Domain representations of spaces derived from dyadic subbases

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Outline of the talk

1. Modified Gray expansion / IM2-machine for real number computation

Generalization to other Hausdorff spaces.

joint works with Yasuyuki Tsukamoto

2. Dyadic subbase
3. Proper dyadic subbase
4. Domain representations derived from proper dyadic subbases
5. Strongly proper dyadic subbase

1. Modified Gray expansion and IM2-machine for real number computation

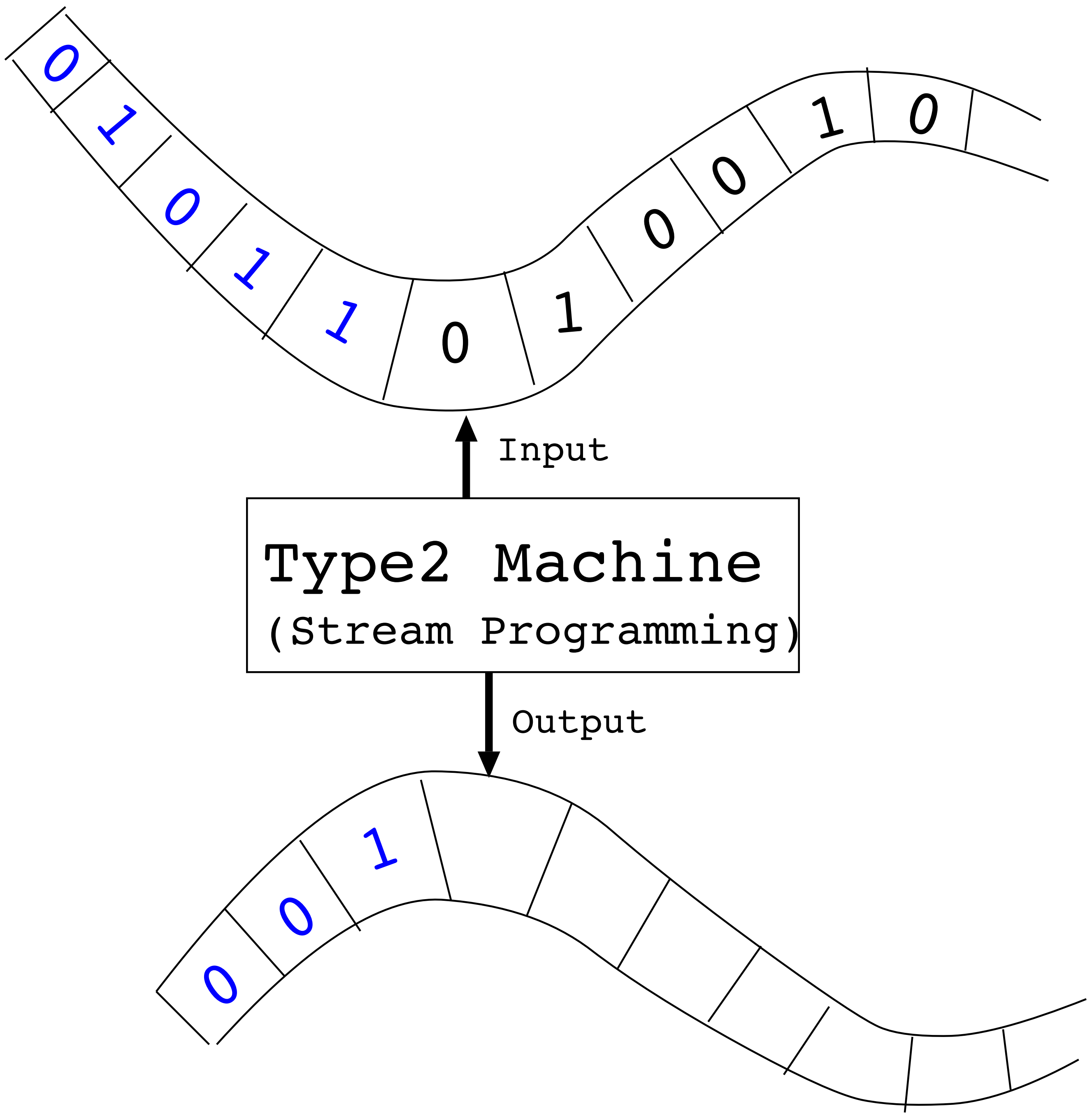
Computation over Topological Spaces

- Stream input/output access over infinite sequences.

  (Type2 machine)

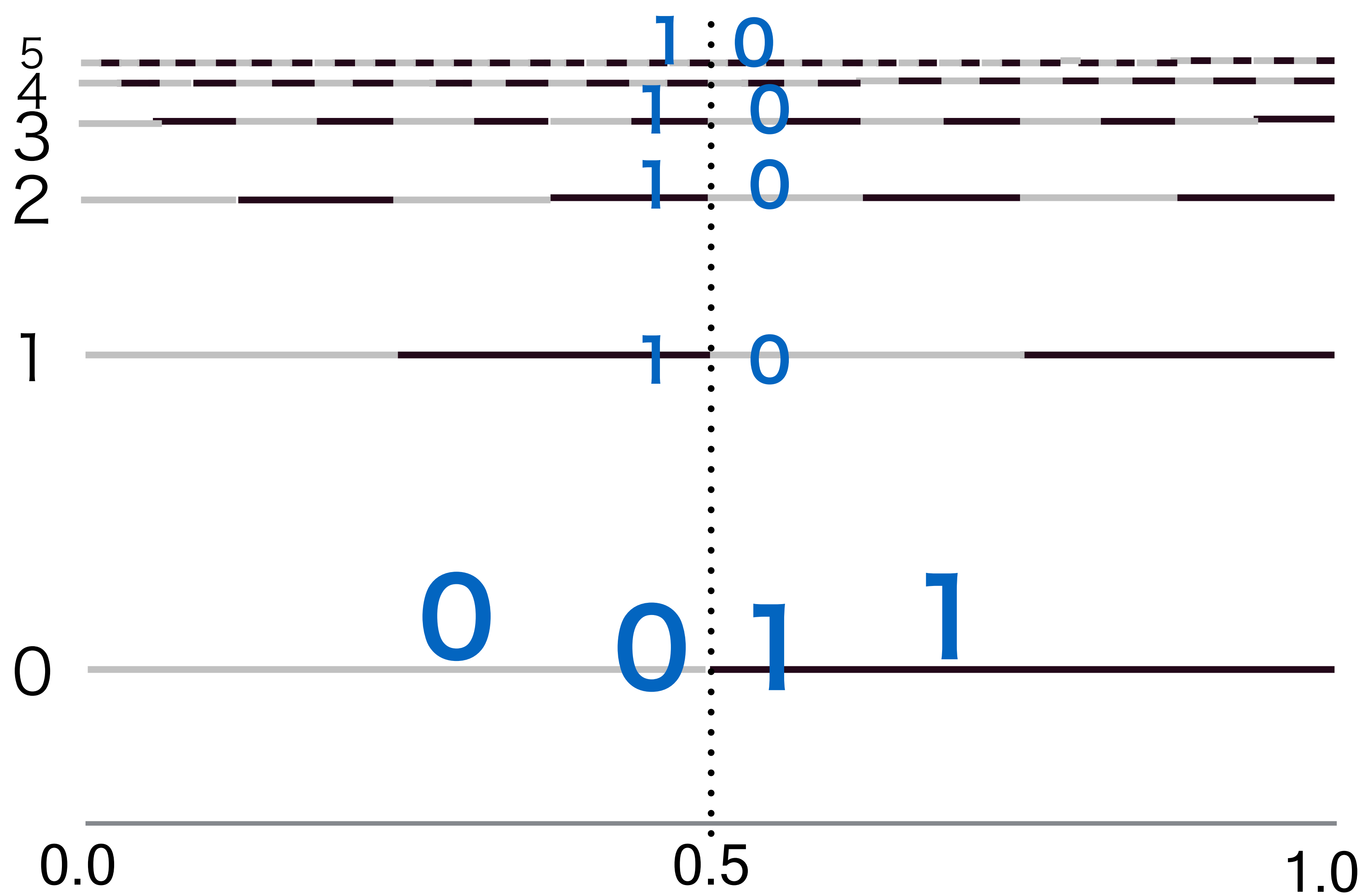
- Real number computation via binary expansion into infinite sequences.

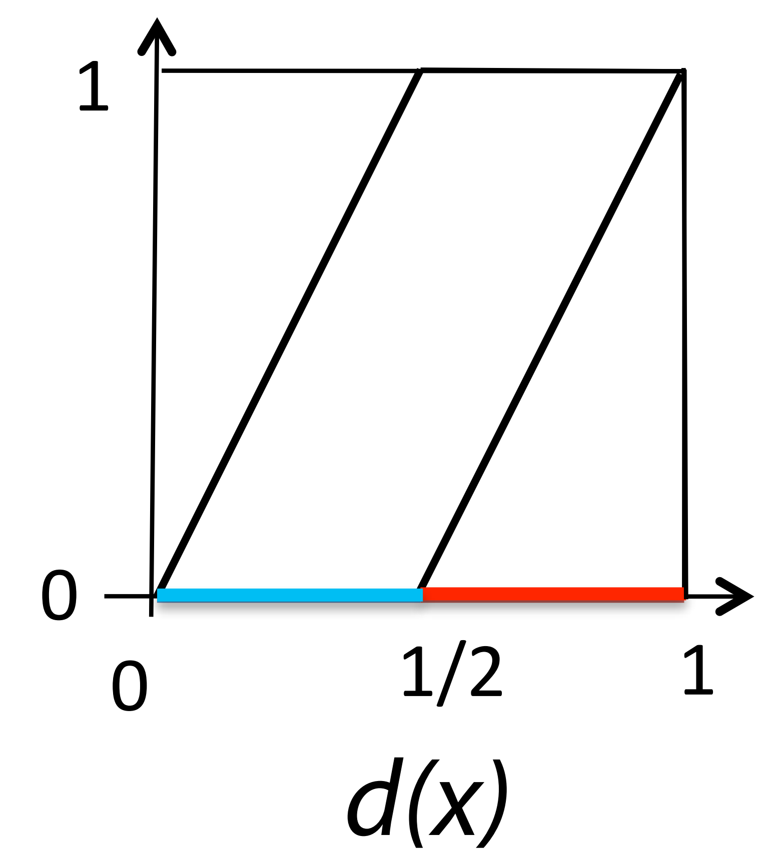
- **Unnatural computation over R.**



Binary expansion

- Coding of the unit interval [0,1] as {0, 1}-sequences.
- The first digit of x is 0 or 1 depending on $x \leqq 1/2$ or $x \geqq 1/2$.
- The rest is the code of $d(x)$ for d the function below.
- d is multiple-valued on 1/2, a value is chosen based on the first digit.
- $\times 3$ function not expressible (0010101..(=1/6)$\times$ 3 = 0111… or 1000…)

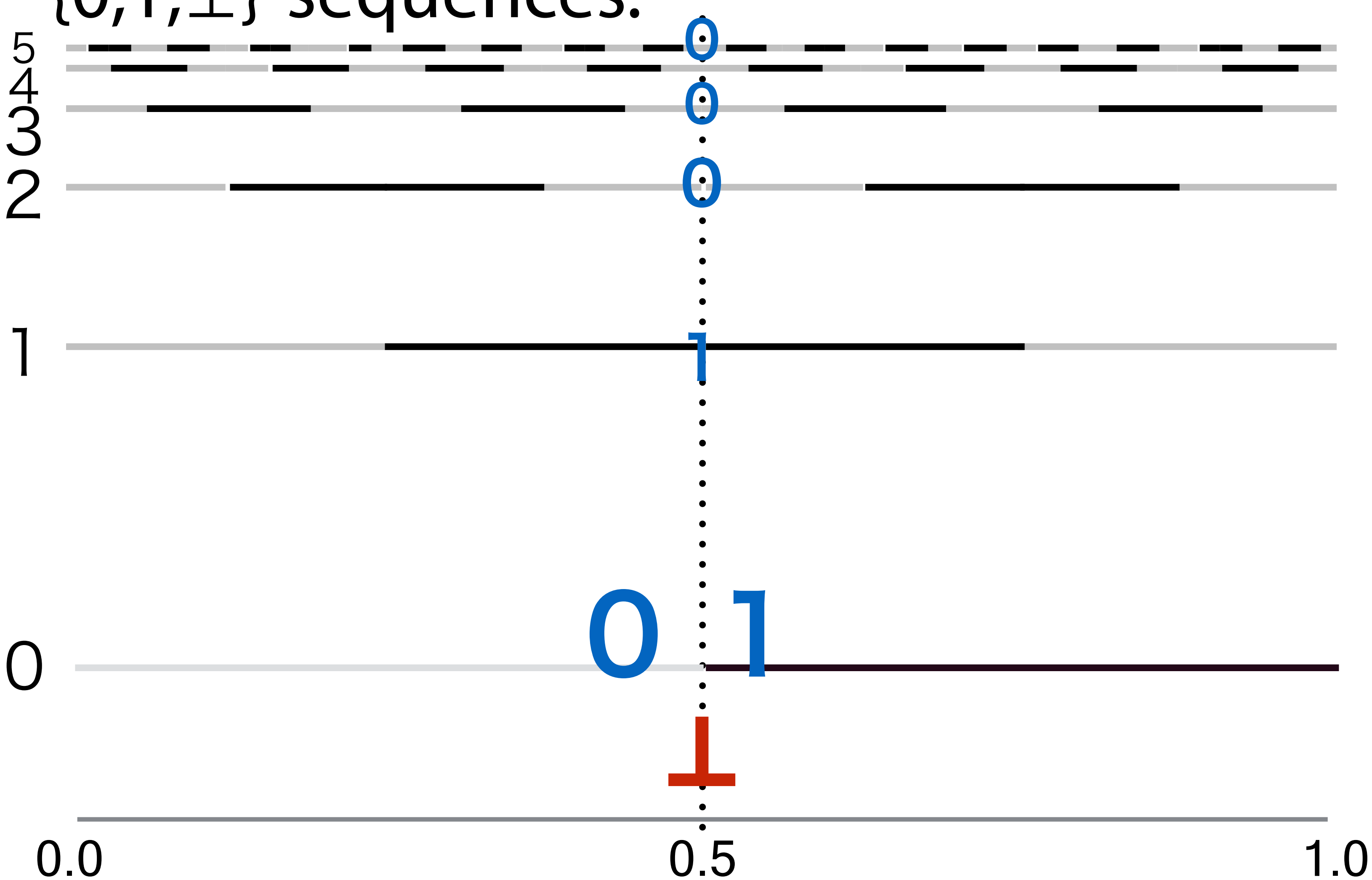


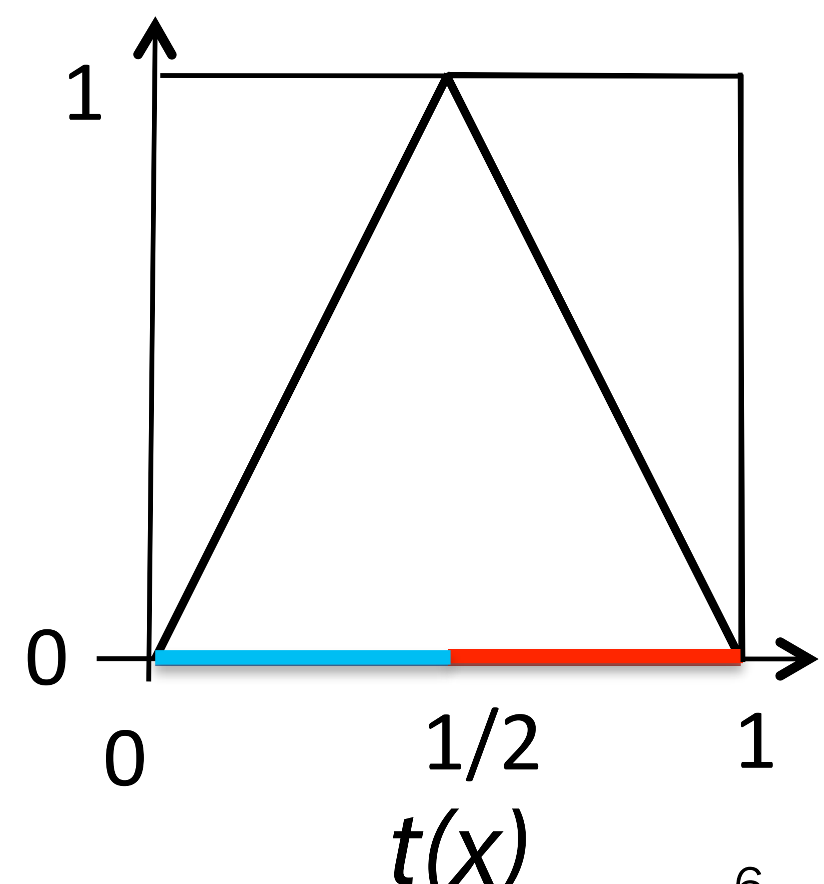


Modified Gray expansion

- The first digit of x is 0 or 1 depending on $x \leqq 1/2$ or $x \geqq 1/2$.
- The rest is the code of $t(x)$ for t the tent function.
- Easily converted from/to Binary Expansion (2-state automaton).
- t is single-valued and continuous; not depending on the first digit.
- Leave the first digit as ⊥ (undefined), and consider expansion into {0,1,⊥}-sequences.

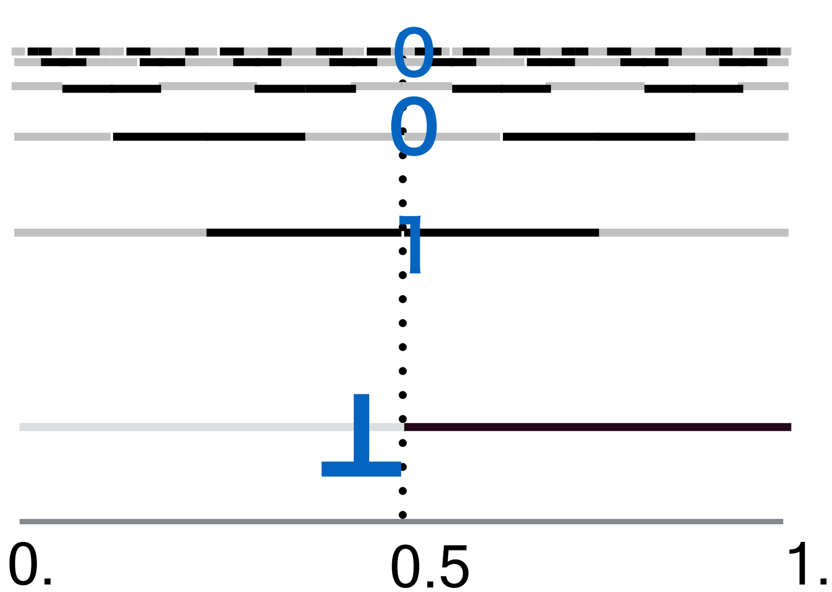
[Gianantonio 1999], [T 2002]

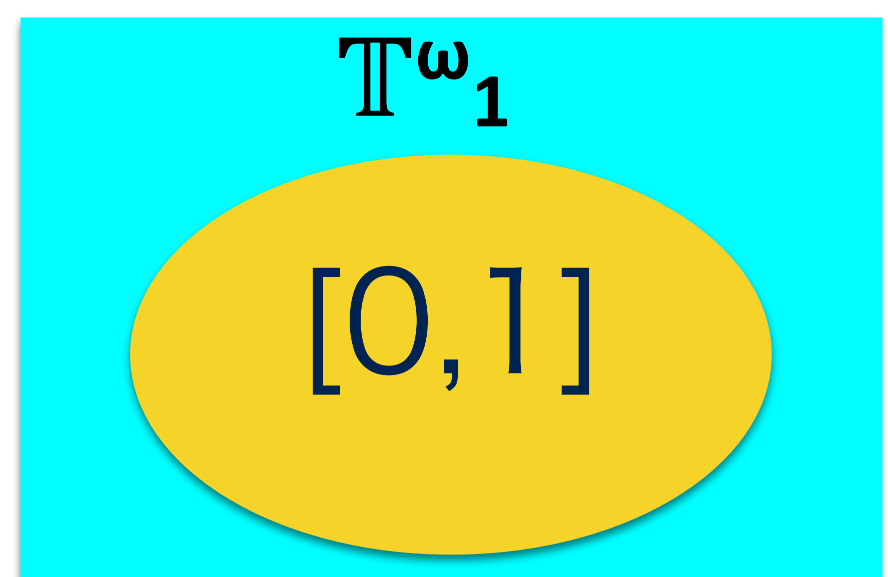




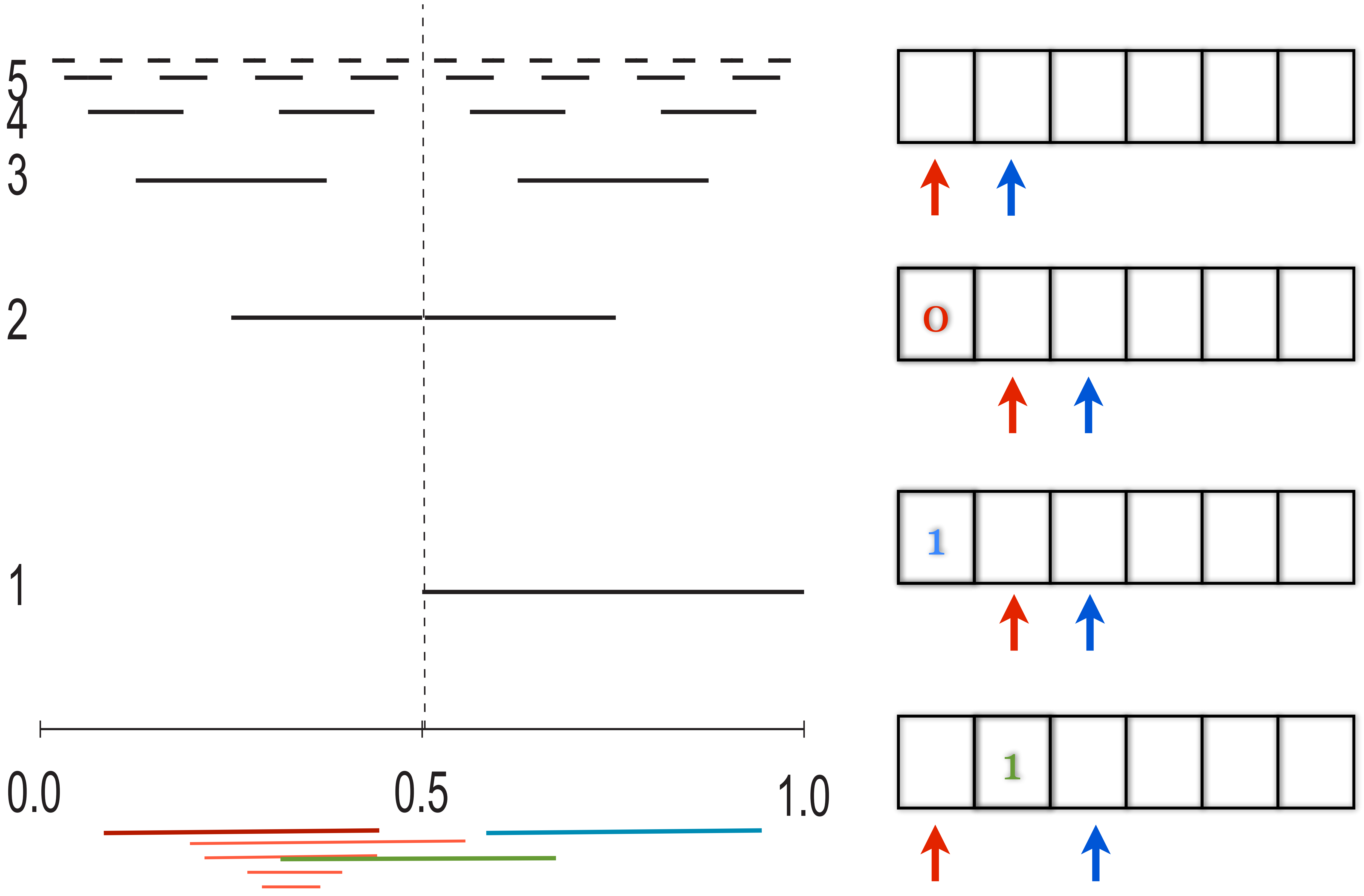
Gray expansion

- $\mathbb{T} = \{0,1,\perp\}$.
- Modified Gray expansion (We simply call it Gray-expansion) assigns a unique $\mathbb{T}$-sequence to each $x \in [0,1]$.
- $\perp$ appears in expansions of dyadic rationals, and we always have $\perp 1000\ldots$
- $1\perp$-sequences : a $\mathbb{T}$-sequence with at most one $\perp$.
- $\mathbb{T}^{\omega}{}_{1}$: the set of $1\perp$-sequences.
- The unit interval $\mathbb{I} = [0,1]$ is topologically embedded in $\mathbb{T}^{\omega}{}_{1}$.
  - Topology on $\mathbb{T}$: $\{0\}, \{1\}, \{0,1,\perp\}$
  - Topology on $\mathbb{T}^{\omega}$: product topology = Scott topology
  - Topology on $\mathbb{T}^{\omega}{}_{1}$: subspace topology

