Effective computability and constructive provability for existence sentences

Makoto Fujiwara

Waseda Institute for Advanced Study (WIAS), Waseda University

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Targets: Existence Statements

Many theorems in ordinary mathematics (Analysis, Algebra, Combinatorics etc.) can be formalized as Π^1_2 sentences having a form

$$\forall f\,(\varphi(f) \to \exists g \psi(f,g))\,,$$

where f and g are (possibly tuples of) functions on natural numbers.

- In computable (recursive) mathematics, effective contents of classical theorems have been investigated.
- In particular, there are many "effectivized" results of classical theorems in combinatorics, e.g.,
    - Brooks's theorem (Schmerl 1982, Carstens/Päppinghaus 1983, Tverberg 1984),
    - Marriage theorem (Kierstead 1983),
    - Dilworth's theorem (Kierstead 1981).

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Two Kinds of Effectivization

- **Non-uniform computability**:
  For any computable f, there exists a computable g.
- **Uniform computability**:
  There exists a uniform algorithm to obtain a witness g for each (computable) f.

Correspondence in Reverse Mathematics

Non-uniform computability
(For any computable f, there exists a computable g.)

Uniform computability
(There exists a uniform algorithm to obtain a witness g for each f.)

Constructive provability

Correspondence in Reverse Mathematics

Non-uniform computability (For any computable f, there exists a computable g.)	$\approx$	RCA_0
Uniform computability (There exists a uniform algorithm to obtain a witness g for each f.)		
Constructive provability	$\approx$	EL_0

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Uniform computability (There exists a uniform algorithm to obtain a witness g for each f.)	$\approx$	**??**
Constructive provability	$\approx$	EL_0

Toward an axiomatization of uniform computability

Investigate how uniform computability for existence statements can be captured by (semi-)intuitionistic provability in (many-sorted) arithmetic!

Formalization of Uniform Computability (F. 2015)

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- $\widehat{\mathsf{E\text{-}HA}}^\omega\restriction$ (resp. $\widehat{\mathsf{E\text{-}PA}}^\omega\restriction$) is the restrictions of $\mathsf{E\text{-}HA}^\omega$ (resp. $\mathsf{E\text{-}PA}^\omega$) to **primitive recursion of type 0** and quantifier-free induction, of which $\mathbf{T_0}$ is the terms.

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Fact. (Kohlenbach 2005)

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Fact. (Kohlenbach 2005)

- RCA_0^ω is a conservative extension of RCA_0.
- That is also the case for $\text{WKL}_0^\omega (:= \text{RCA}_0^\omega + \text{WKL})$ and $\text{ACA}_0^\omega (:= \text{RCA}_0^\omega + \text{ACA})$.

Uniform Provability in $\mathbf{\Gamma}$:

1. There exists a term t^1 s.t.

$$\mathbf{\Gamma} \vdash \forall f \left(\varphi(f) \to t|f \downarrow \wedge \psi(f, t|f) \right),$$

   where

$$\alpha(\beta) := \begin{cases} \alpha(\bar{\beta}n) - 1 \text{ where n is the least } n' \text{ s.t. } \alpha(\bar{\beta}n') \neq 0. \\ \uparrow \text{ if there is no such } n'. \end{cases}$$

$$\alpha|\beta := \lambda n.\ \alpha(\langle n \rangle ^\frown \beta).$$

2. There exists a (Gödel prim. rec.) term $t^{1\to1} \in \mathbf{T}$ s.t.

$$\mathbf{\Gamma} \vdash \forall f \left(\varphi(f) \to \psi(f, tf) \right).$$

3. There exists a (Kleene prim. rec.) term $t^{1\to1} \in \mathbf{T_0}$ s.t.

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Proposition 1. (F. 2015)

Let $\forall f\,(\varphi(f) \to \exists g \psi(f,g))$ be a $\mathcal{L}(\mathsf{EL}_0)$-formula such that $\varphi(f)$ is purely universal and $\psi(f,g)$ is equivalent to some formula $\forall w^\rho \exists s^0 \psi_{qf}(f,g,w,s)$ over EL_0.
There exists a term t^1 such that

$$\mathsf{RCA}_0(+\mathrm{WKL}) \vdash \forall f\,(\varphi(f) \to t|f\downarrow \wedge \psi(f, t|f))$$

if and only if

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On the Proof.

