openness is essentially due to "one-coloredness" of partitions.

Main Result

Let $\mathbb{X}$ and $\mathbb{Y}$ be separable metrizable uniform spaces with linear admissible representations, and $f : \mathbb{X} \to \mathbb{Y}$.

Recall: every separable metrizable space has the standard representation, hence has linear admissible representations.

Th. If $\mathbb{X}$ is chain-connected,

- f : stably realizable ⇔ it is continuous.
- f : linearly realizable ⇔ it is uniformly continuous.

Cor. If $f : \mathbb{X} \to \mathbb{Y}$ is linearly realizable then it is uniformly continuous on each chain-connected component.

Conversely, every uniformly continuous function is linearly realizable. To complete the corollary, we need to reduce the components by identifying some of them.

I. Review: Coherent Spaces

II. Coherence as Uniformity

III. Linear Admissibility

IV. Concluding Comments

Towards Linear Realizability

- A linear combinatory algebra (LCA) U_{lin} is defined from the "universal coherence space".
- Coherent Representations = Modest Sets over U_{lin}
- **Mod**(U_{lin}) is a model of linear logic.
- I'm essentially a realist...but still a bit a *dreamer*
    - so I *imagine* there's a kind of "Linear Analysis" as the decomposition of Computable Analysis.
        - An analogy of the discovery of Linear Logic.
        - Every mathematical space has "admissible" representations in some sense, and functions are all linearly computable...

Towards Complexity in Analysis

- It seems hopeless that the category of ***linear admissible*** representations is monoidal closed because function spaces are not separable metrizable.
- Nonetheless, I still believe that linearity is strongly related to uniform structures because:

Th (Férée-Gomaa-Hoyrup' 13). For any real functional $F : \mathcal{C}[0,1] \to \mathbb{R}$

Theorem

The following are equivalent:

- F *is computable by a polynomial time machine doing only one oracle query*
- $\forall f, F(f) = \phi(f(\alpha))$ *where:*
  - $\alpha \in Poly(\mathbb{R})$
  - $\phi \in Poly(\mathbb{R} \to \mathbb{R})$
  - ϕ *is uniformly continuous*

linear in our terminology

F is "uniformly continuous" w.r.t. a kind of uniformity on C[0,1]:

$$\|f - g\| < \epsilon \iff \exists \alpha \in [0,1].\ |f(\alpha) - g(\alpha)| < \epsilon$$

We can explain this phenomenon in our model:

$$(\mathbf{I} \multimap \mathbf{R})^{\perp} \quad = \quad \mathbf{I} \otimes \mathbf{R}^{\perp}$$

Uniformity on C[0,1] a point in [0,1] & a uniformity of R

A Possible Way: Quasi-Uniformity

- In some sense, separable metrisability of linear admissible representations seems *inevitable:*
  - Because of countability of coherence spaces.
    - We must have a countable basis of countable coverings.
- But... the Hausdorff property seems not necessary.
    - just used for well-definedness of the representation.
- The use of non-Hausdorff metric seems a possible answer.
- Every second-countable T_1-space is separable *quasi-metrizable.*
  - It is very likely that they have the standard representations for quasi-uniform spaces.