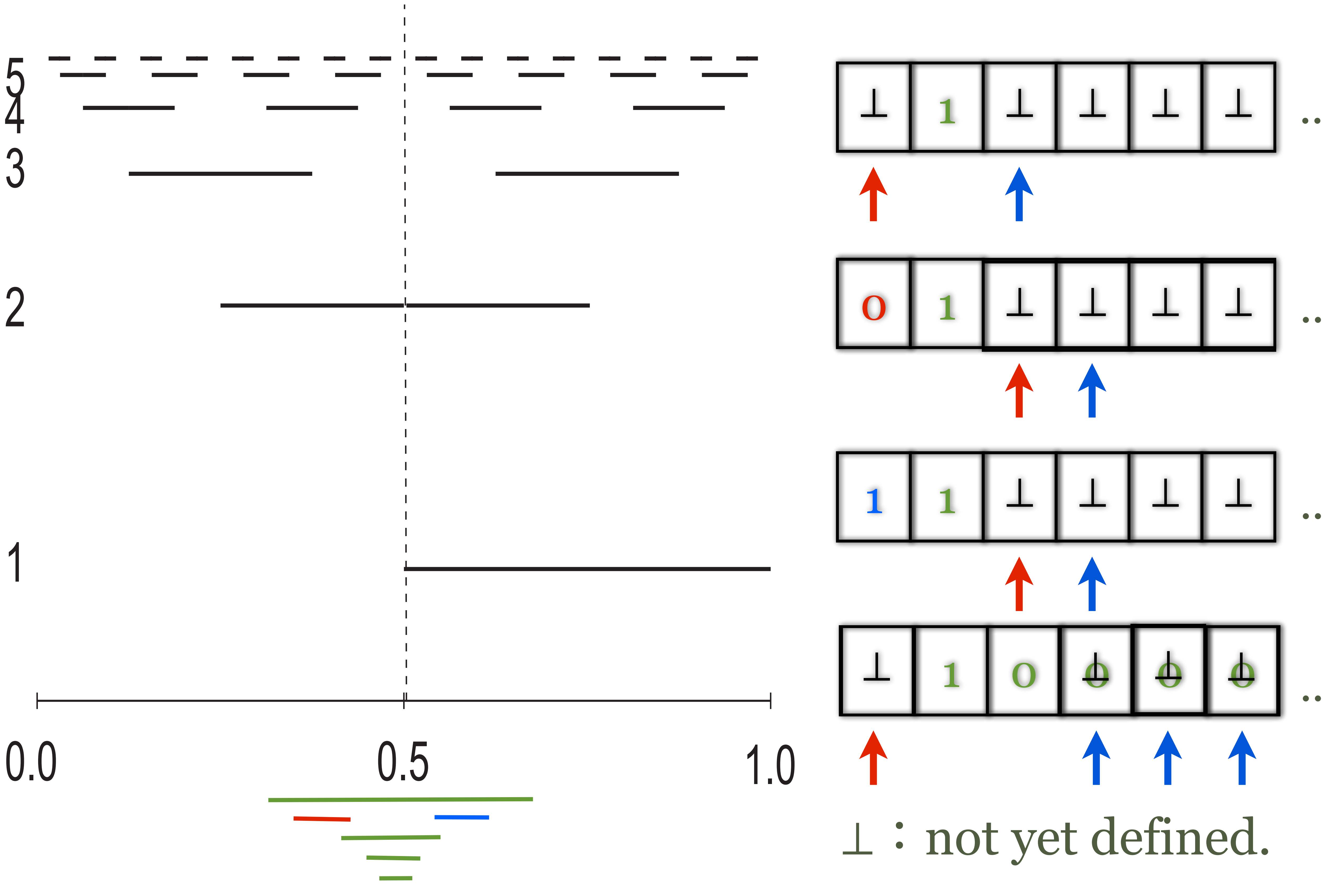




How to output Gray expansion?

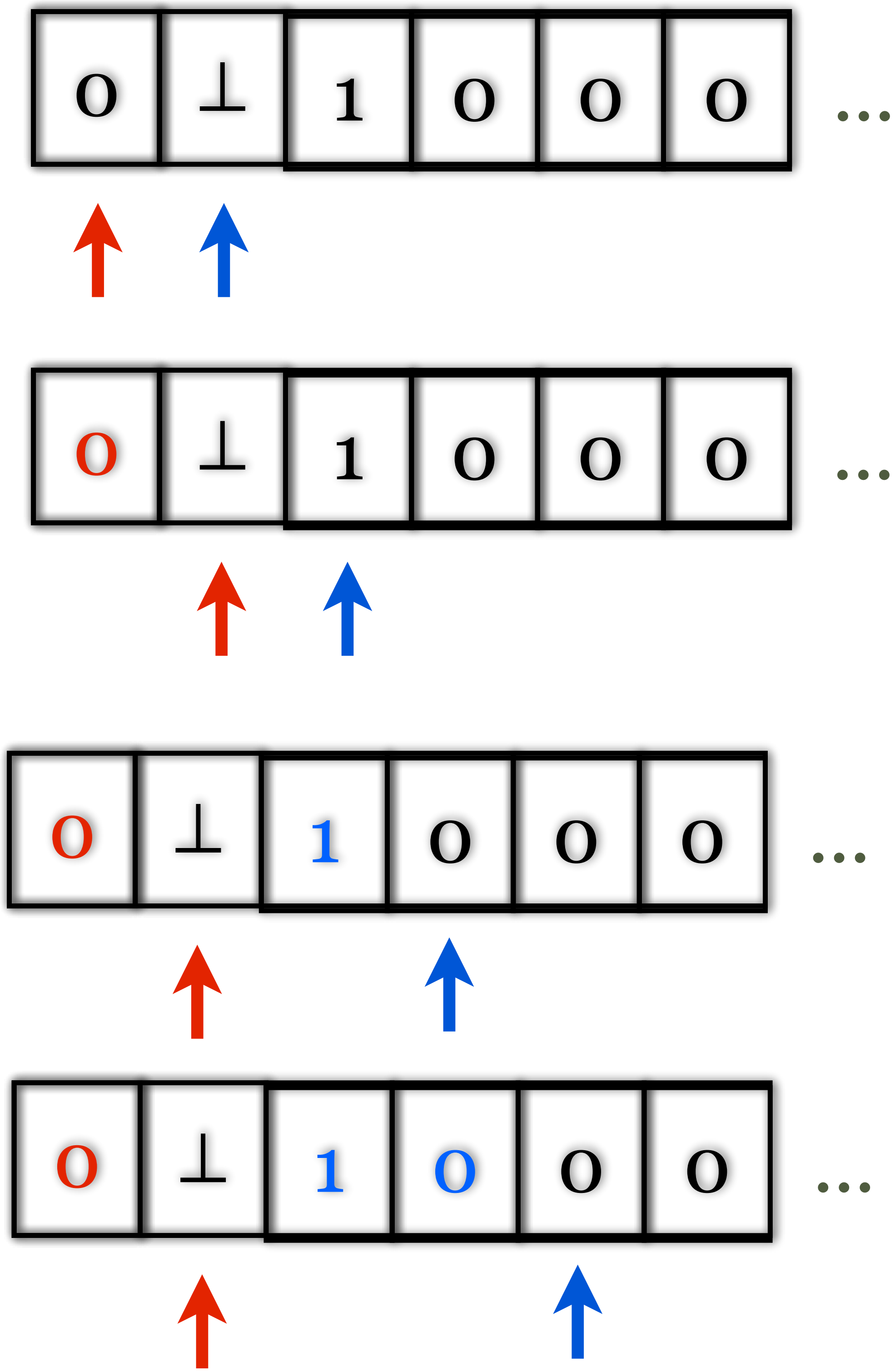


How to output Gray expansion?

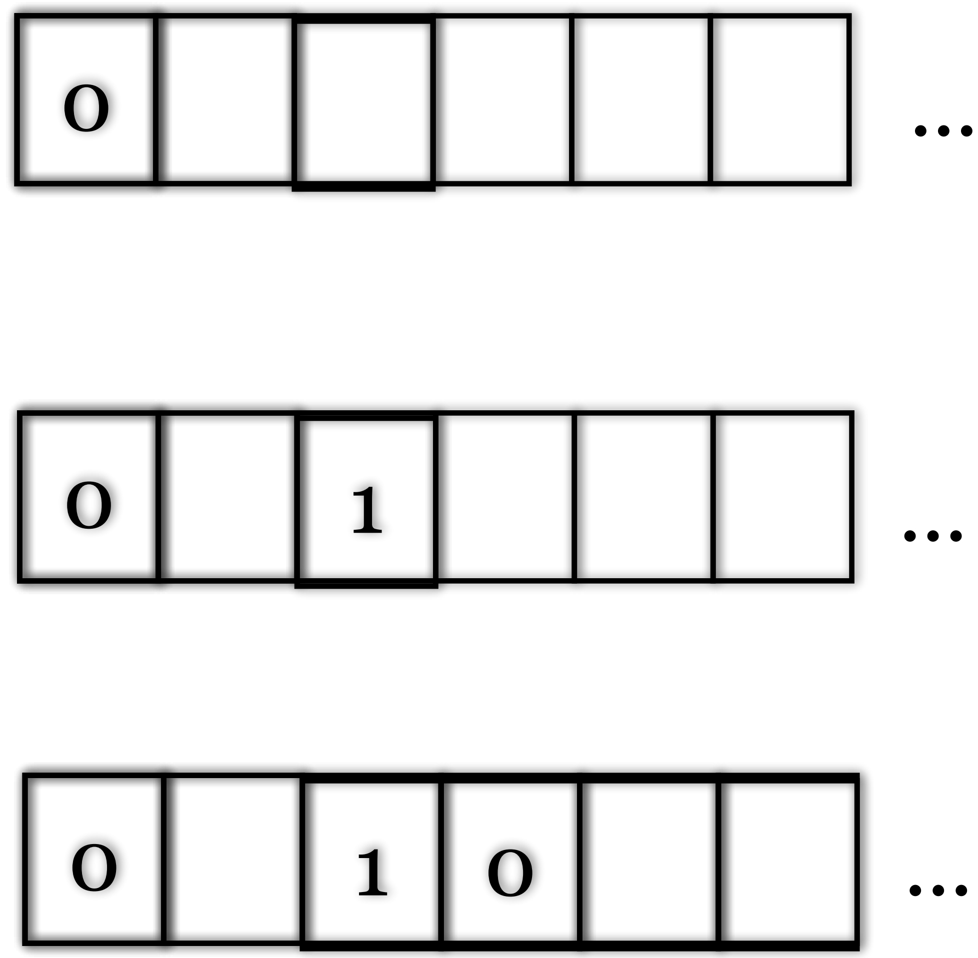


⊥ : not yet defined.

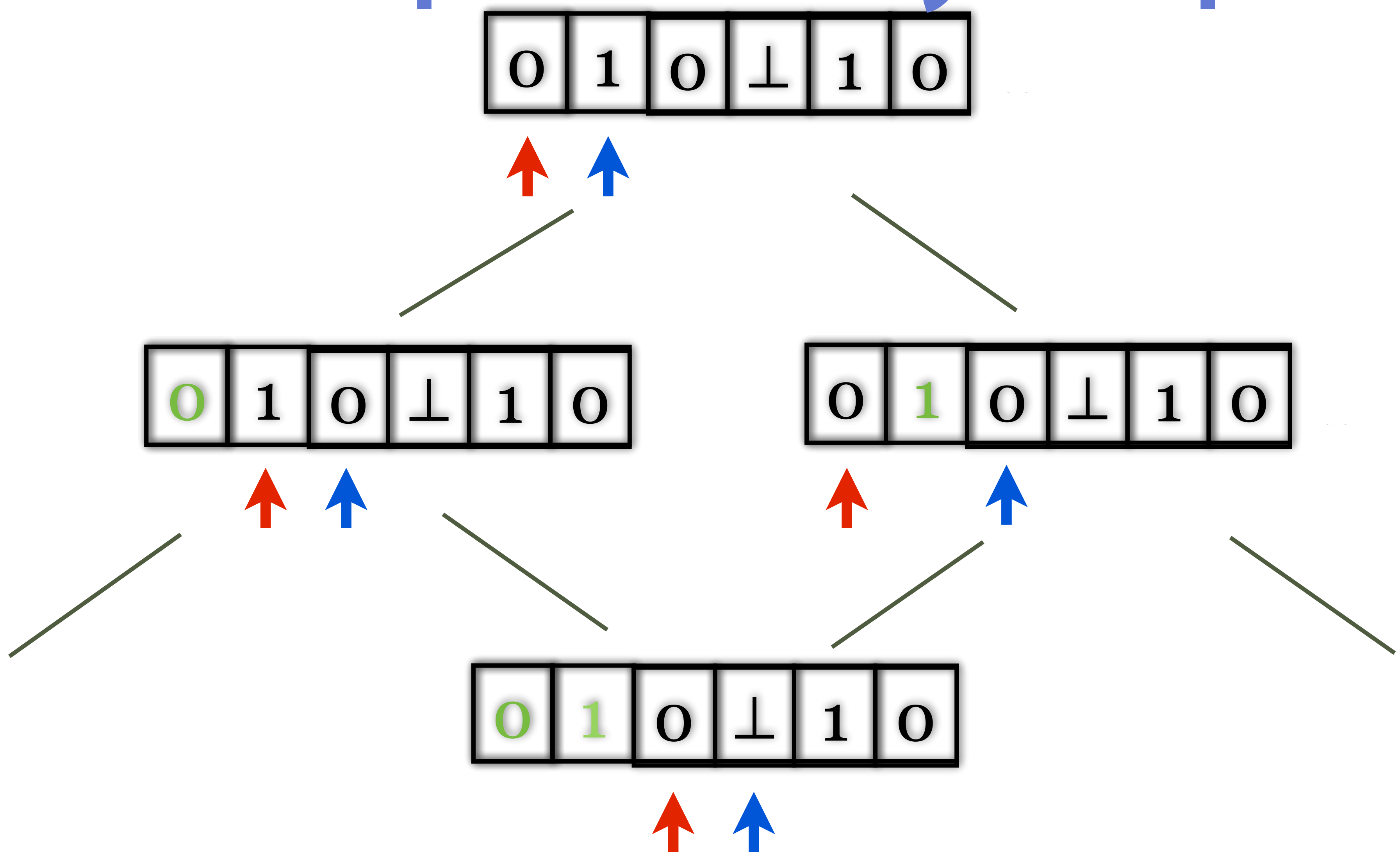
How to input Gray expansion?



The sequence input by a machine



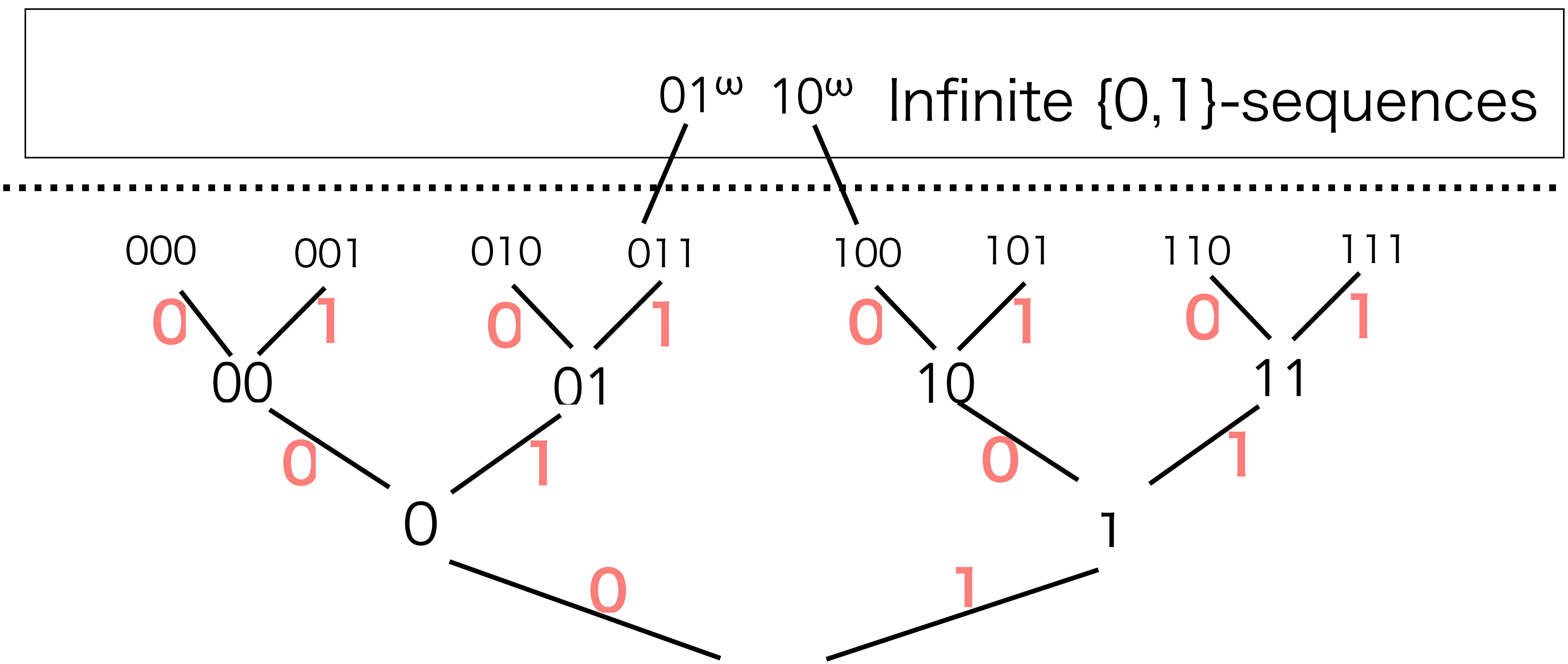
How to input Gray expansion?