- IF direction is by the function realizability (Dorais 2014).
- ONLY IF direction is by Kuroda's negative translation and the (monotone) Dialectica interpretation.

Application.

For example, Kierstead's effective marriage theorem EMT has the required syntactical form in Proposition 1 and uniformly provable in RCA_0, then it follows that EMT is provable in EL_0.

Remark.

- It is known that many existence theorems are formalized as a Π^1_2 formula of the syntactical form in Proposition 1.
- The analogous results for EL, RCA instead of EL_0, RCA_0 also hold.

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- $\Pi^0_1\text{-AC}^{0,0}$:

  $\forall\alpha^1(\forall x^0\exists y^0\forall z^0\alpha(x,y,z)=0\rightarrow\exists\beta^1\forall x,z\,\alpha(x,\beta(x),z)=0)$

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Question.

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Question.

How is uniform provability in classical systems with $\Pi^0_1\text{-AC}^{0,0}$ characterized?

Fact.

The negative translation of $\Pi^0_1\text{-AC}^{0,0}$ is intuitionistically derived from $\Pi^0_1\text{-AC}^{0,0}$ and $\Sigma^0_2\text{-DNS}^0$:

$$\forall\alpha^1(\forall x^0\neg\neg\exists y^0\forall z^0\alpha(x,y,z)=0\to\neg\neg\forall x\exists y\forall z\alpha(x,y,z)=0).$$

Uniform $\Rightarrow$ Intuitionistic

Proposition 2.

Let $\forall f\, (\varphi(f) \to \exists g \psi(f,g))$ be a $\mathcal{L}(\mathsf{EL}_0)$-formula such that $\varphi(f) \in \mathcal{A}$ and $\psi(f,g) \in \mathcal{B}$.
If there exists a term $t^{1\to 1} \in \mathbf{T_0}$ such that

$$\widehat{\mathsf{E\text{-}PA}}^{\omega}\!\restriction + \Pi^0_1\text{-}\mathrm{AC}^{0,0} \vdash \forall f\, (\varphi(f) \to \psi(f, tf)),$$

then $\quad \mathsf{EL}_0 + \Pi^0_1\text{-}\mathrm{AC}^{0,0} + \Sigma^0_2\text{-}\mathrm{DNS}^0 \vdash \forall f\, (\varphi(f) \to \exists g \psi(f,g))\,.$

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The classes $\mathcal{A}$, $\mathcal{B}$ of formulas are defined simultaneously by

- P, $A_1 \wedge A_2$, $A_1 \vee A_2$, $\forall x A_1$, $\exists x A_1$, $B_1 \to A_1$ are in $\mathcal{A}$;
- P, $B_1 \wedge B_2$, $\forall x B_1$, $A_1 \to B_1$ are in $\mathcal{B}$;

where P, A_i, B_i range over prime formulas, formulas in $\mathcal{A}$, $\mathcal{B}$ respectively.

Uniform $\Leftarrow$ Intuitionistic

Proposition. (Hirst/Mummer 2011)

Let $\forall f\,(\varphi(f) \to \exists g \psi(f,g))$ be a $\mathcal{L}(\widehat{\text{E-HA}}^{\omega}\restriction)$-formula such that $\varphi(f)$ is $\exists$-free and $\psi(f,g) \in \Gamma_1$. If

$$\widehat{\text{E-HA}}^{\omega}\restriction + \mathrm{AC} + \mathrm{IP}^{\omega}_{\mathrm{ef}} \vdash \forall f(\varphi(f) \to \exists g \psi(f,g)),$$

then there exists a term $t^{1\to1} \in \mathbf{T_0}$ such that

$$\widehat{\text{E-HA}}^{\omega}\restriction \vdash \forall f\,(\varphi(f) \to \psi(f, tf))\,.$$

- AC: the axiom of choice in all finite types.
- $\mathrm{IP}^{\omega}_{\mathrm{ef}}$: independence of premise scheme for $\exists$-free formulas.