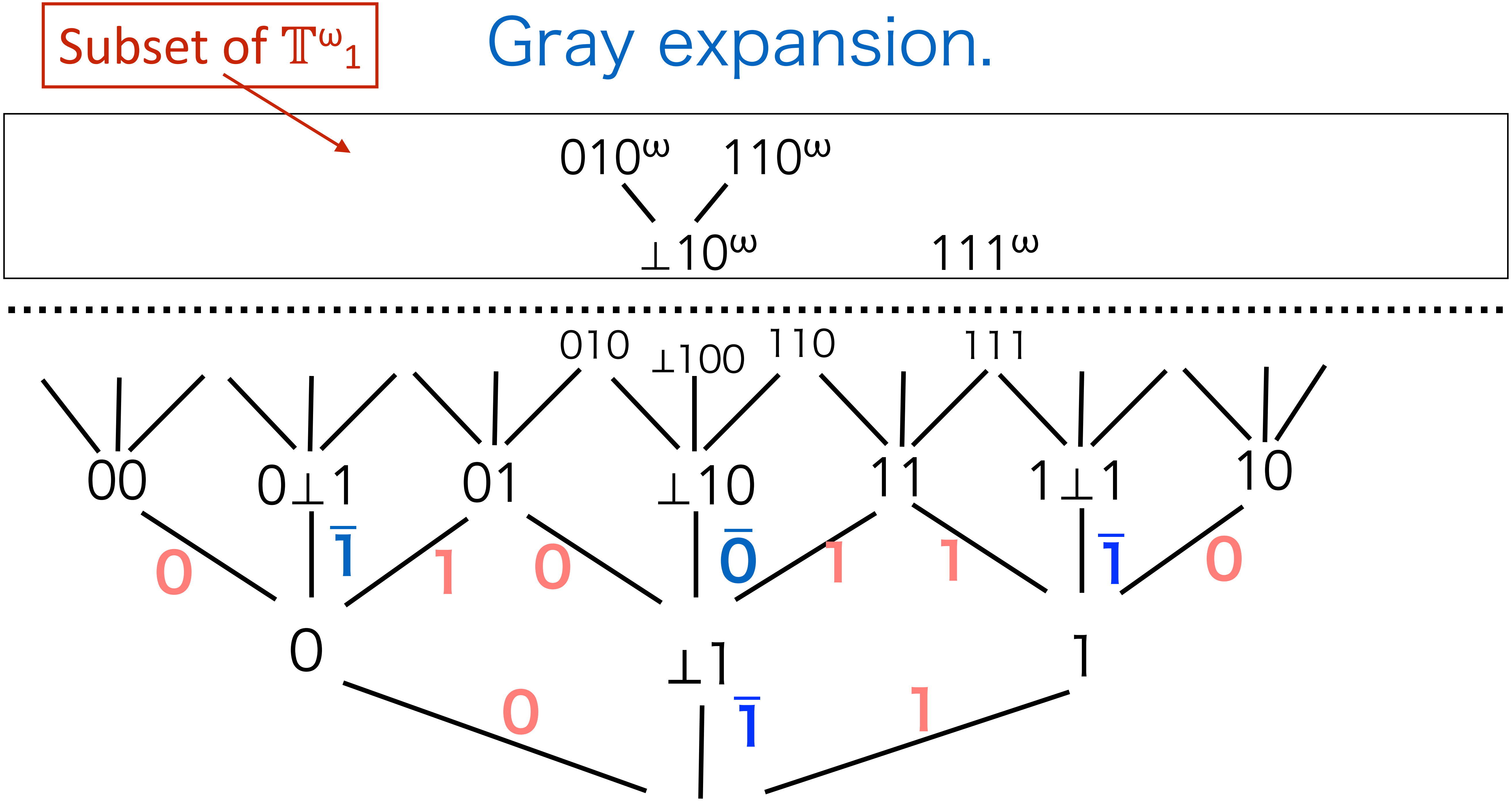
- Indeterministic (i.e. nondeterministic) behavior.
- It should input 000.. , not ⊥000.. for the input 000…
- A program should be written so that it can input all the digits to identify a point. (⊥1 is valid, but ⊥0 is not)

Finite/infinite-time state of a tape for usual stream {0,1} -output

- One way access from left to right.

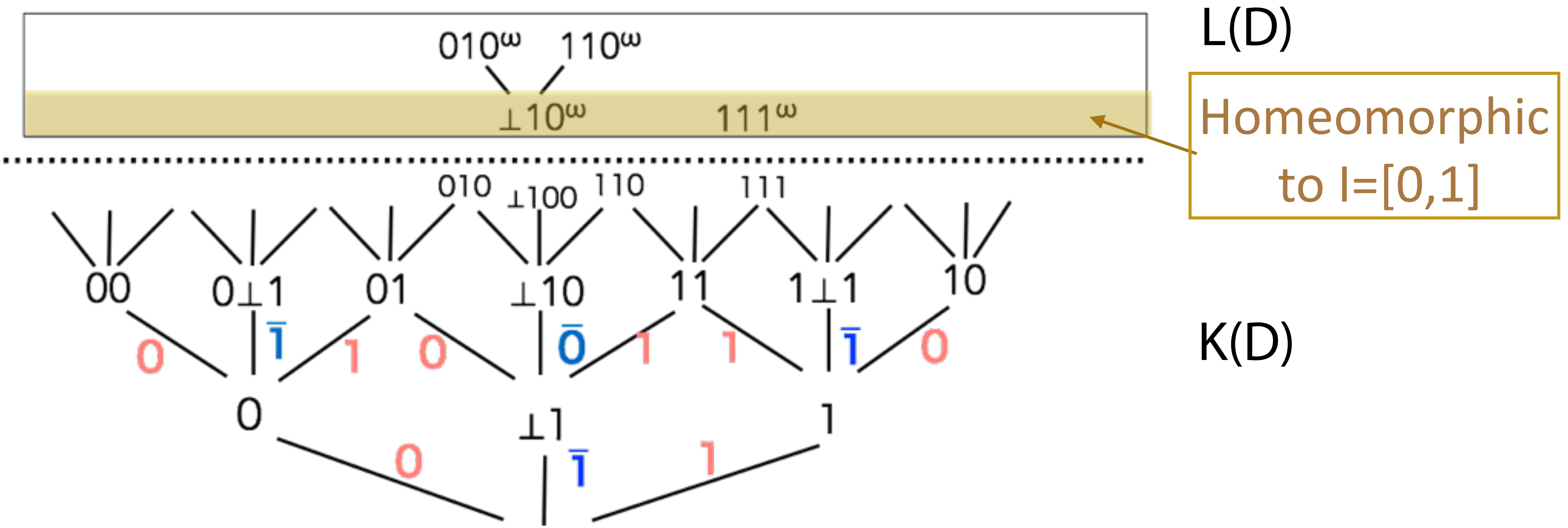


Finite/infinte-time state of a tape for the Gray expansion.



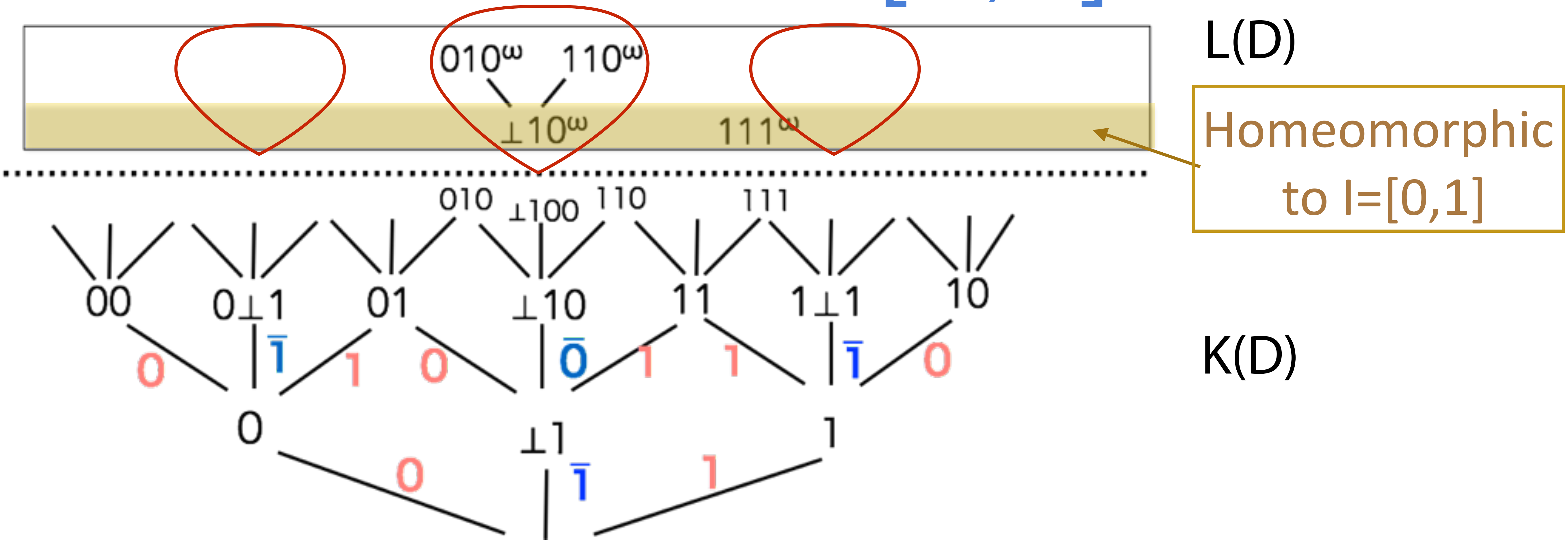
- $\bar{0}, \bar{1}$: output of 0 or 1 from the blue head.
- ⊥0 unnecessary.
  - The set of limit of finite-time observation (ideal completion of finite-time state) subset of $\mathbb{T}^{\omega}{}_1$.

Domain of [0,1]



- Let L(D) be the set of limit (i.e., non-compact) elements.
- Scott domain (algebraic bounded complete dcpo).
- [0,1] is homeomorphic to the set of minimal elements of L(D).
- All the increasing sequences following K(D) are identifying points of [0,1].
- It ensures that an IM2-machine can “input” enough information to identify a point, if an IM2-machine program is written following the structure of K(D).

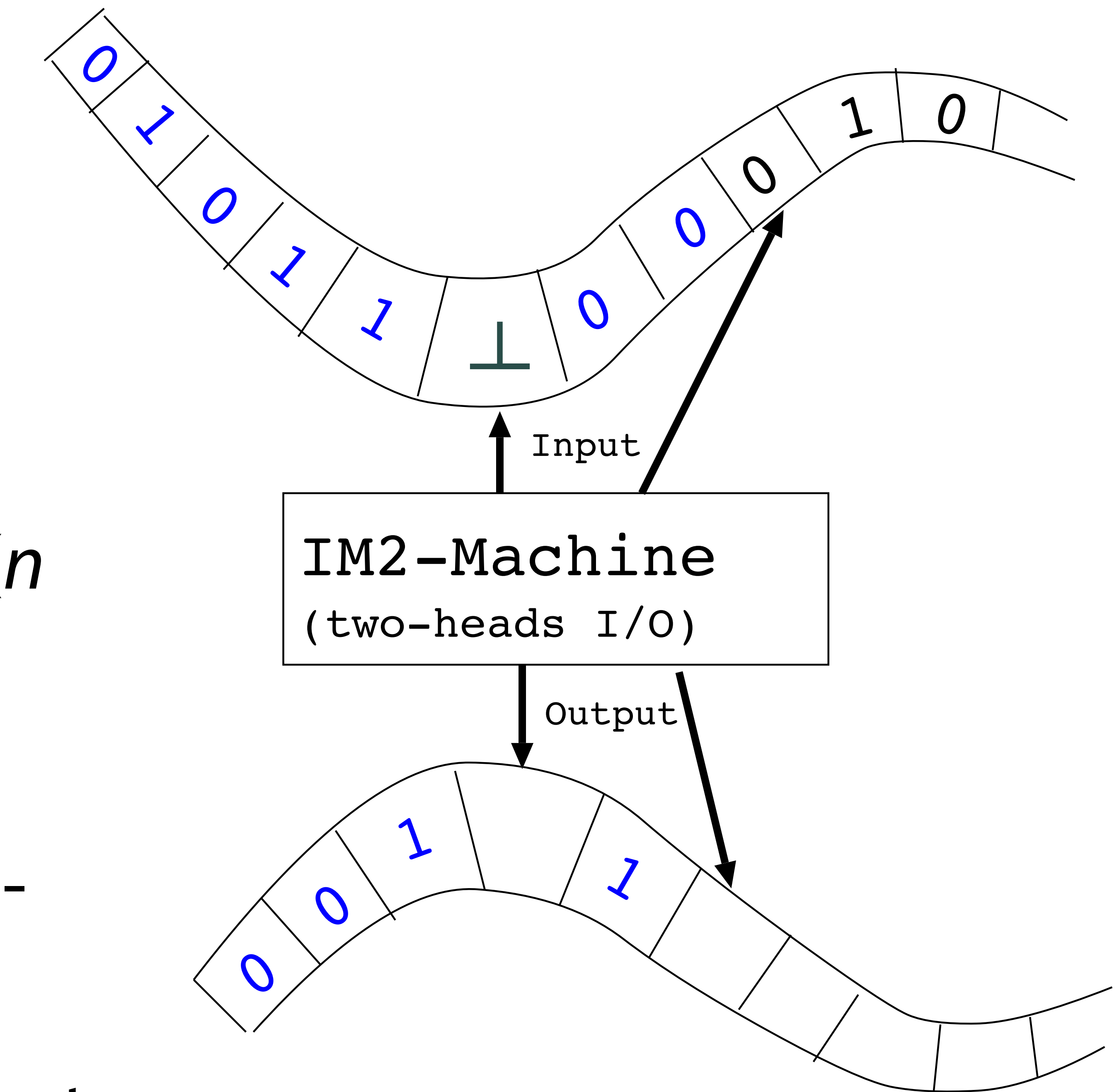
Domain of [0,1]



- [0,1] is a retract of L(D). (∀ p ∈ L(D). ∃ x ∈[0,1], φ(x) ⊑ p).
- We have another representation that uses the whole L(D), by considering that 010^ω, 110^ω, $\bot 10^\omega$ are all representing 1/2.
- It corresponds to considering the state 01 not an open interval (1/4, 1/2) but the closed interval [1/4, 1/2], which is more natural for programming.

IM2-machine

- Two-head access to input/output tapes.
- Indeterministic behavior depending on how to input when both input heads have values.
- More generally, defined as an (*n* +*1*)-head machine which can access n⊥-sequences.
- Thus, we can compute over n⊥-sequences.
- Easily implemented in concurrent logic programming languages.



IM2-machine + Gray-code

Type2-machine + Signed-digit expansion and IM2-machine + Gray-expansion induce the same computability notion on [0,1].

[0,1]

admissible representation

$\{0,1\}^\omega$

Type2-machine

$\{0,1,\perp\}^\omega$

[0,1]

IM2 machine

- A machine is operating directly on points.
- "Extensional" representation.
- Computation which reflect the structure of the space.

Generalization of Gray-expansion to other spaces.

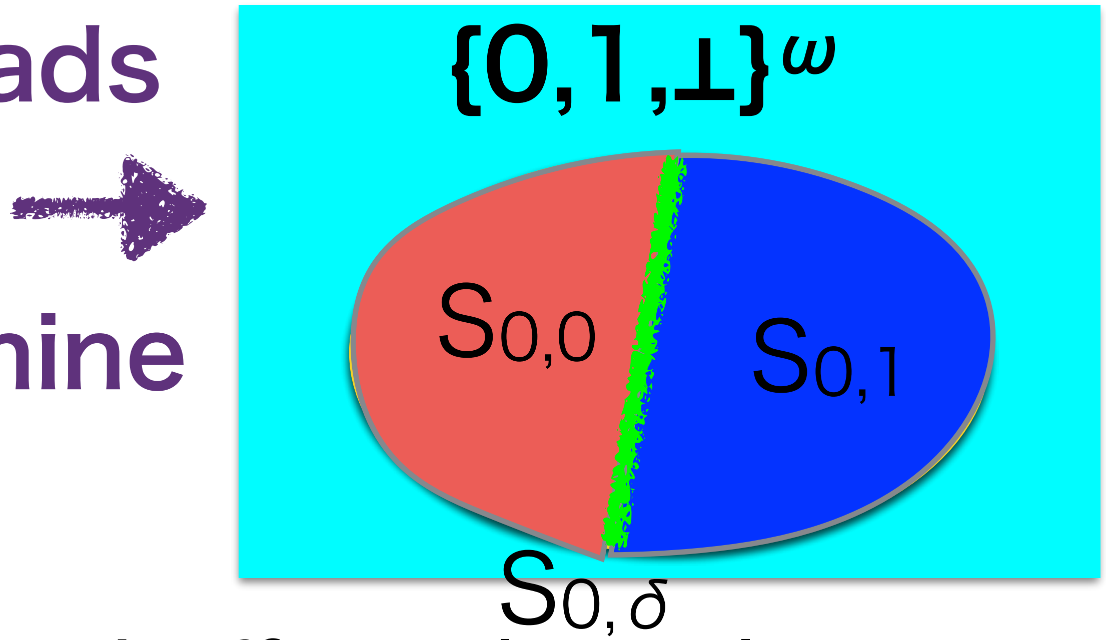
2. Dyadic subbase

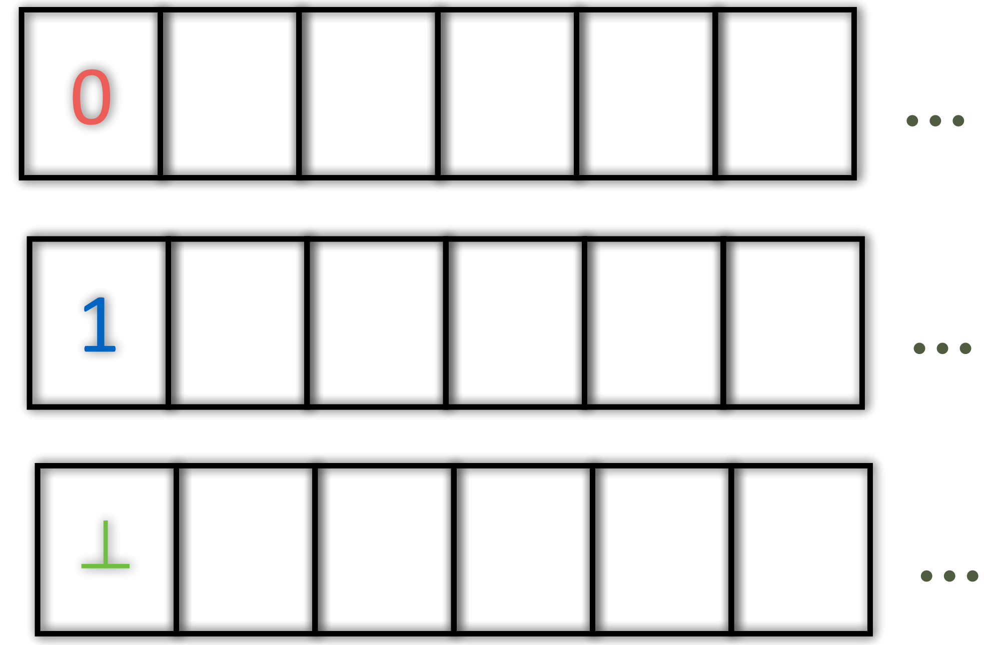
Generalization of Gray-embedding

finite (not bounded)

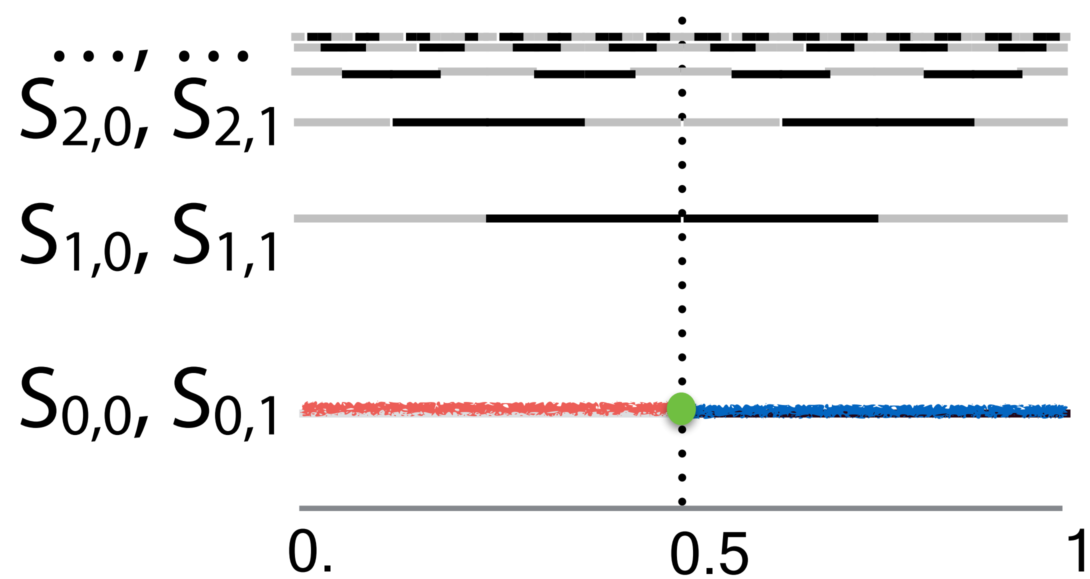
~~2~~-heads

IM2 machine





X: Hausdorff Topological space

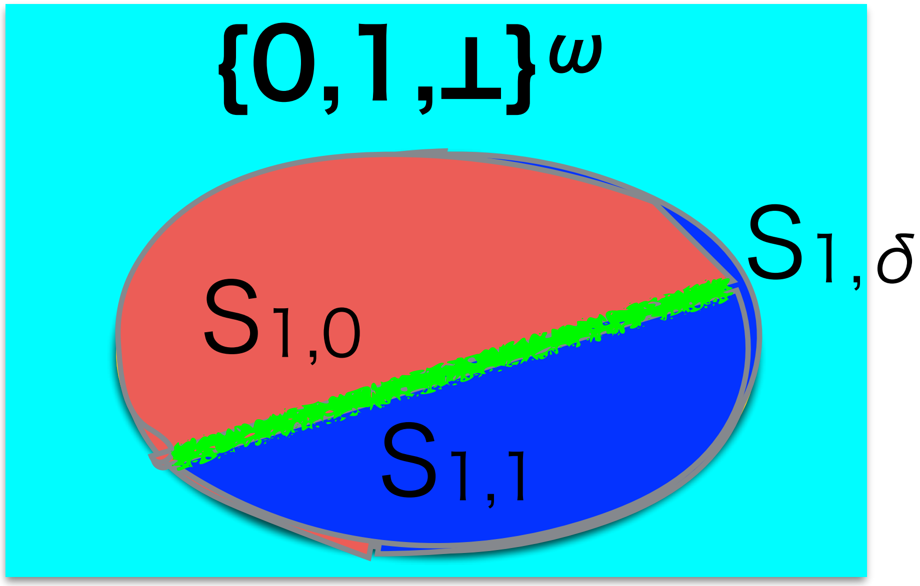


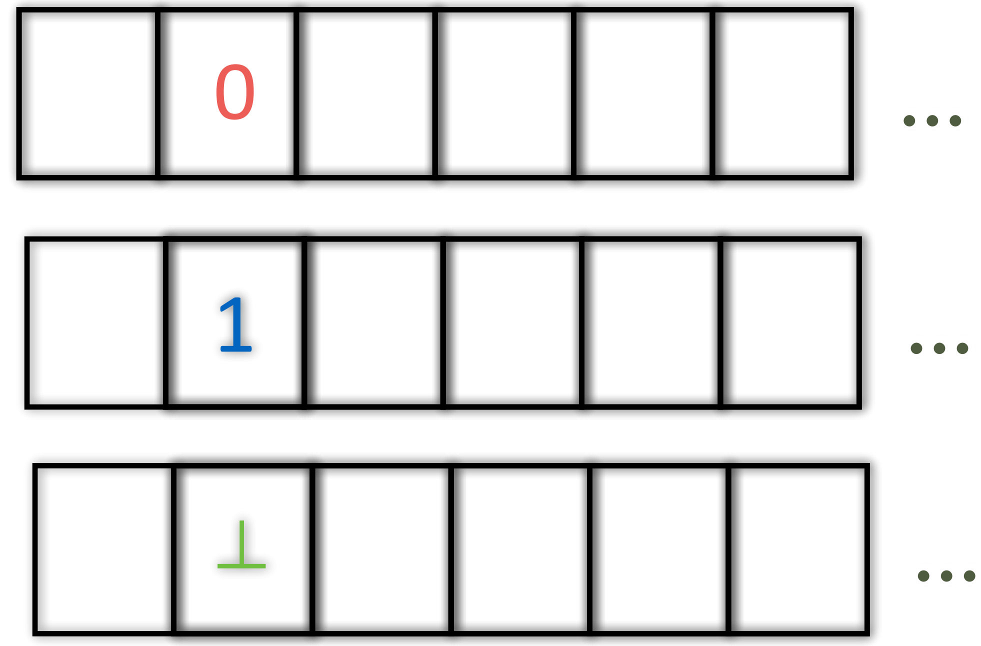
Generalization of Gray-code

finite (not bounded)

~~2~~-heads

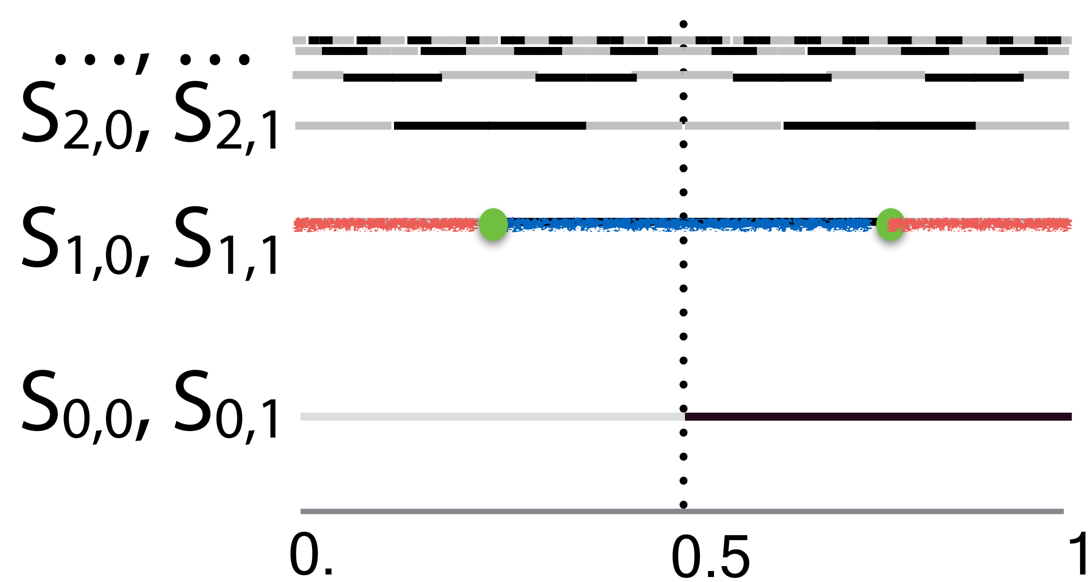
IM2 → machine





X: Hausdorff Topological space

- Disjoint open sets $S_{n,0}$ and $S_{n,1}$.
- In order to identify a point $x \in X$, $S_{n,0}$ and $S_{n,1}$ are used but $S_{n,\delta}$ are not used.
- $\{S_{n,a} \mid n \in N, a \in \{0,1\}\}$ forms a subbase of X .



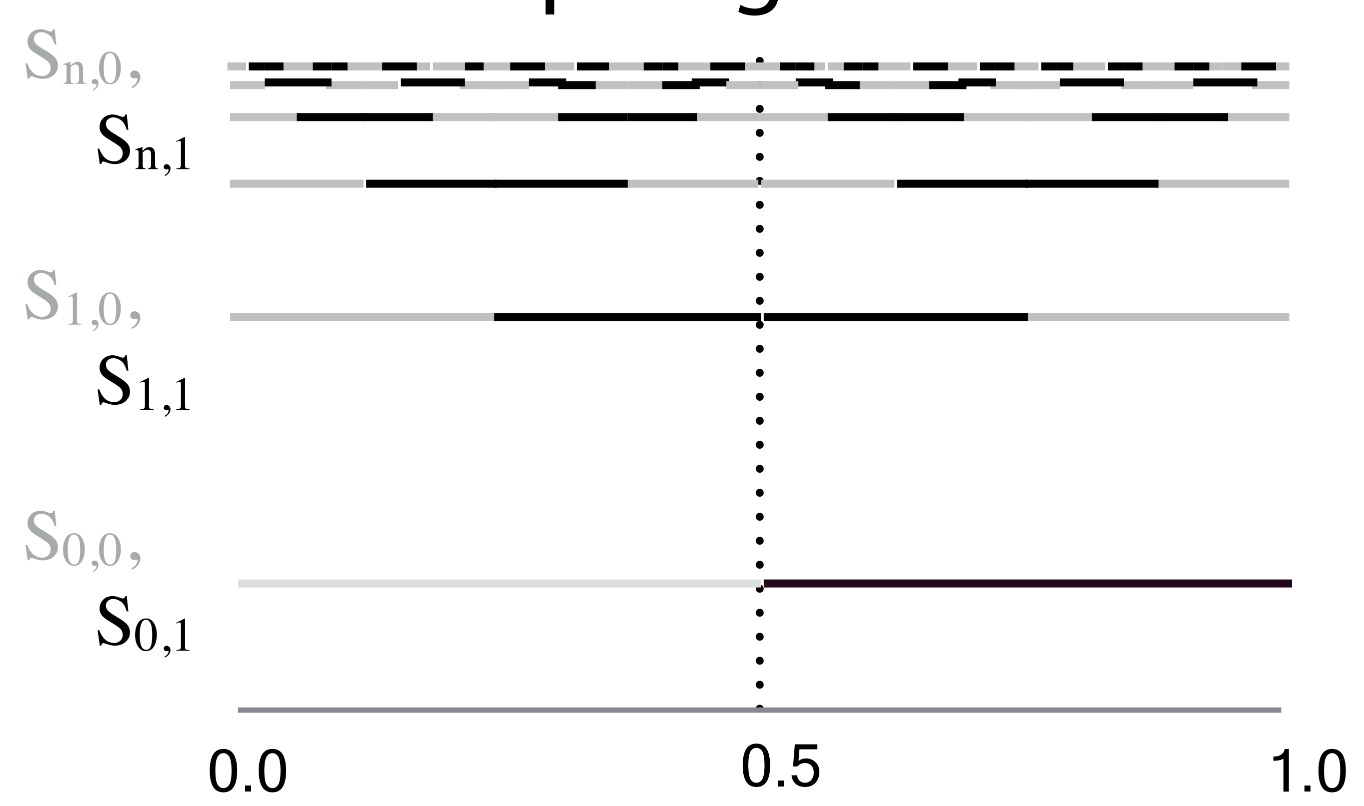
Dyadic subbase

- Let (X, O) be a Hausdorff space.

- **Definition 1.** A dyadic subbase of X is a map $S : \omega \times \{0, 1\} \to O$ such that
  - $\{S_{n,a} \mid n \in N, a \in \{0,1\}\}$ is a subbase of X ,
  - $S_{n,0} \cap S_{n,1} = \varnothing$ for all $n \in N$.

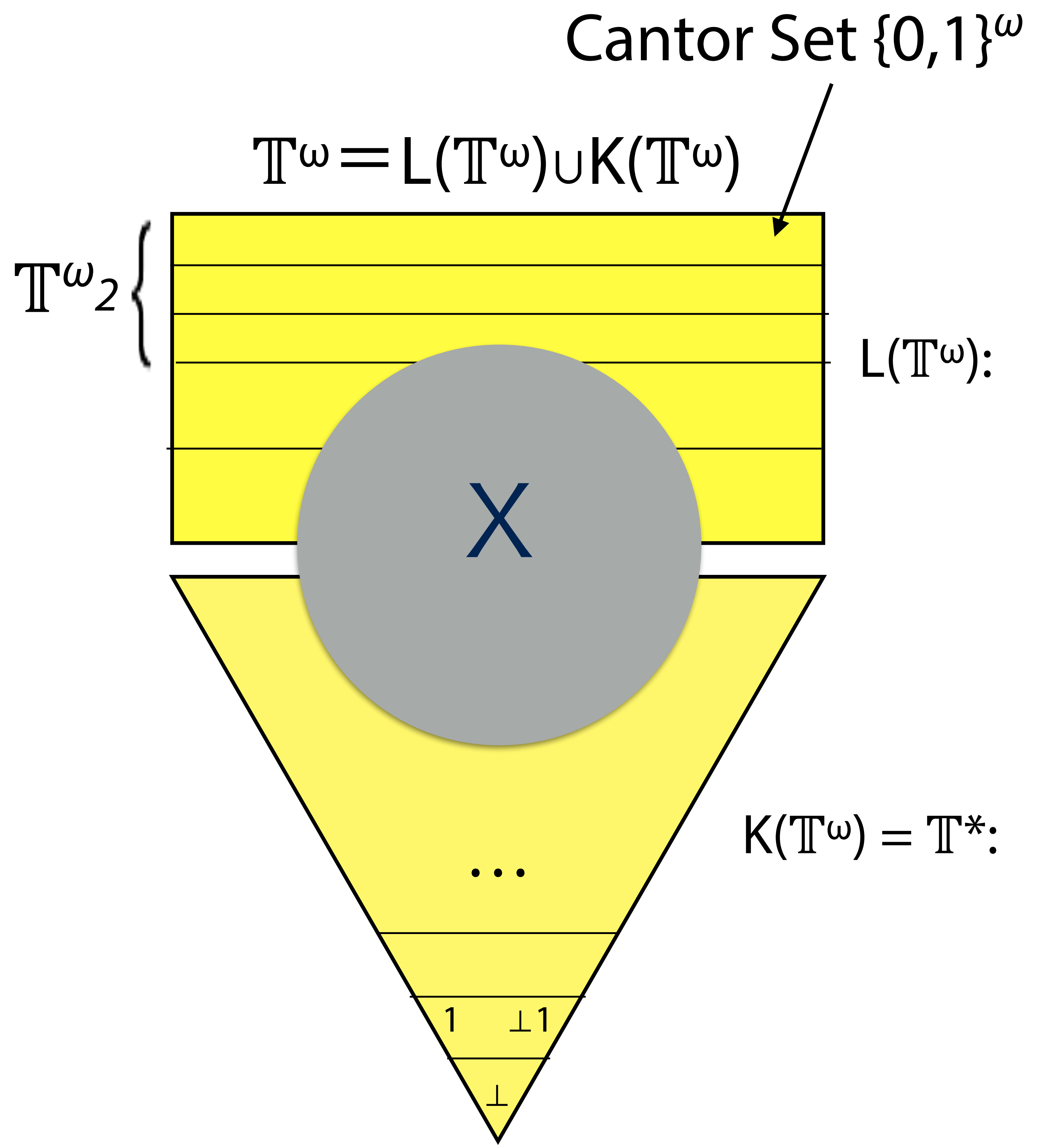
- We define $S_{n,\delta} = X \setminus (S_{n,0} \cup S_{n,1})$.
- A dyadic subbase S corresponds to the topological embedding $\varphi_S : X \to \mathbb{T}^\omega$,

$$\varphi_S(x)(n) = \begin{cases} 0 \ (x \in S_{n,0}) \\ 1 \ (x \in S_{n,1}) \\ \bot \ (x \in S_{n,\delta}) \end{cases} .$$

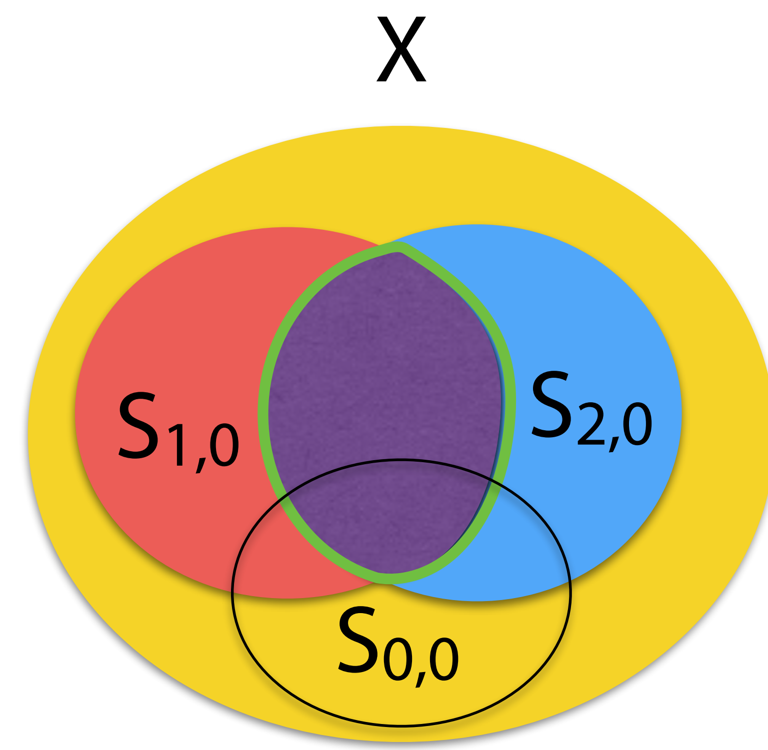


The domain $\mathbb{T}^\omega$

- Order on $\mathbb{T}$: $\bot \sqsubseteq 0$, $\bot \sqsubseteq 1$
- $\mathbb{T}^\omega$: Scott domain (algebraic bounded complete dcpo.)
- $\mathbb{T}^* = K(\mathbb{T}^\omega)$: The set of compact elements of $\mathbb{T}^\omega$.
  - Finite number of 0, 1.
  - We write $0\bot10$ for $0\bot10\bot^\omega$
  - Inner bottoms.
- $L(\mathbb{T}^\omega)$: The set of limit (i.e., non-compact) elements of $\mathbb{T}^\omega$.
  - Infinite number of 0, 1.
- Stratified as in the figure.



$S(p)$ and $\overline{S}(p)$



- For a dyadic subbase S and $p \in \mathbb{T}^*$, define

$$S(p) = \bigcap_{k \in \mathrm{dom}(p)} S_{k,p(k)},$$

- Here, $\mathrm{dom}(p) = \{k : p(k) \neq \perp\}$.
- $S(p)$: the set of points which satisfy the specification p.
- $\{S(p) : p \in \mathbb{T}^*\}$ forms the base of X generated by S.

$$\bar{S}(p) = \bigcap_{k \in \mathrm{dom}(p)} X \setminus S_{k,1-p(k)} = \bigcap_{k \in \mathrm{dom}(p)} S_{k,p(k)} \cup S_{k,\delta}$$

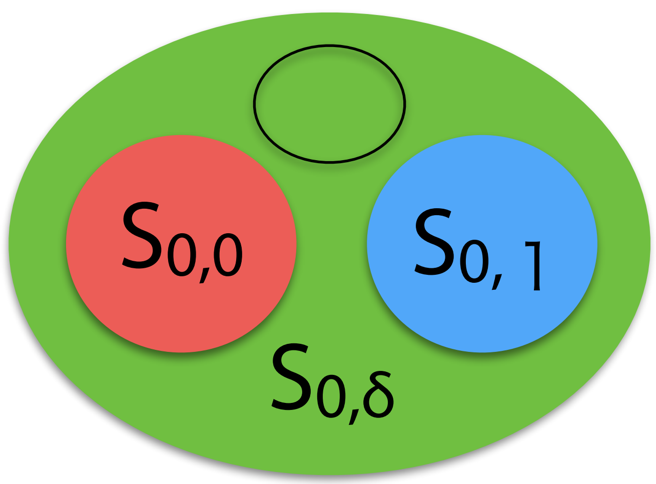
- $\overline{S}(p)$: the set of points which satisfy the specification p, where $p(k) = a$ means $x \in S_{k,a} \cup S_{k,\delta}$.
- We want $\overline{S}(p)$ to be the closure of $S(p)$.

3. Proper dyadic subbase

Proper Dyadic Subbase

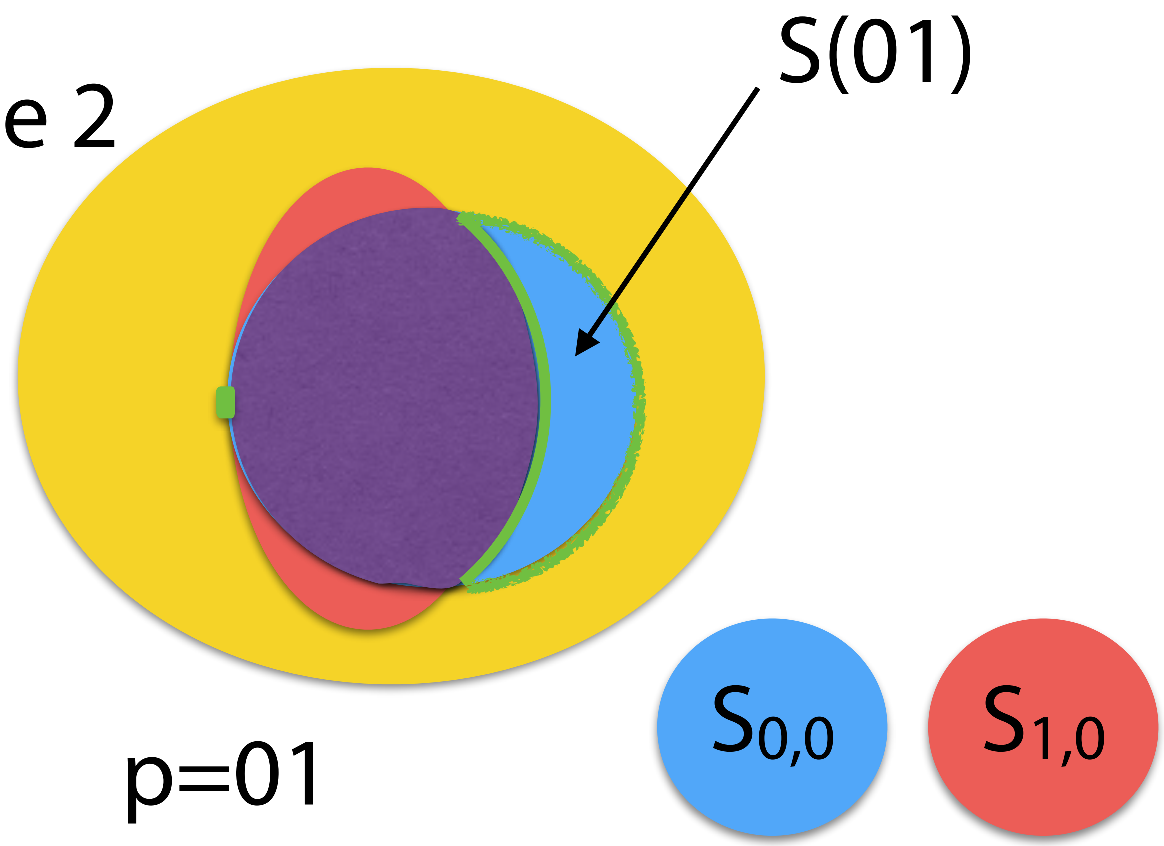
- **Definition 2.** A dyadic subbase S of X is proper if $\overline{S}(p) = \mathrm{cl}\, S(p)$ for every $p \in \mathbb{T}^*$.
- Examples of non-proper dyadic subbases.

Eample 1



p=0

Eample 2

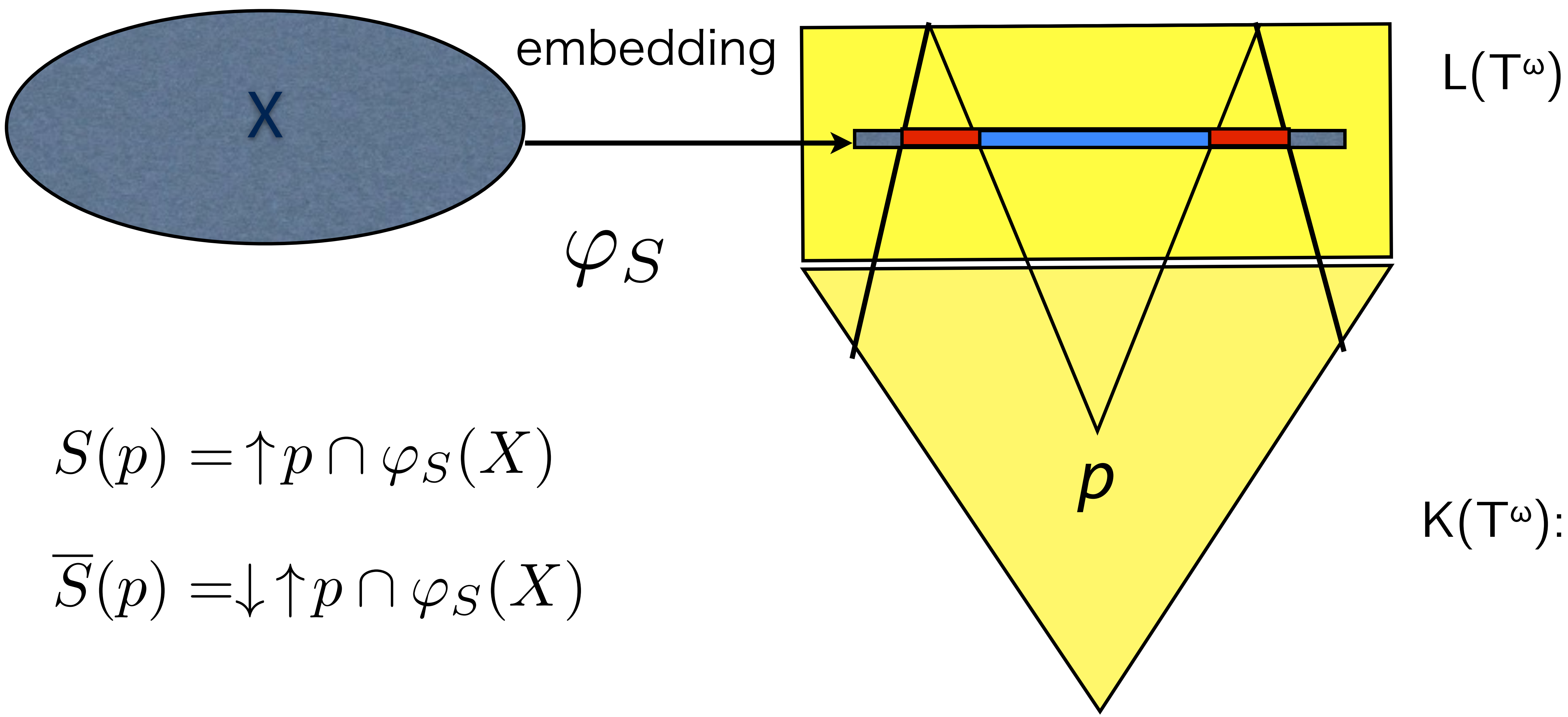


p=01

- If S is proper, then $S_{n,0}$ and $S_{n,1}$ are exteriors of each other and $S_{n,\delta}$ is their common boundary.
- If S is proper, then $\varphi_S(X) \subseteq L(D_S)$.
- Proper —- Boundaries are orthogonal.