The proof is by the **modified realizability interpretation**.

Uniform $\Leftarrow$ Intuitionistic

Lemma.

The modified realizability interpretation of Σ^0_2-DNS^0 is intuitionistically derived from Σ^0_2-DNS^0 and Π^0_1-$\mathrm{AC}^{0,0}$.

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Lemma.

The modified realizability interpretation of $\Sigma^0_2\text{-DNS}^0$ is intuitionistically derived from $\Sigma^0_2\text{-DNS}^0$ and $\Pi^0_1\text{-AC}^{0,0}$.

Proposition 3.

Let $\forall f\,(\varphi(f) \to \exists g \psi(f,g))$ be a $\mathcal{L}(\widehat{\text{E-HA}}^{\omega}{\restriction})$-formula such that $\varphi(f)$ is $\exists$-free and $\psi(f,g) \in \Gamma_1$. If

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Combining Proposition 2 and 3

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Lemma.

1. $\exists$-free $= \mathcal{A} \cap \mathcal{B}$.
2. $\mathcal{B} \cap \Gamma_1 = \mathcal{C}$.

The class $\mathcal{C}$ of formulas is defined by

$$P,\ C_1 \wedge C_2,\ \forall x C_1,\ Q \rightarrow C_1 \text{ are in } \mathcal{C}$$

where P, Q, C_i range over prime formulas, $\exists$-free formulas, formulas in $\mathcal{C}$ respectively.

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Proposition.

- $\mathcal{A} \cap \exists$-free $= \exists$-free.
- $\mathcal{B} \cap \Gamma_1 = \exists$-free.

Theorem.

Let $\forall f\,(\varphi(f) \to \exists g \psi(f,g))$ be a $\mathcal{L}(\mathsf{EL}_0)$-formula such that $\varphi(f)$ and $\psi(f,g)$ are $\exists$-free.
There exists exists a term $t^{1\to 1} \in \mathbf{T_0}$ such that

$$\widehat{\mathsf{E\text{-}PA}}^{\omega}{\restriction} + \Pi^0_1\text{-}\mathrm{AC}^{0,0} \vdash \forall f\,(\varphi(f) \to \psi(f, tf))$$

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Remark.

The analogous result for $\mathbf{T}$, E-PA$^{\omega}$, EL instead of $\mathbf{T_0}$, $\widehat{\mathrm{E\text{-}PA}}^{\omega}{\restriction}$, EL_0 also holds.

Appendix

Proposition.

Let $\forall f\,(\varphi(f) \to \exists g\psi(f,g))$ be a $\mathcal{L}(\mathsf{EL}_0)$-formula such that $\varphi(f)$ is purely universal and $\psi(f,g)$ is equivalent to $\forall w^\rho \exists s^\tau \psi_{qf}(f,g,w,s)$ over EL_0 $(\rho,\tau \in \{0,1\})$.
There exists a term $t^{1\to1} \in \mathbf{T_0} + \mathbf{B_{0,1}}$ such that

$$\widehat{\mathsf{E\text{-}PA}}^{\omega}\restriction + \Pi^0_1\text{-AC}^{0,0} + \mathrm{BR}_{0,1} \vdash \forall f\,(\varphi(f) \to \psi(f,tf))$$

if and only if

$$\mathsf{EL}_0 + \mathrm{AC}^{0,0} + \mathrm{MP} + \Sigma^0_2\text{-DNS}^0 + \mathrm{BR}_{0,1} \vdash \forall f\,(\varphi(f) \to \exists g\psi(f,g))\,.$$

Fact. (Kohlenbach 1999)

$t^{1\to1} \in \mathbf{T_0} + \mathbf{BR_{0,1}} \Leftrightarrow t^{1\to1} \in \mathbf{T}$.

References

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Thank you for your attention!