$S(p)$ and $\overline{S}(p)$ are order-theoretic

If S is a dyadic subbase of X,



$$S(p) = \uparrow p \cap \varphi_S(X)$$

$$\overline{S}(p) = \downarrow\uparrow p \cap \varphi_S(X)$$

Properness connect this order-theoretic notion with closure, which is a topological notionof X.

Proper Dyadic Subbase

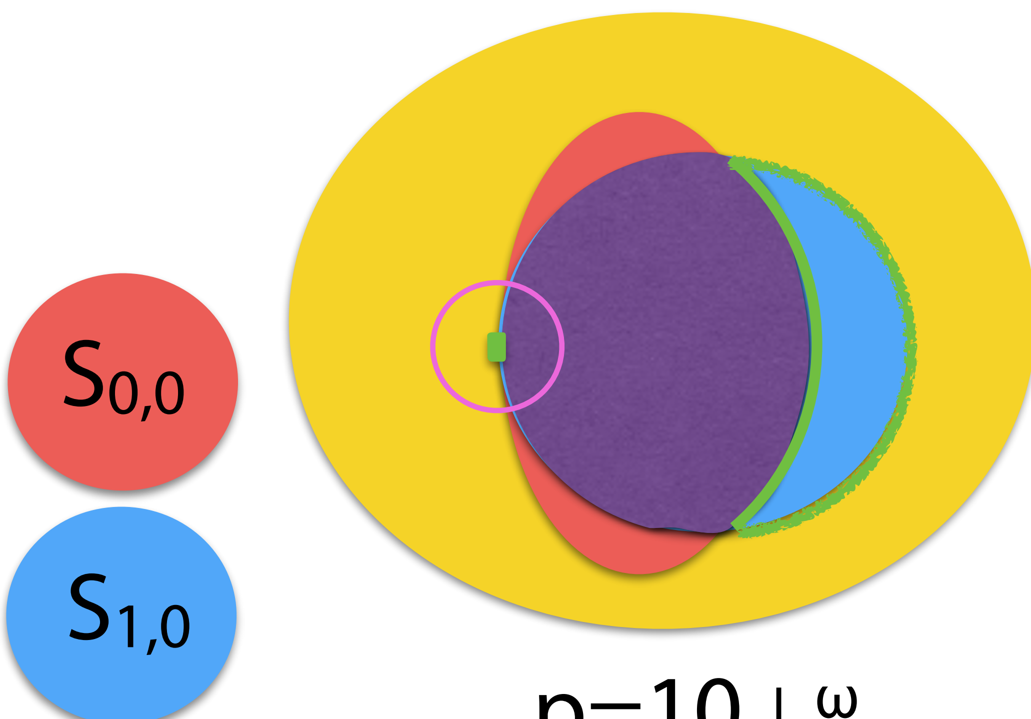
- **Theorem 1.** Every separable metric space has a proper dyadic subbase. In particular, if X is a separable metric space with dimension n, then X has a proper dyadic subbase of degree n. [Ohta, T, Yamada 2013]

- dimension —- small inductive dimension (= large inductive dimension = covering dimension.)

- degree of S —- the supremum of the number of bottoms appearing in $\varphi_S(x)$. It is the number of extra heads required to access the space by an IM2-machine.
- Connecting a property of a space (dimension) and a structure of a machine (number of heads).

Independent Subbse

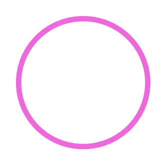
- **Definition 3.** A dyadic subbase S of X is independent if $S(p) \neq \emptyset$ for every $p \in \mathbb{T}^*$.
- **Proposition 1.** An independent subbase is proper.



$p=10\bot^{\omega}$

(Proof)

Suppose that $x \in \overline{S}(p)$.



Let $q \in \mathbb{T}^*$ has arbitrary small $S(q) \ni x$.

Then, p and q are compatible. Since $S(p) \cap S(q) = S(p \sqcup q) \neq \emptyset$, the point x is in the closure of S(p).

- **Theorem 3.** Every dense in itself separable metric space has an independent subbase. [Ohta, T, Yamada 2010]

Possibility to separate points w.r.t. S.

- Since X is Hausdorff, every pair of points can be separated by a pair of open sets.
- **Proposition2.** If S is a proper dyadic subbase of X, every pair of points can be separated by one component of a dyadic subbase.
- If S is not proper, it may be the case that

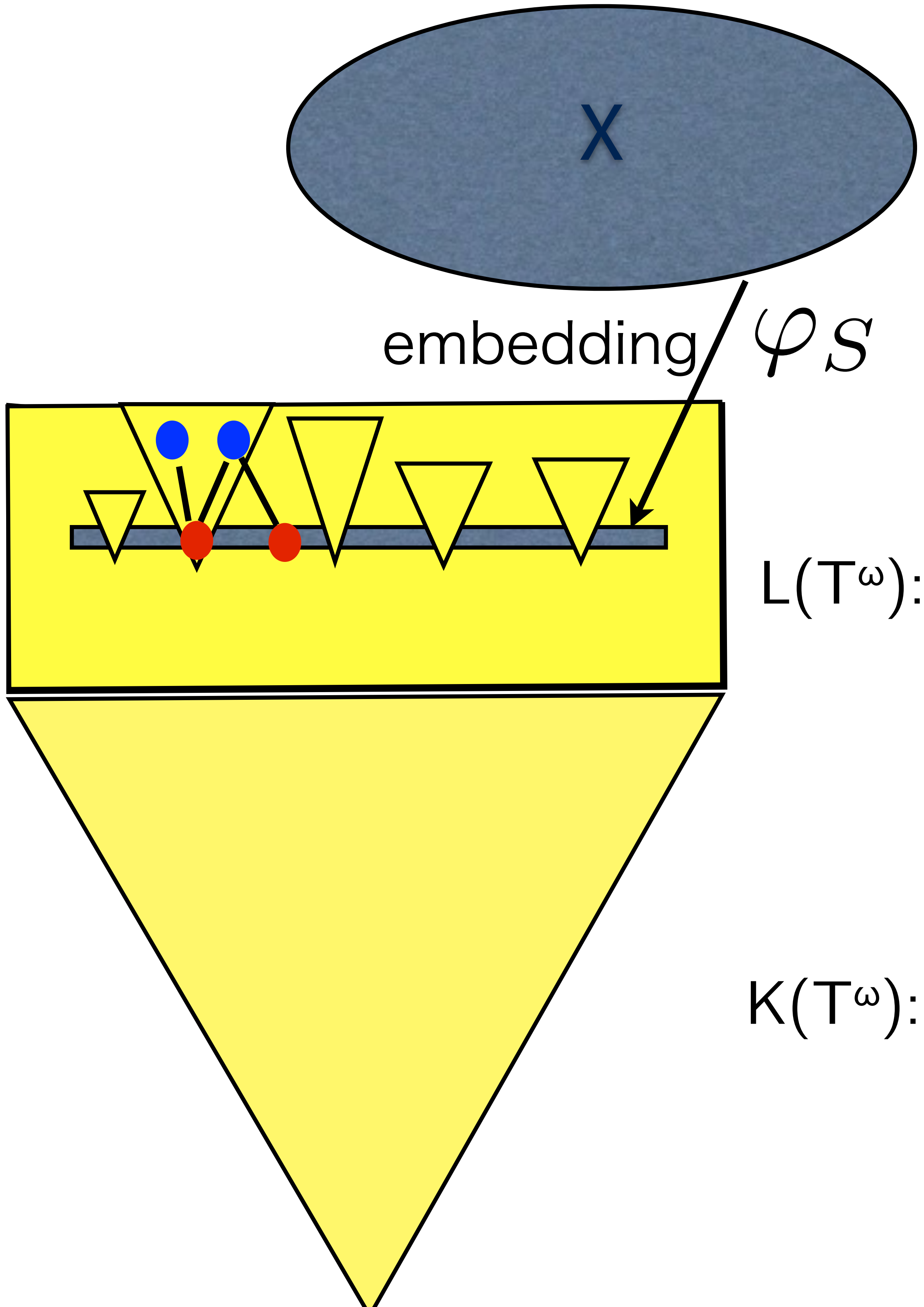
n	0	1	2	3	4	...
$\varphi_S(x)(n)$	$\perp$	1	1	0	1	
$\varphi_S(y)(n)$	1	$\perp$	$\perp$	0	1	

- If S is proper, for $x \neq y$, we always have an index i such that

n			...	i	...	
$\varphi_S(x)(n)$			1	0	0	
$\varphi_S(y)(n)$			$\perp$	1	0	

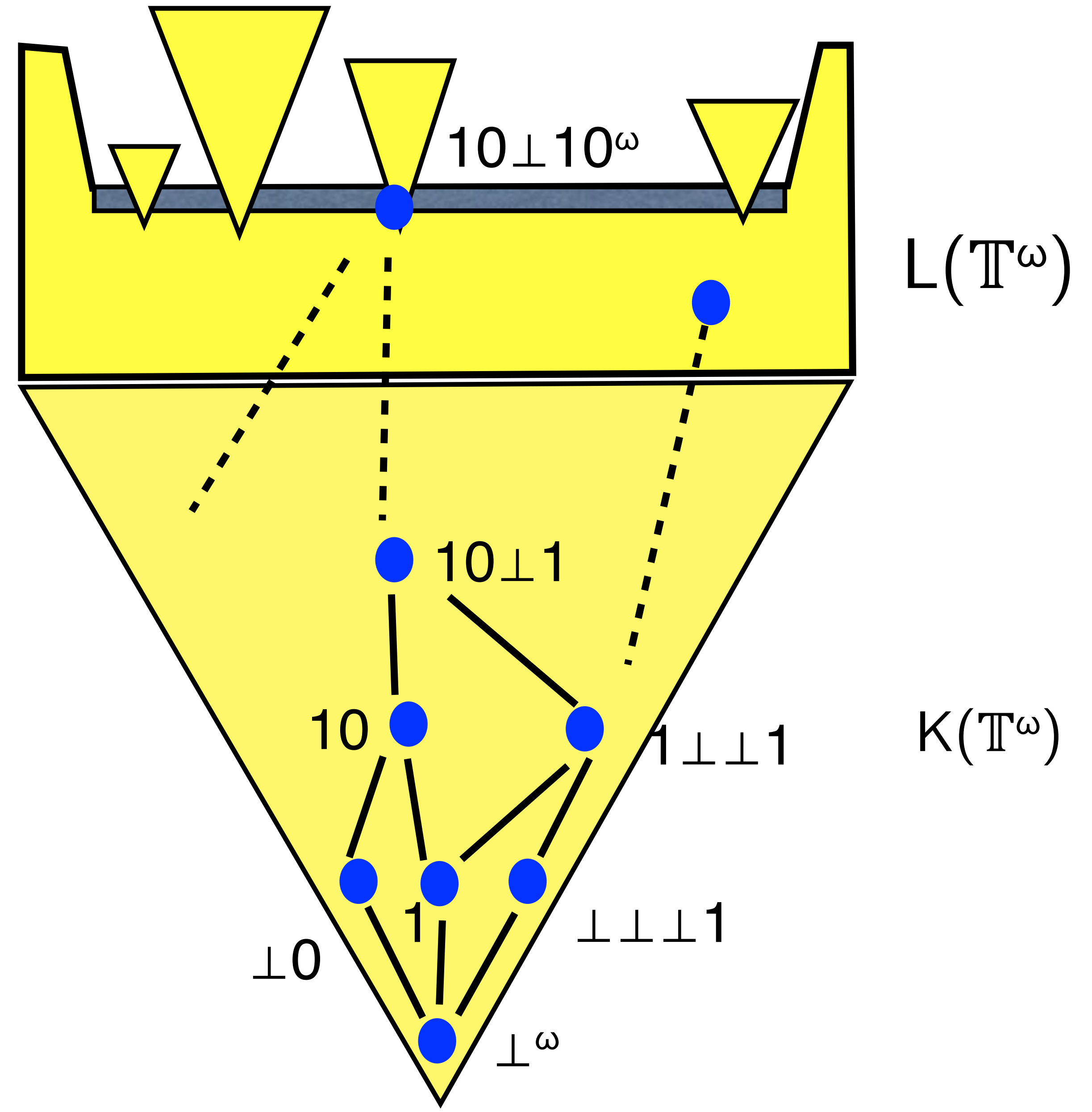
Retract Structure on the domain

- If S is not proper, it may be the case that $\varphi_S(x) \sqsubseteq p$ and $\varphi_S(y) \sqsubseteq p$ for $x \neq y$.
- **Proposition 3.** If S is proper, for each p $\in \uparrow\varphi_S(X)$, there is a unique point $x \in$ X such that $\varphi_S(x) \sqsubseteq p$.
- We denote this x by $\rho(p)$.
- **Proposition 4.** X is regular iff the map ρ is continuous. In this case, $\rho: \uparrow\varphi_S(X) \to X$ is a retraction.



Increasing Sequences in T$^{\omega}$

- Every increasing sequence in K($\mathbb{T}^{\omega}$) identifies an element of L($\mathbb{T}^{\omega}$) .
- Only interested in sequences identifying φ_S(X).
- Consider a subset of K($\mathbb{T}^{\omega}$) so that the set of limit is more close to φ_S(X), just as the case of Gray-expansion.

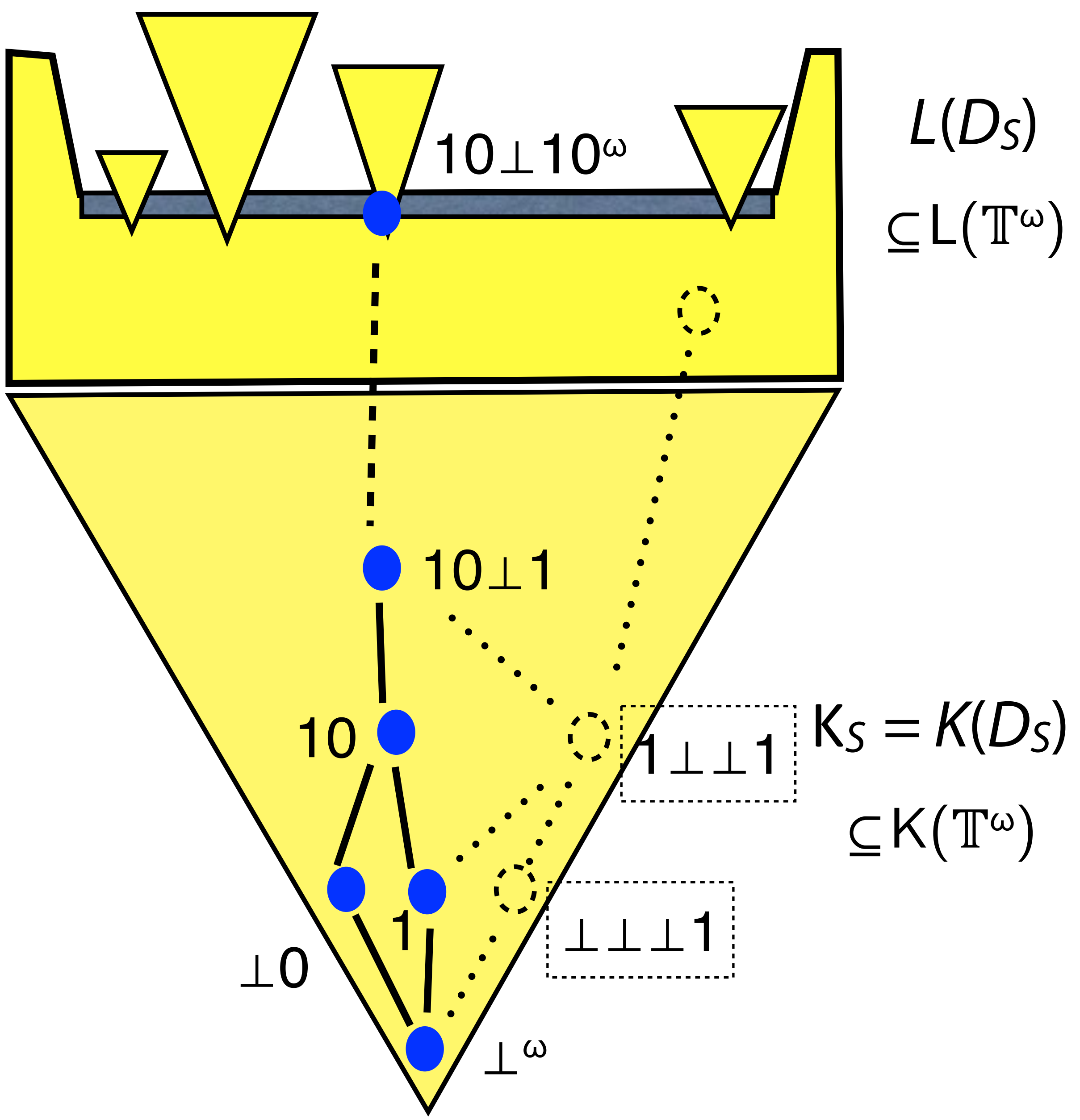


4. Proper dyadic subbase and domain representations

[T, Tsukamoto]

Restricting the set of finite states

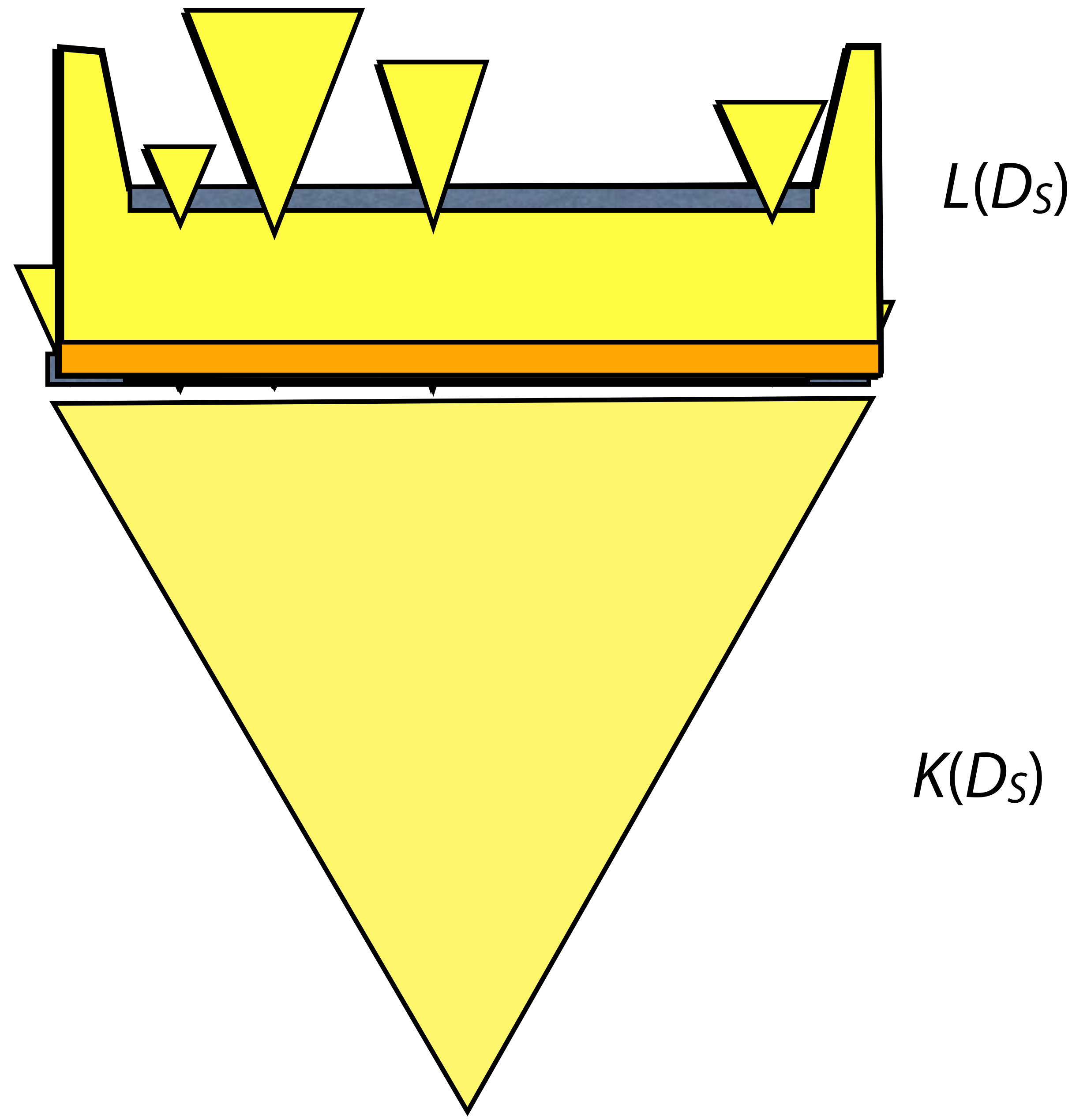
- $p|_m = p_0 p_1 \ldots p_{m-1} \perp^\omega$
- $K_S = \{p|_m : p \in \varphi_S(X), m \in \mathbb{N}\}$.
- D_S : Ideal completion of K_S.
- $\varphi(X) \subseteq L(D_S)$.



Properties of X and properties of the corresponding embeddings.

Let S be any proper d. s. of X.

- **Theorem 4.** If X is strongly nonadhesive, $L(D_S)$ has the set of minimal elements.
- **Theorem 5.** If X is regular, then $X \subseteq \min(L(D_S))$.
- **Theorem 6.** If X is compact, $X = \min(L(D_S))$.

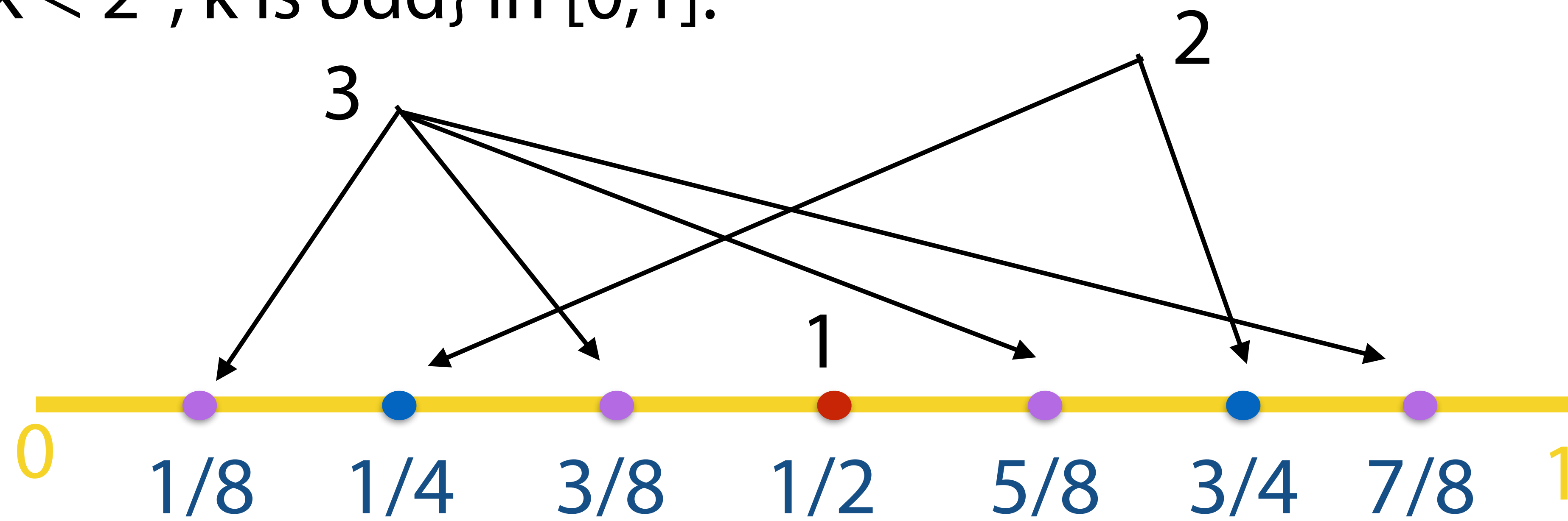


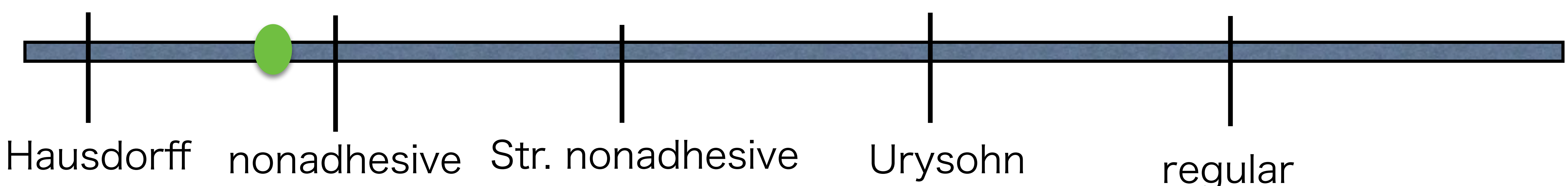
Strongly nonadhesive space

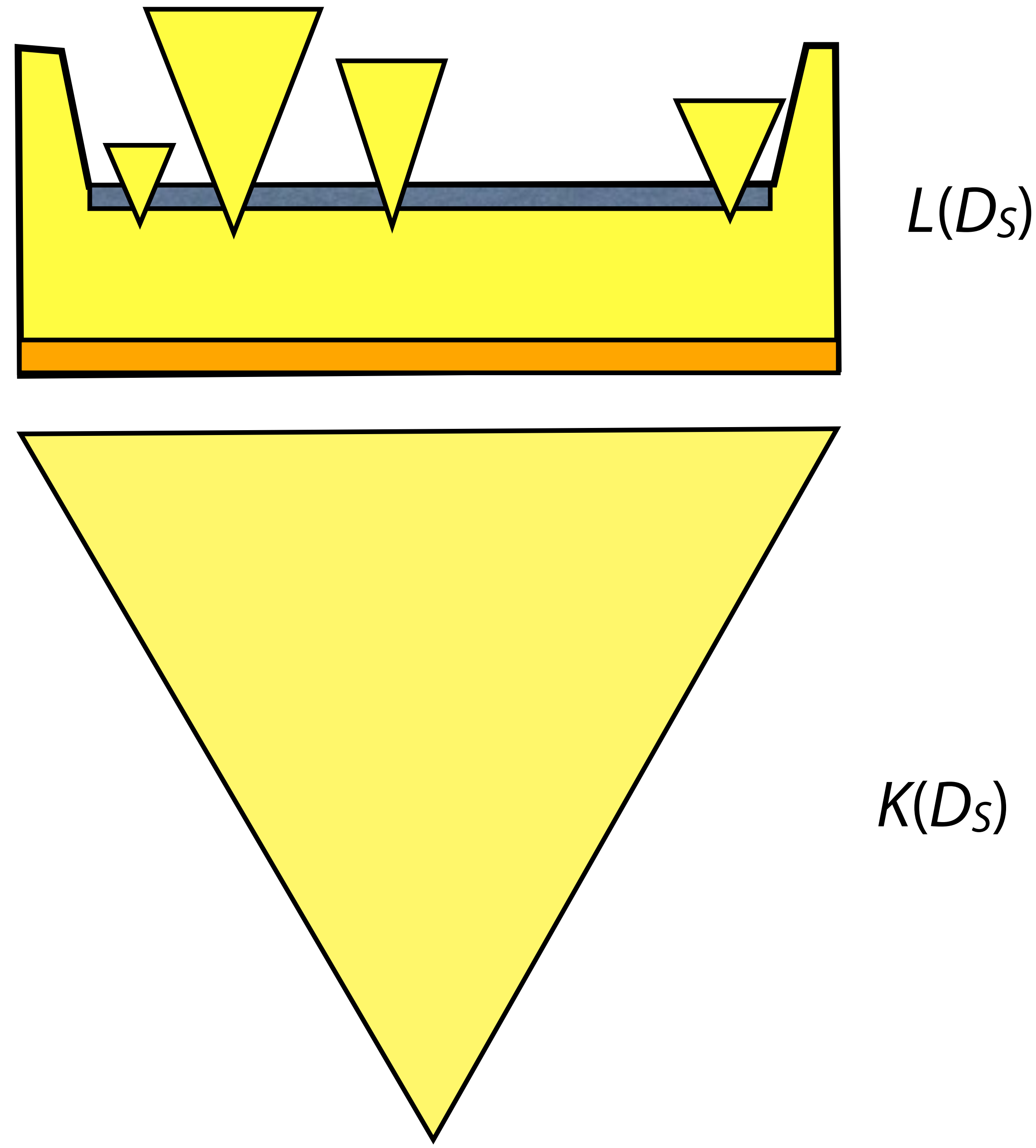
- **Definition.** (1) We say that a Hausdorff space X is adhesive if X has at least two points and closures of any pair of non-empty open sets have non-empty intersection.
- (2) We say that X is nonadhesive if it is not adhesive.
- (3) We say that X is strongly nonadhesive if every open subspace is nonadhesive.
- A space is called Urysohn (or completely Hausdorff) if any two distinct points can be separated by closed neighbourhoods. A regular space is always Urysohn.
- **Proposition 2.** Every Urysohn space is strongly nonadhesive.
- **Proposition 3.** There exists an adhesive Hausdorff space.

Example of an adhesive Hausdorff space

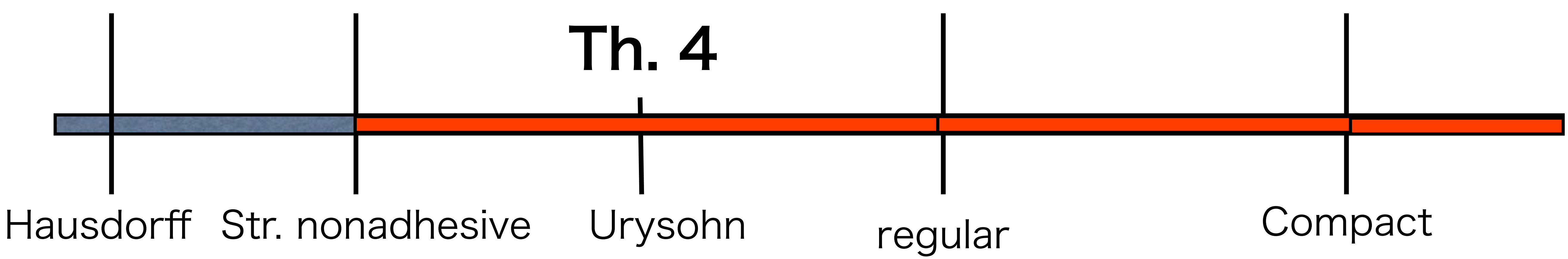
- Let P be the set of dyadic irrational numbers in [0,1].
- $X = P \cup \mathbb{N}$ for $\mathbb{N} = \{1,2,3\ldots\}$.
- Neighbourhood base of $x \in P$ is $U \cap P$ for U a nhd. of x in [0,1].
- Neighbourhood base of $n \in \mathbb{N}$ is $\{n\} \cup (U \cap P)$ for U a nhd. of $\{k/2^n : 0 < k < 2^n, k \text{ is odd}\}$ in [0,1].







Theorem 4. $\min(L(D_S))$ exists if X is strongly nonadhesive.



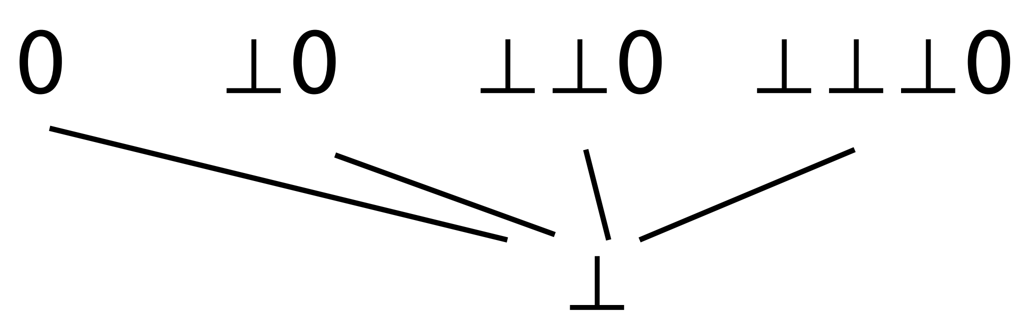
Proof of Theorem 4.

- **Theorem 4.** If X is strongly nonadhesive, $L(D_S)$ has the set of minimal elements.

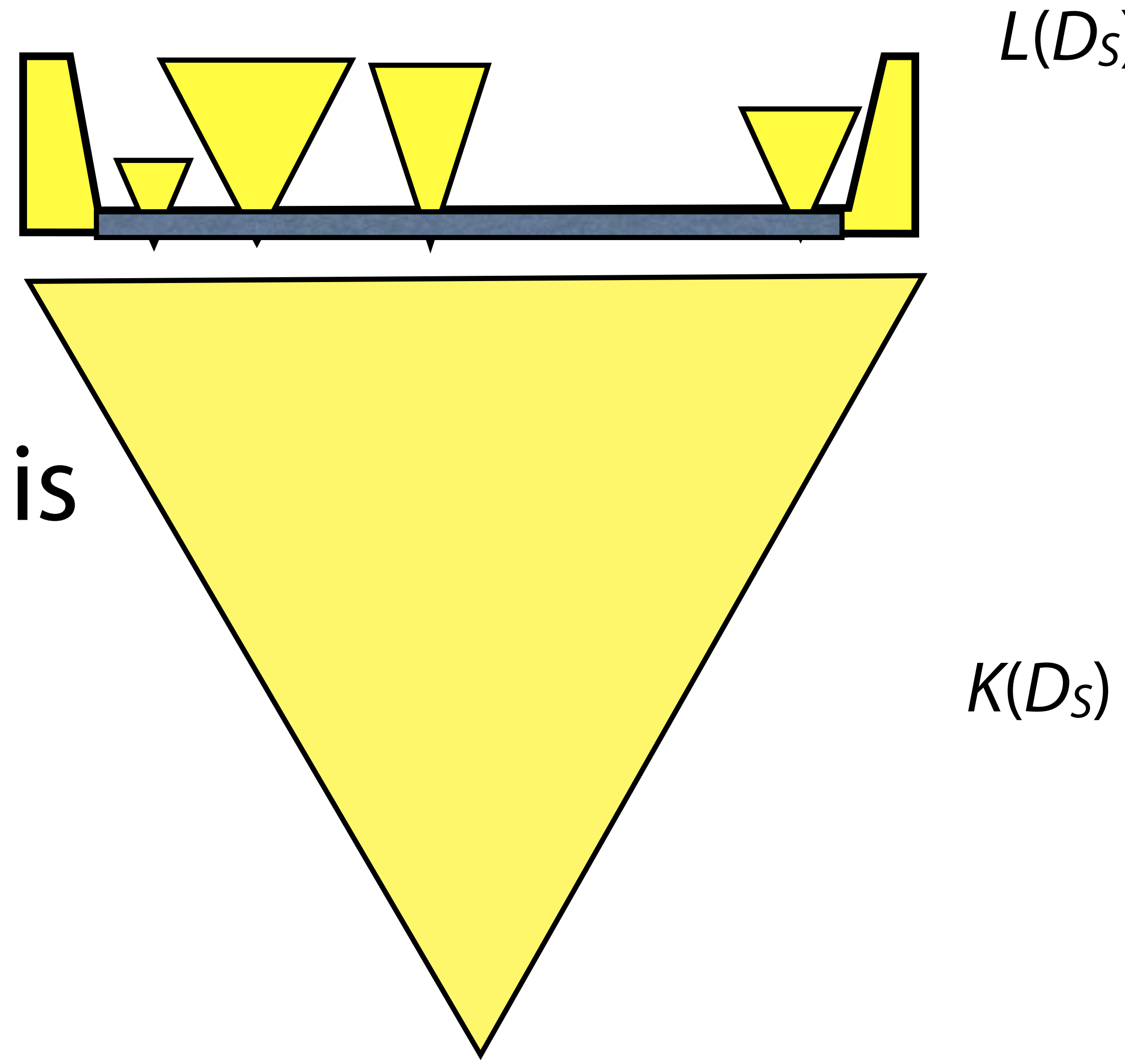
  - Since X is nonadhesive, only finite number of elements of K_S has one digit (0 or 1).

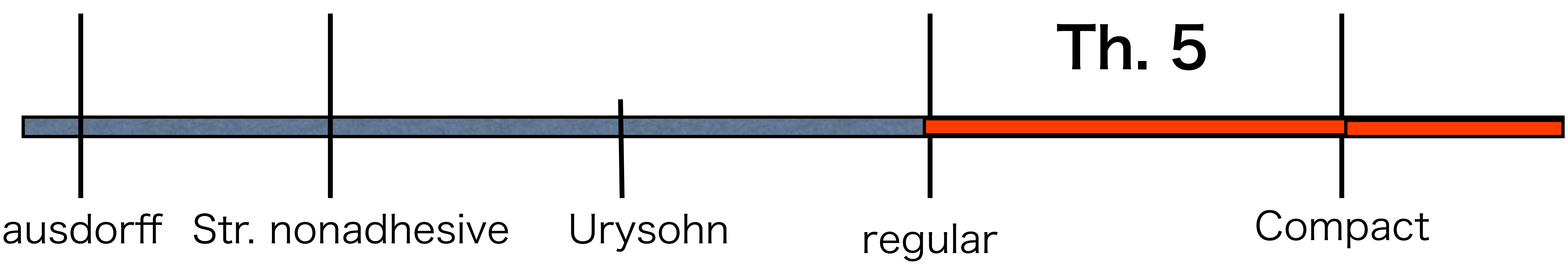
    (If S(p) and S(q) do not intersect in their closures, there is no point x such that $\varphi_S(x) = \perp^n\ldots$ for n the maximal length of p and q.)

  - We can show that K_S is finite-branching by applying this to the subspace S(e) with the dyadic subbase restriction of S to S(e),
  - As the limit of a finite-branching poset, $L(D_S)$ has a set of minimal elements.



- **Theorem 5.** Suppose that *X* is regular. If p ∈ L(D) and p is compatible with φ(x) in $\mathbb{T}^\omega$, then φ(x) ⊑ p. In particular, φ(X) ⊆ min(L(D)).





Exact version of S

- We consider $\{0,1,\delta\}^\omega$, instead of $\mathbb{T}^\omega = \{0,1,\perp\}^\omega$.
- δ : exactly on the boundary.
- We define $S_{k,\delta}$ as the common boundary of $S_{k,0}$ and $S_{k,\delta}$.
- For $p \in \{0,1,\delta\}^\omega$, we define *S(p)* as

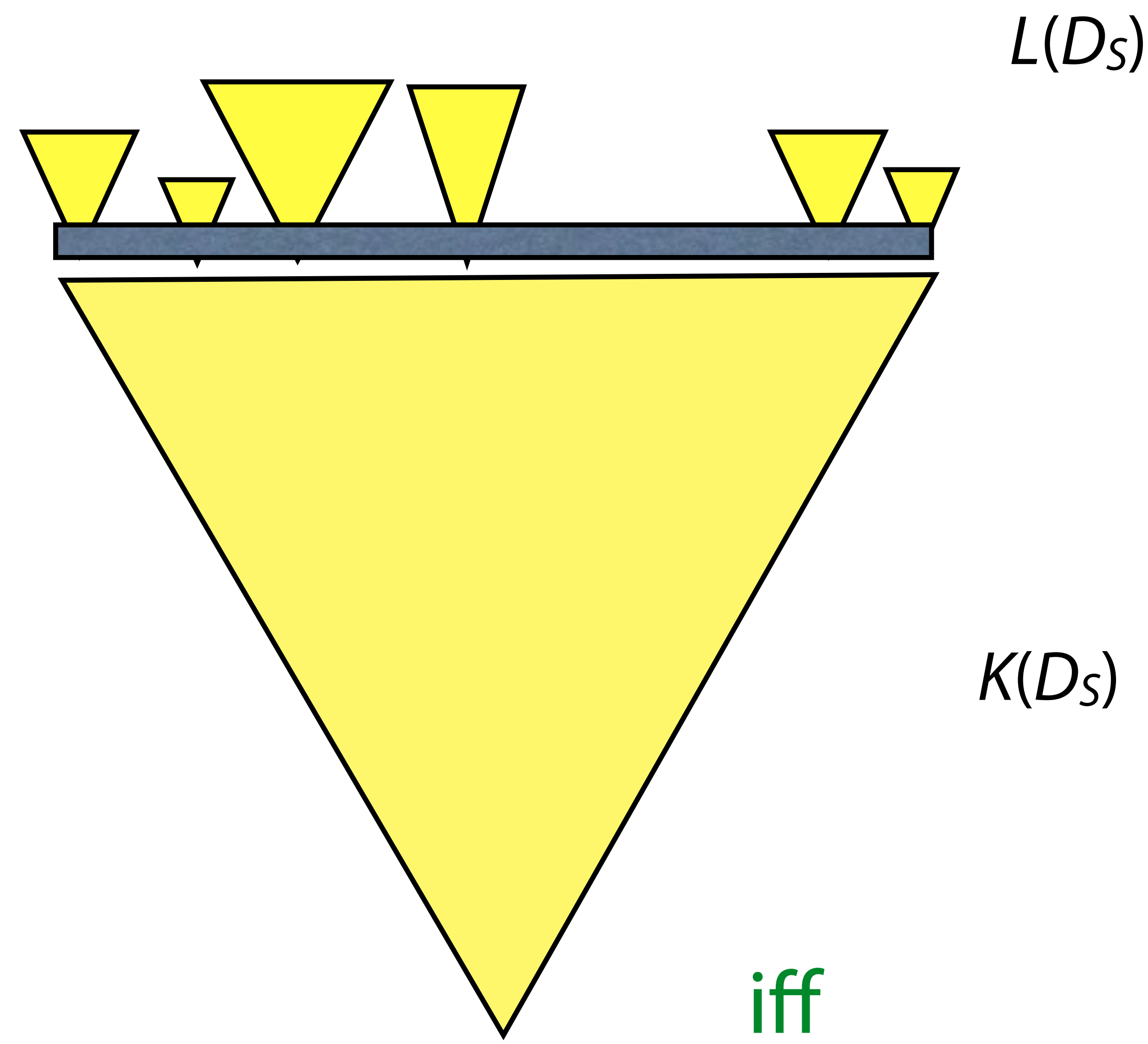
$$S(p) = \bigcap_{k<len(p)} S_{k,p(k)}$$

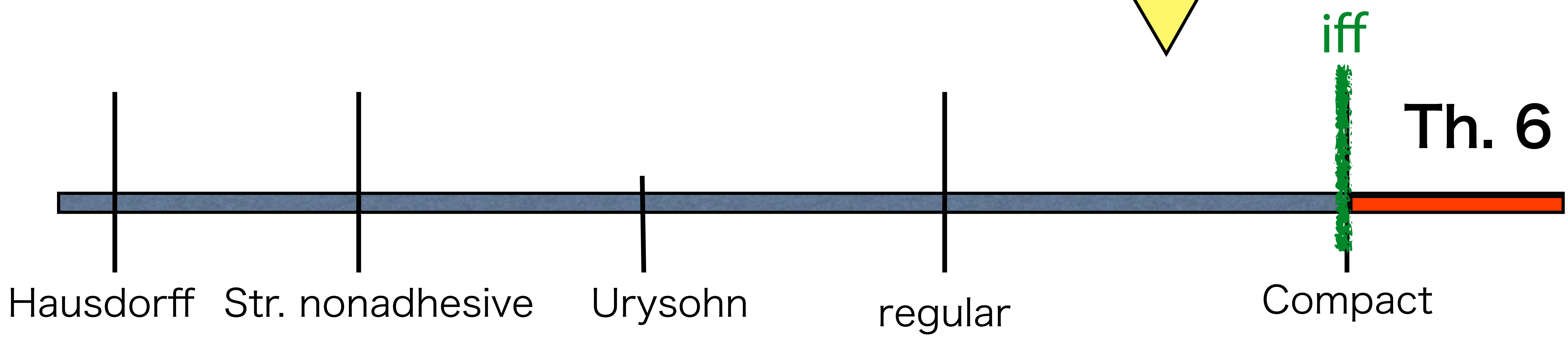
- For a sequence $p \in \{0,1,\perp\}^*$, we denote by $p^\delta \in \{0,1,\delta\}^*$ the sequence obtained by replacing inner bottoms with δ.
- For example, for $p = 01\perp1\perp^\omega$, $p^\delta = 01\delta1$.
- $K_S = \{ p \in \mathbb{T}^\omega \mid S(p^\delta) \neq \emptyset \}$.

Proof of Theorem 5.

- **Theorem 5.** Suppose that *X* is regular. If $p \in L(D)$ and p is compatible with $\varphi(x)$ in $\mathbb{T}^\omega$, then $\varphi(x) \sqsubseteq p$. In particular, $\varphi(X) \subseteq \min(L(D))$.
  - (proof) Assume that $\varphi(x)(m) = 0$, and prove that $p(m) = 0$.
  - Since X is regular and S is proper, $\overline{S}(\varphi(x)|_n) \subseteq S_{m,0}$ for some $n > m$.
  - Since $p \in L(D)$, $p|_n \in K(D)$. Therefore, $S(p|_n{}^\delta)$ is not empty. Let $y \in S(p|_n{}^\delta)$. Since $p|_n$ is compatible with $\varphi(x)|_n$, $y \in \overline{S}(\varphi(x)|_n)$.
  - Therefore, $\varphi(y)(m) = 0$. Thus, $p(m) = 0$.

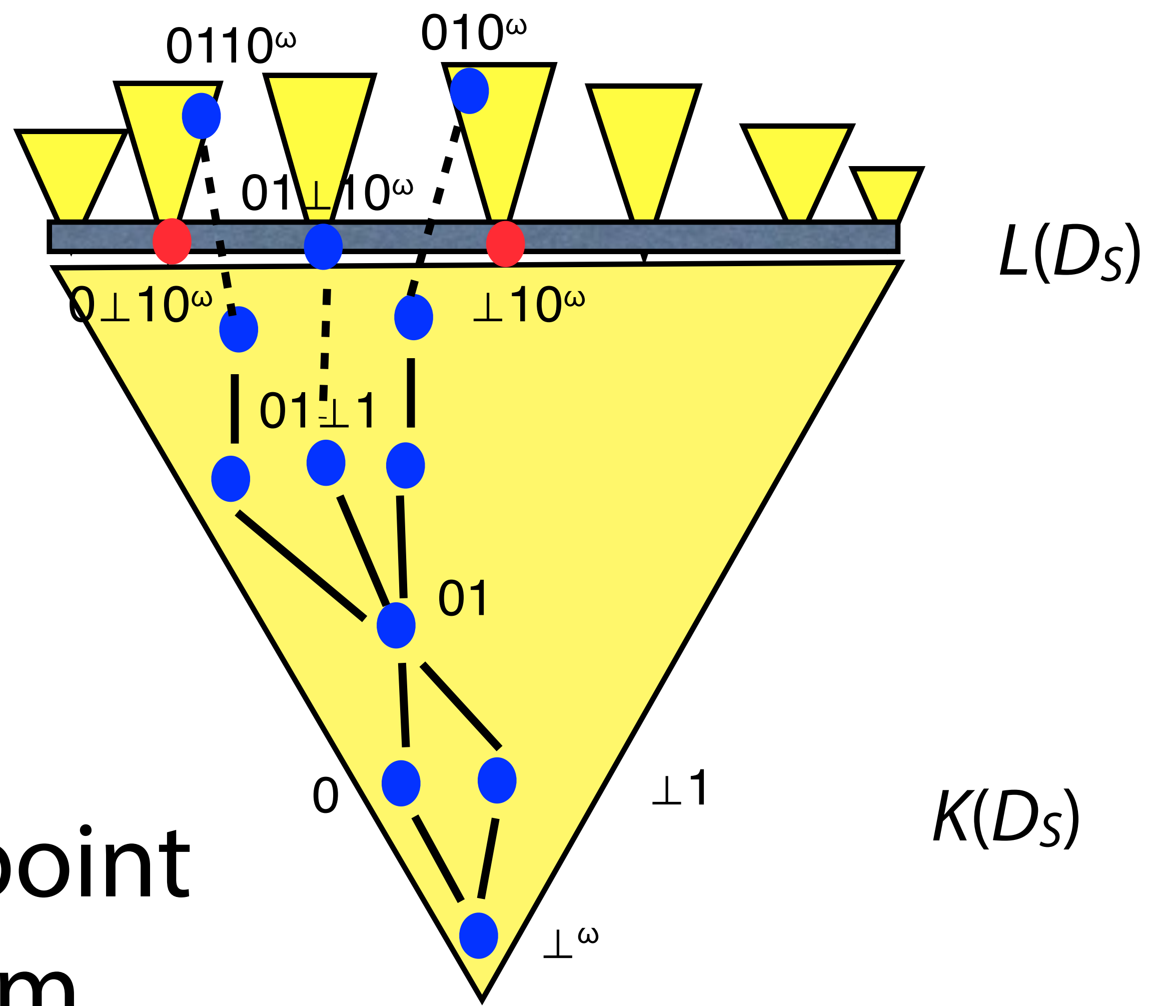
- **Theorem 6.** If X is compact, then $X = \min(L(D_S))$.





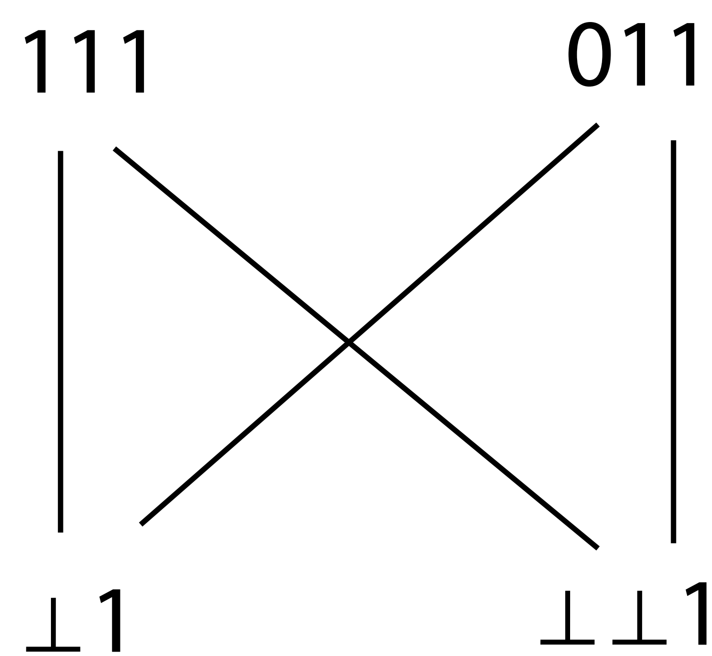
Proof of Theorem 6.

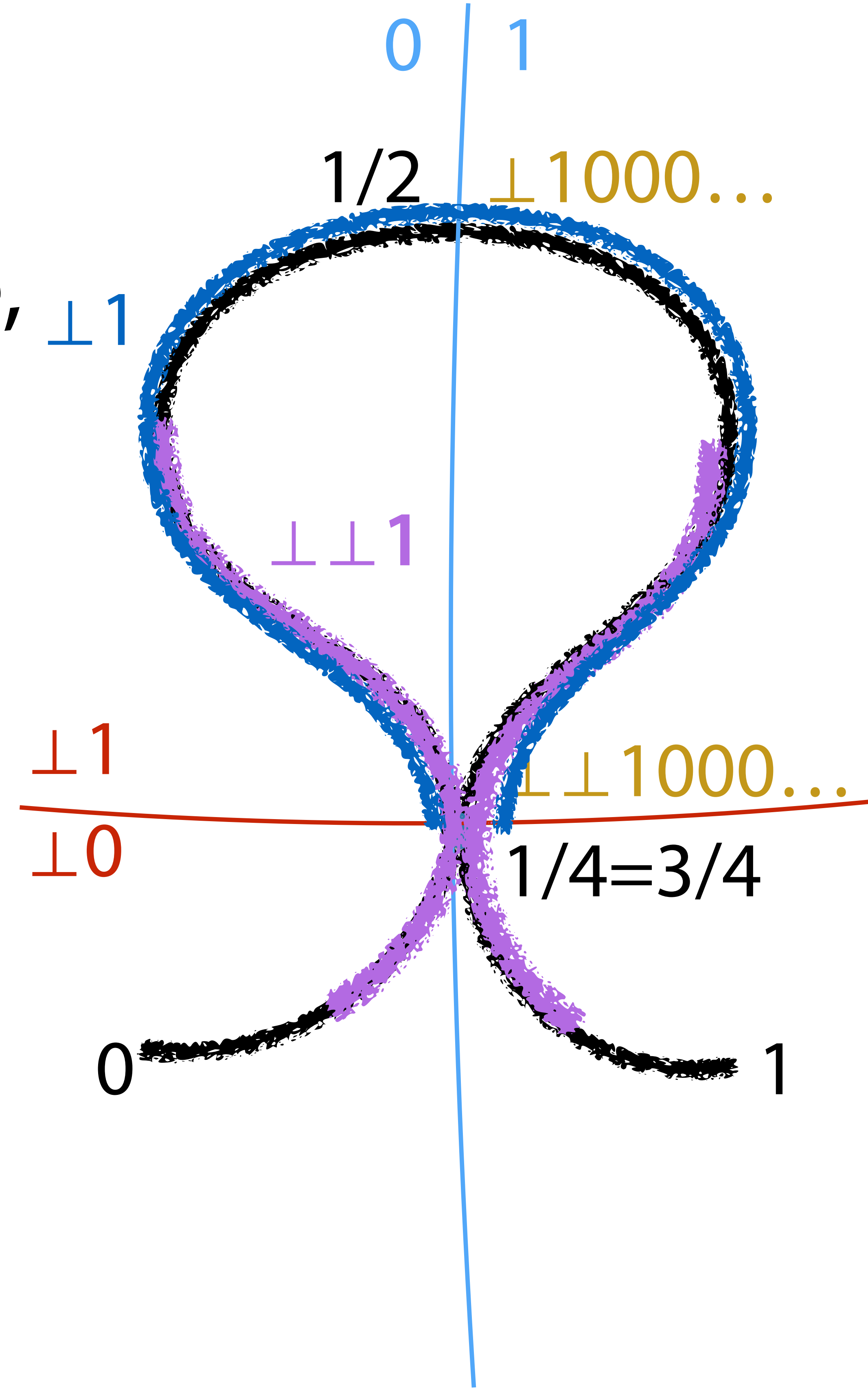
- **Theorem 6.** If X is compact, then X =min(L(D_S)) .
  - (proof) compactness of min(L(D_S)) .
- If X is compact, the poset K_S determine the space X.
- all incr. seq. in K(D_S) identify a point of X through the retraction from L(D_S) to min(L(D_S)).



D_S is not bounded complete

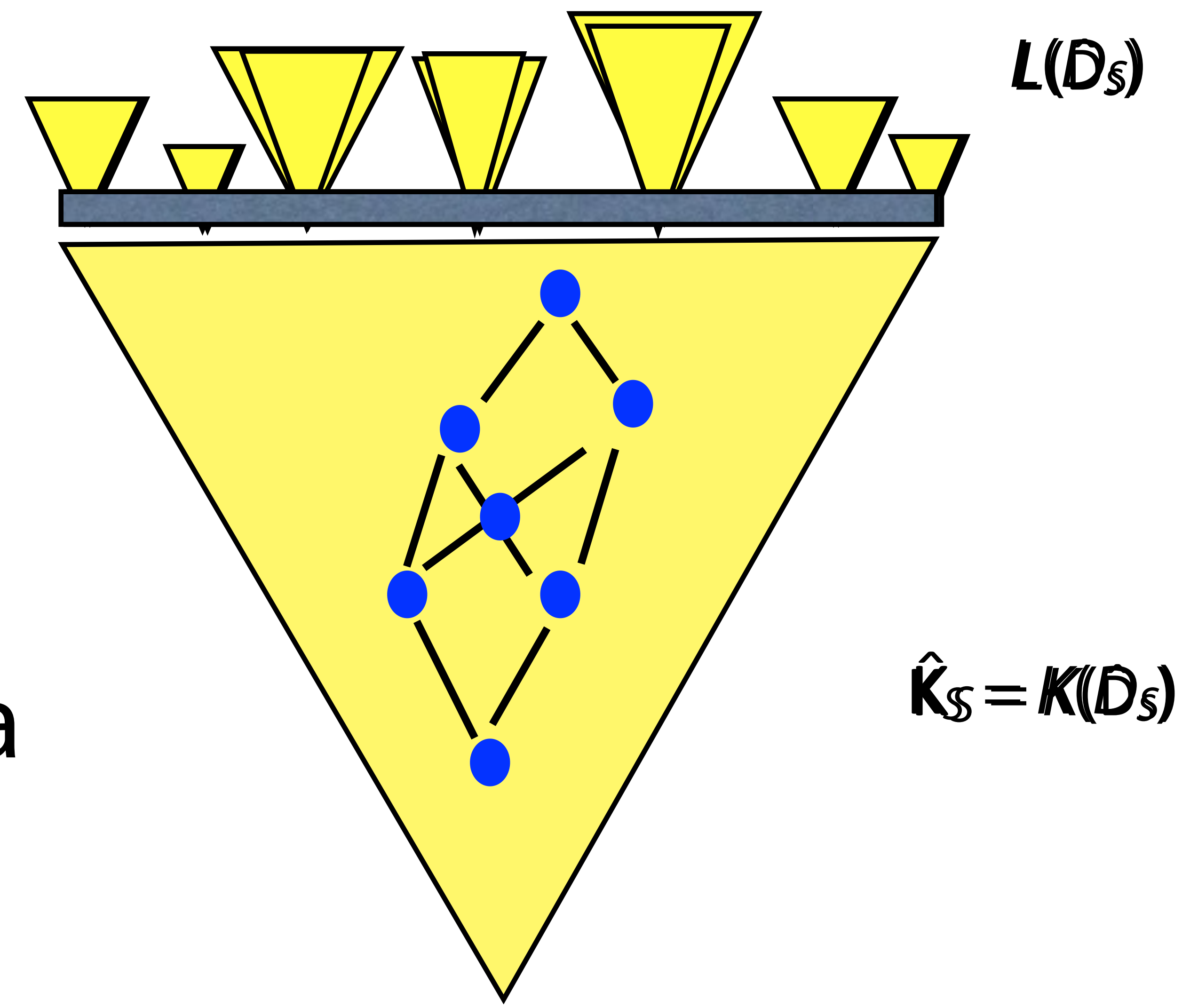
- Even if S is proper and X is compact, D_S may not be bounded complete. Therefore, D_S may not be a Scott domain.
- Example: [0,1] with the Gray code, with identification 1/4=3/4.





$\hat{D}_S$: bounded complete modification.

- $K_S = \{p|_m : p \in \varphi_S(X), m \in \mathbb{N}\}$.
- D_S is not bounded complete, in general.
- $\hat{K}_S = \{p|_m : p \in \uparrow\varphi_S(X), m \in \mathbb{N}\}$.
- $\hat{D}_S$: Ideal completion of $\hat{K}_S$.
- **Theorem.** $\hat{D}_S$ is bounded complete (and therefore is a Scott domain).
- $\hat{D}_S$ also satisfies Theorem 4 to 6.



Exact version of S and $\bar{S}$

- We consider $\{0,1,\delta\}^\omega$, instead of $\mathbb{T}^\omega = \{0,1,\perp\}^\omega$.
- δ : exactly on the boundary.
- We define $S_{k,\delta}$ as the common boundary of $S_{k,0}$ and $S_{k,\delta}$.
- For $p \in \{0,1,\delta\}^\omega$, we define *S(p)* as

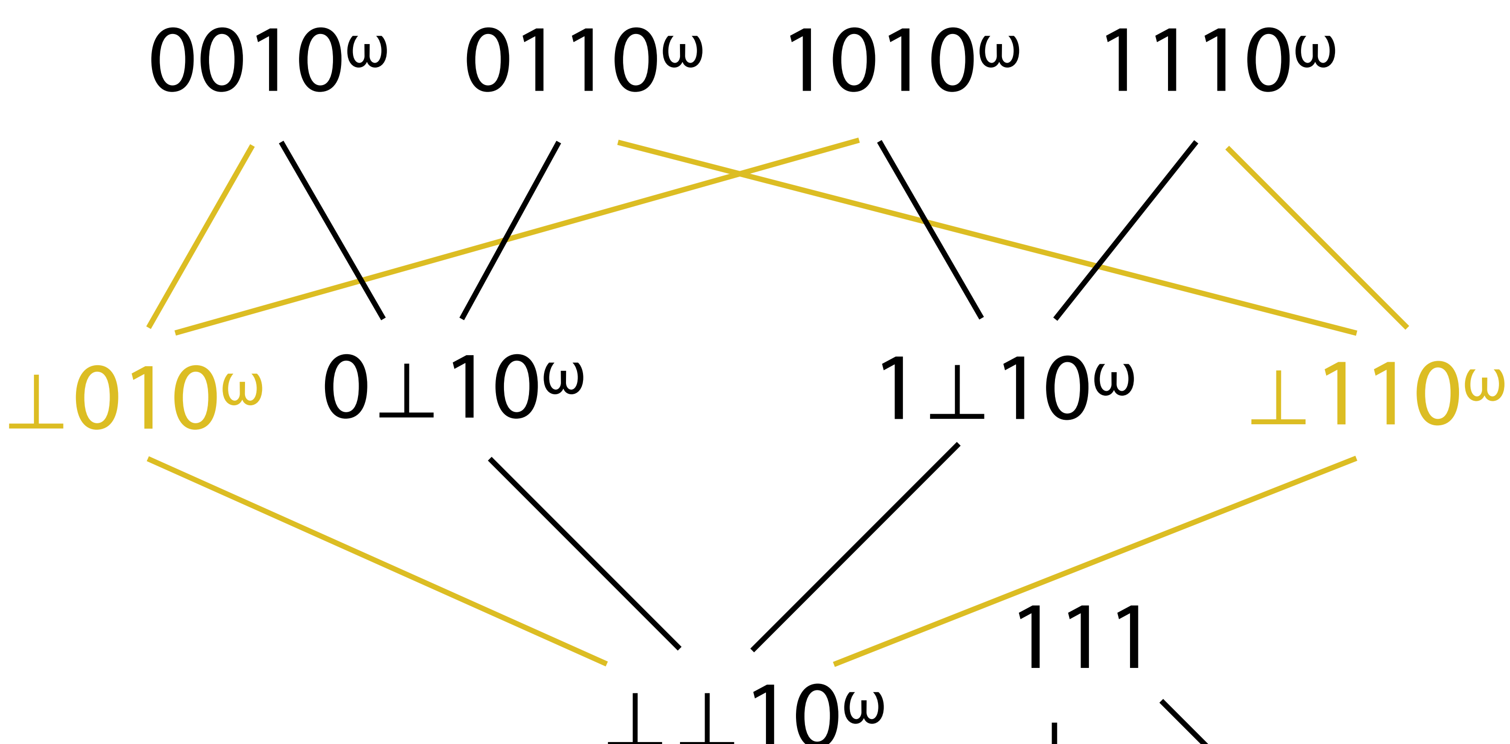
$$S(p) = \bigcap_{k<len(p)} S_{k,p(k)}$$

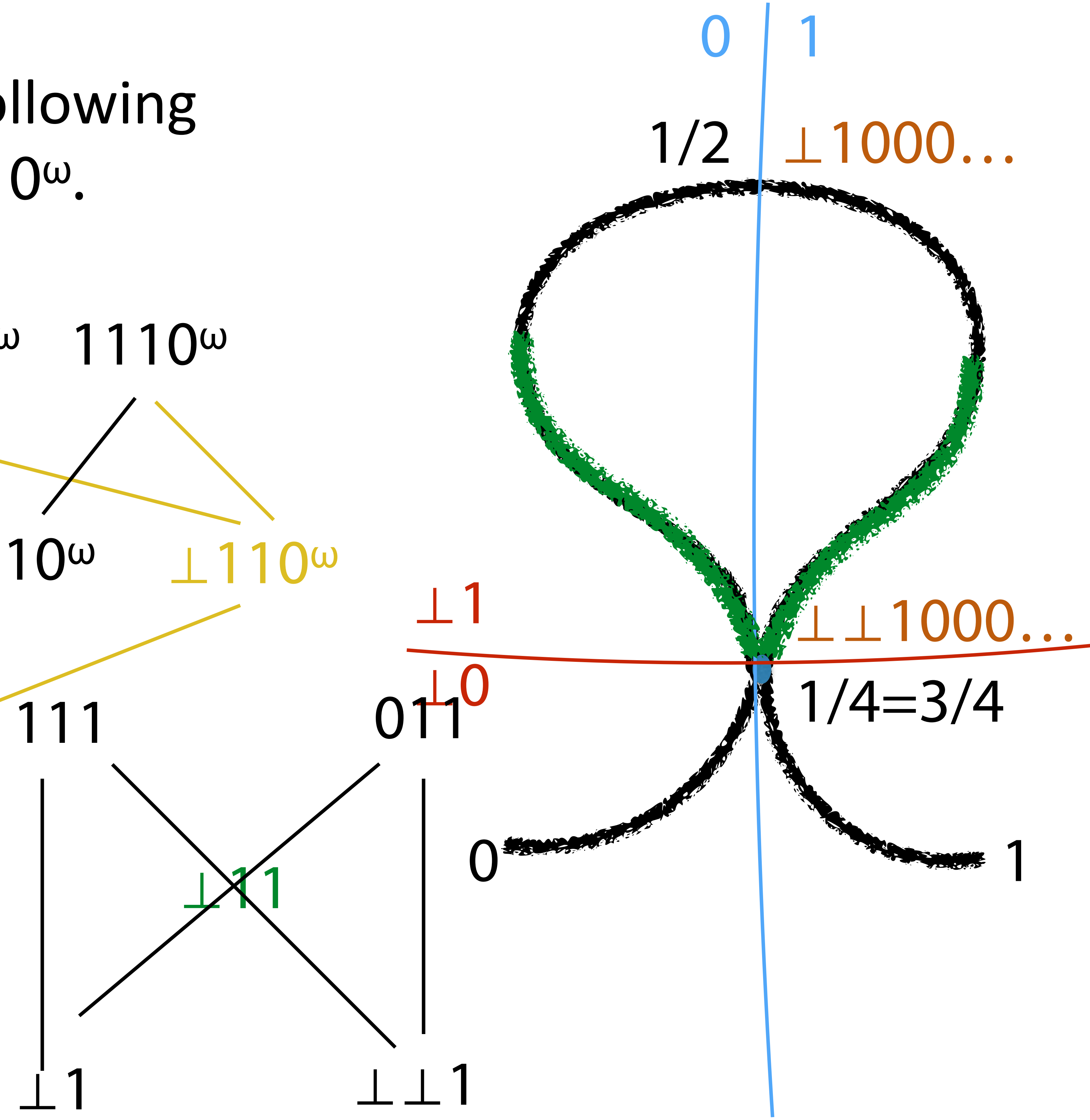
$$\bar{S}(p) = \bigcap_{k<len(p)} \mathrm{cl}\ S_{k,p(k)} = \bigcap_{k<len(p)} (S_{k,p(k)} \cup S_{k,\delta})$$

- For a sequence $p \in \{0,1,\perp\}^*$, we denote by $p^\delta \in \{0,1,\delta\}^*$ the sequence obtained by replacing inner bottoms with δ.
- For example, for $p = 01\perp1\perp^\omega$, $p^\delta = 01\delta1\perp^\omega$.
- $K_S = \{ p \in \mathbb{T}^\omega \mid S(p^\delta) \neq \emptyset\}$. $\hat{K}_S = \{ p \in \mathbb{T}^\omega \mid \bar{S}(p^\delta) \neq \emptyset\}$.

Example: $\hat{D}_S$ *is* bounded complete

- In D_S , we have the following elements above $\perp\perp10^\omega$.

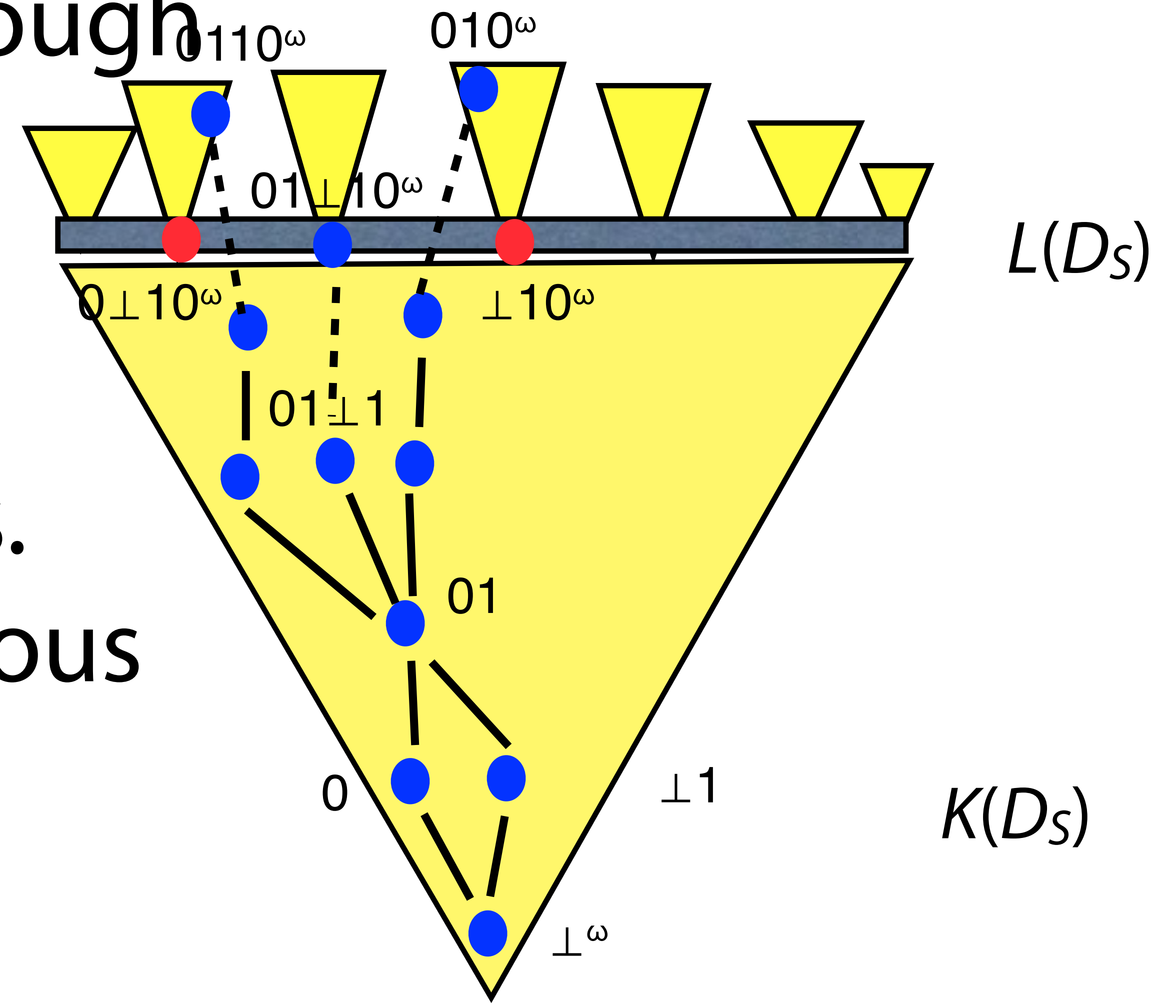




- In $\hat{D}_S$, we have more elements.

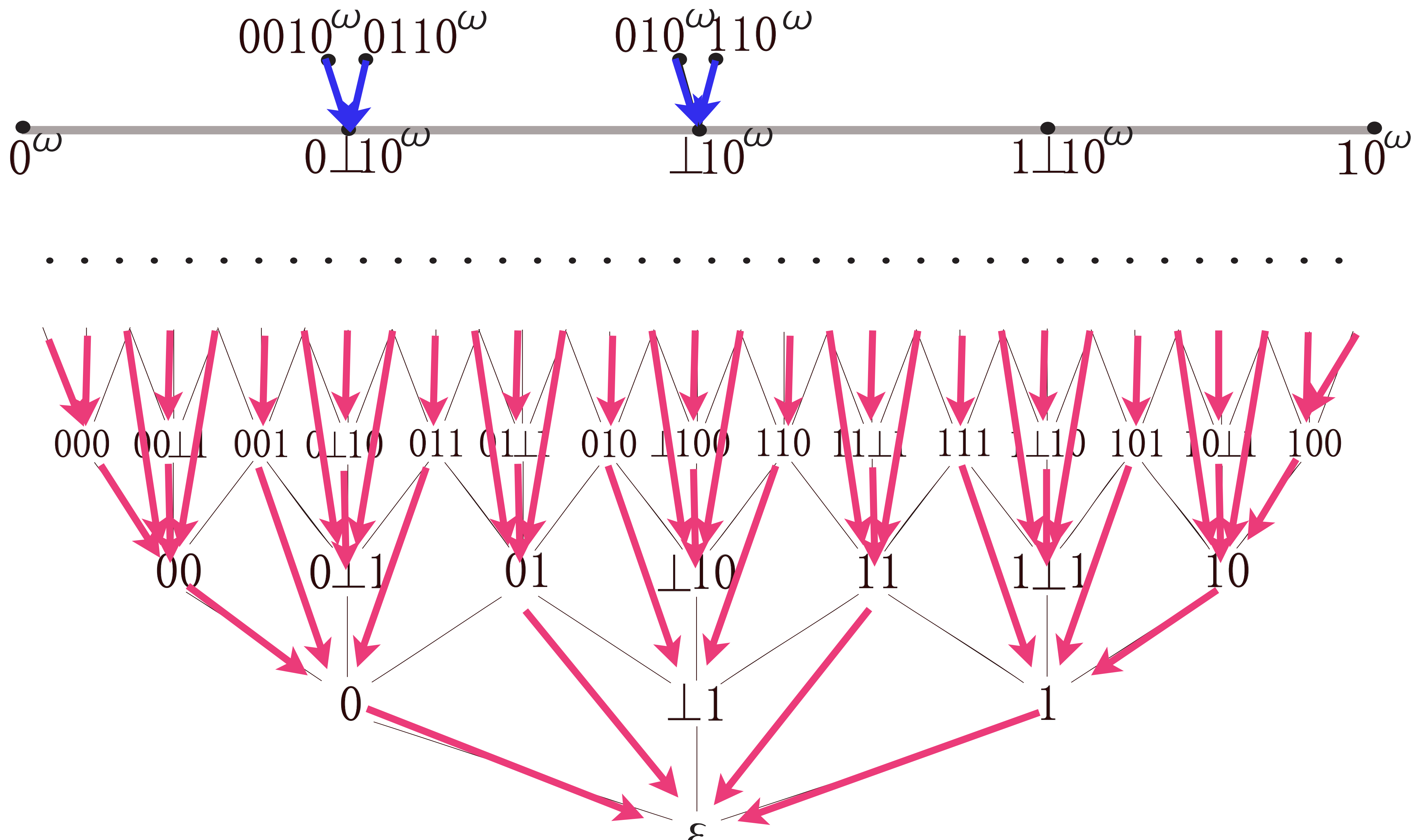
Extending the retraction to $\hat{D}_S$

- If X is compact, all incr. seq. in $K(\hat{D}_S)$ identify a point of X through the retraction ρ from $L(\hat{D}_S)$ to $\min(L(\hat{D}_S))$.
- The retraction ρ is continuous. Can we extend it to a continuous function from $\hat{D}_S$ to $\hat{D}_S$?



n	0	1	2	3	4	...
Input	0	1		1	0	
Output	0	1		1		

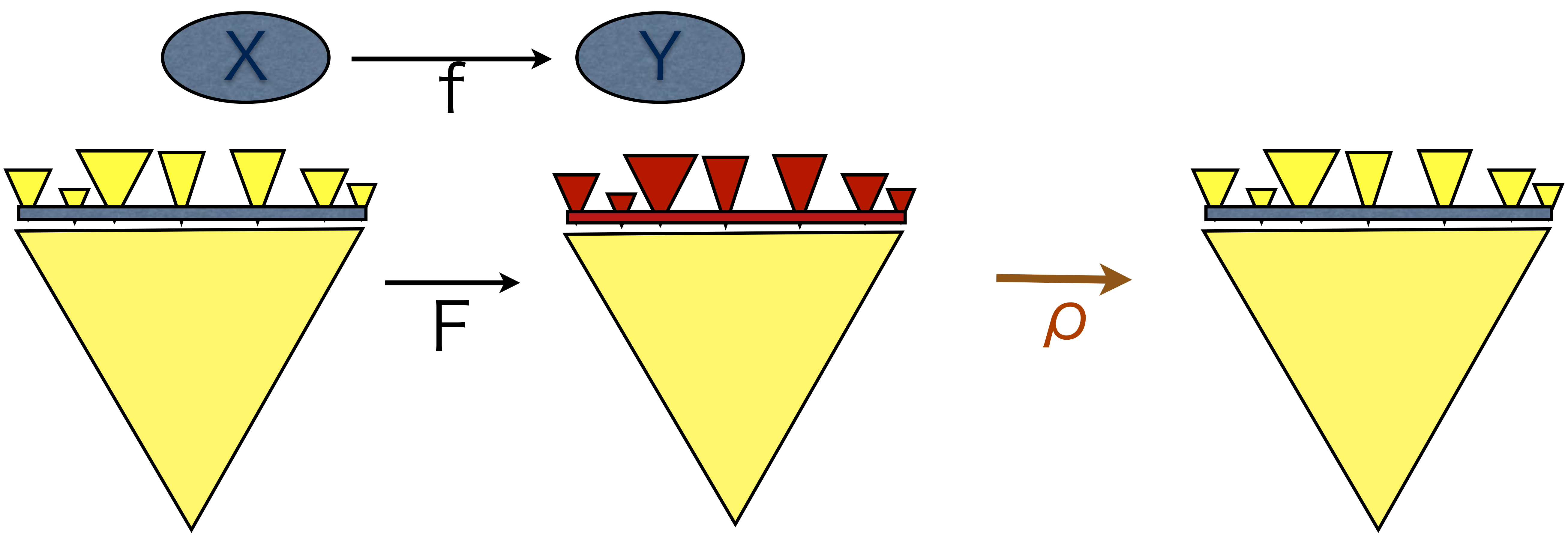
The domain D_G ($= \hat{D}_G$) of [0,1]



- ρ can be extended to a map from $\hat{D}_S$ to $\hat{D}_S$ as $\rho(p) = \mathrm{glb}(\{q \mid q \sqsubsetneq p$ s.t. there is no element between q and p$\})$.
- This map is computable if K_S is decidable as a subset of $\{0,1,\perp\}^*$.

Application to computation

- To write a program f from X to Y with an IM2-machine, one can define a function F, instead.



5. Strongly proper dyadic subbase

[Tsukamoto, T]

Strongly proper dyadic subbase

- We consider $\{0,1,\delta,\bot\}^\omega$, instead of $\mathbb{T}^\omega = \{0,1,\bot\}^\omega$.
- For $p \in \{0,1,\delta,\bot\}^\omega$, we define $S(p)$ and $\bar{S}(p)$ as

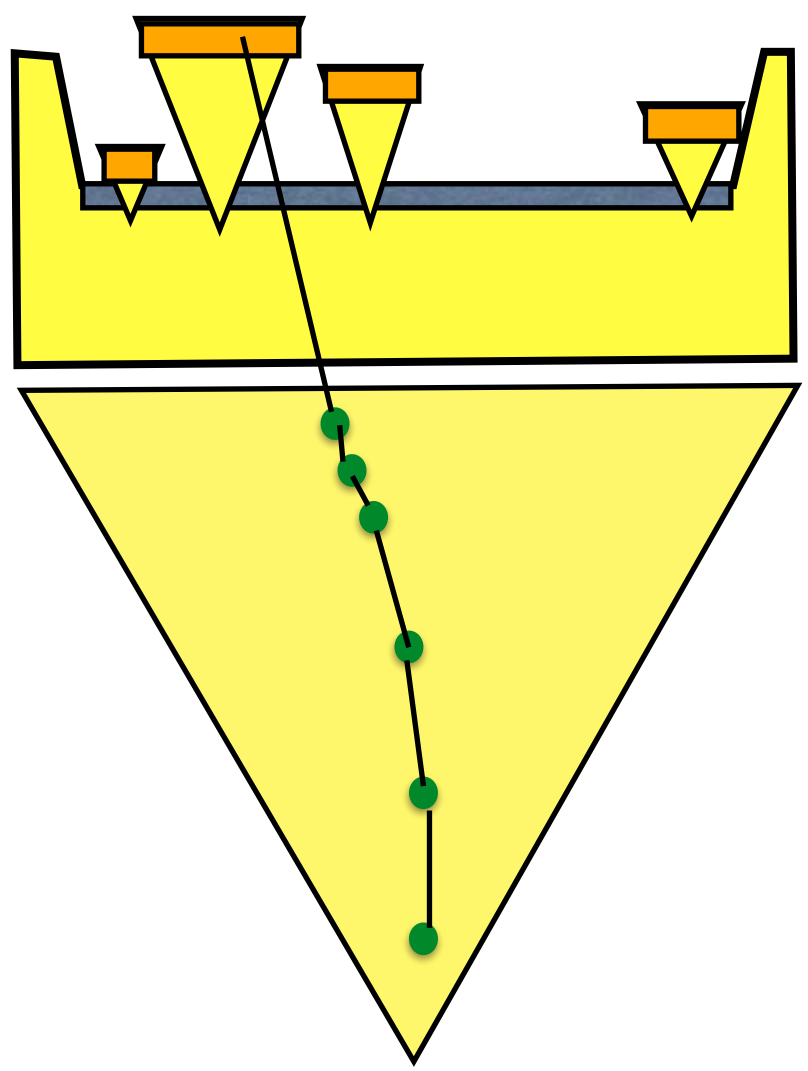
$$S(p) = \bigcap_{k \in \mathrm{dom}(p)} S_{k,p(k)},$$

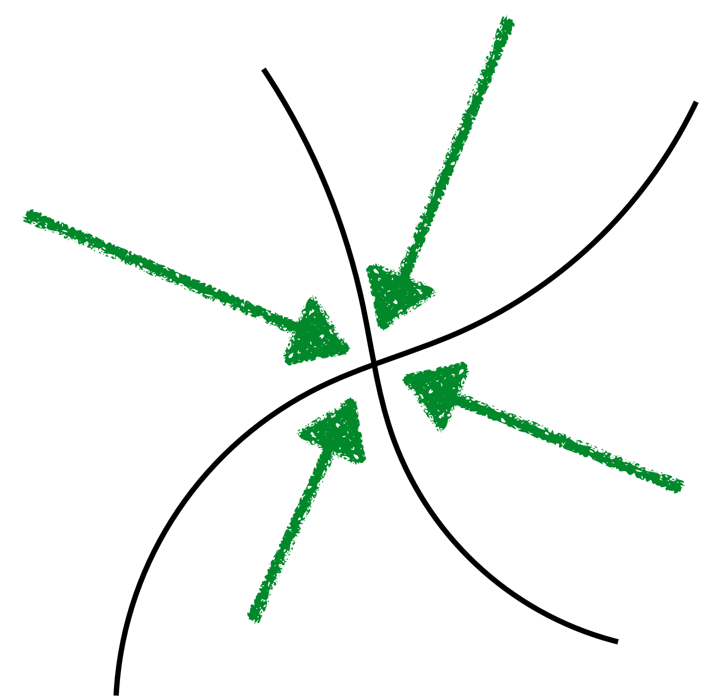
$$\bar{S}(p) = \bigcap_{k \in \mathrm{dom}(p)} \mathrm{cl}\, S_{k,p(k)}$$

- **Definition.** A dyadic subbase S is strongly proper if $\bar{S}(p) = \mathrm{cl}\, S(p)$.
- Recall that S is proper if $\bar{S}(p) = \mathrm{cl}\, S(p)$ for $p \in \{0,1,\bot\}^\omega$.

Properness and approximation.

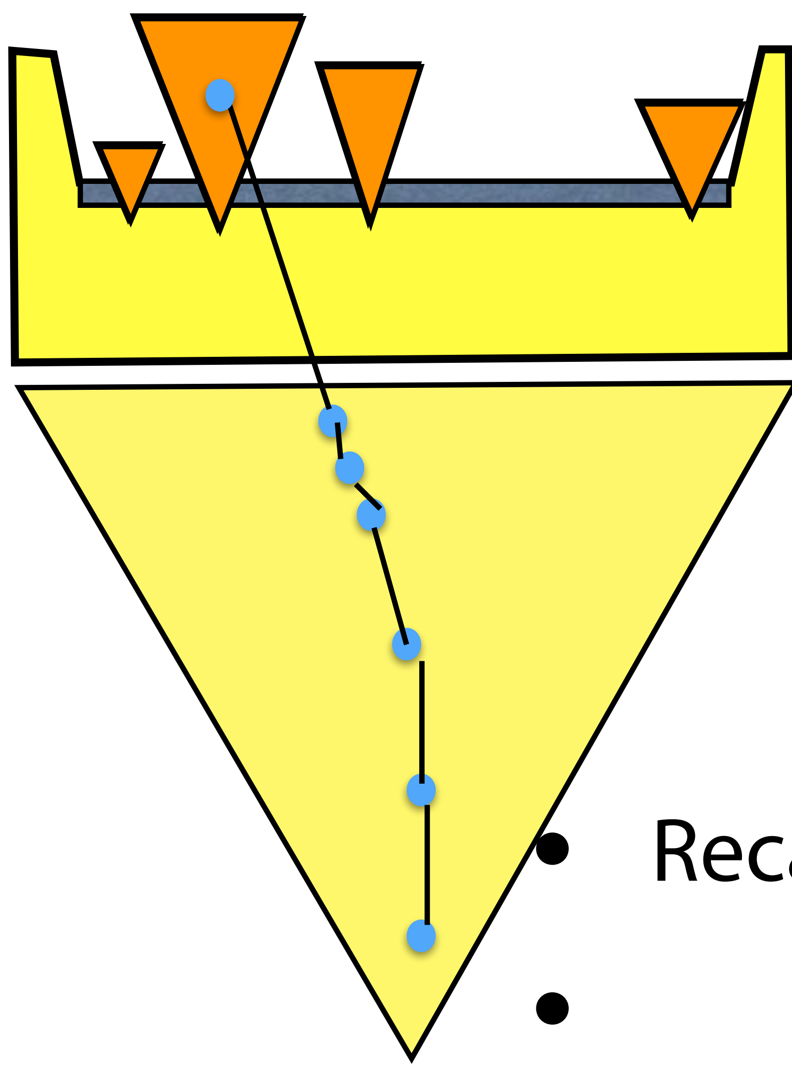
- Let X be a Hausdorff space and S be a dyadic subbase of X.
- Proposition. S is proper if and only if $\uparrow\varphi_S(x) \cap 2^\omega \subseteq D_S$ for $\forall x \in X$. That is, $\overline{S}(p) \neq \emptyset$ implies $S(p) \neq \emptyset$ for $p \in \{0,1\}^*$.
- Ex. $\varphi_S(x) = \bot\bot p$ for $p \in \{0,1\}^\omega$, then $00p, 01p, 10p, 11p \in D_S$.

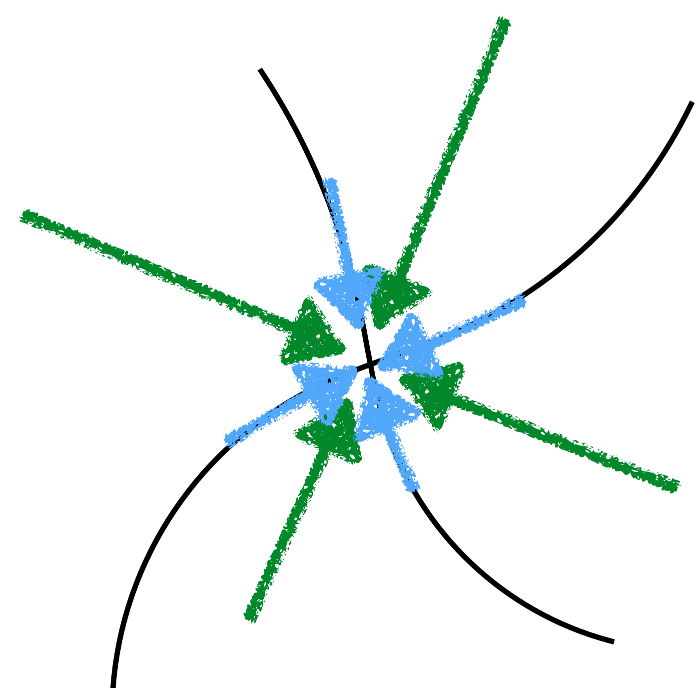




Strongly Properness and approx.

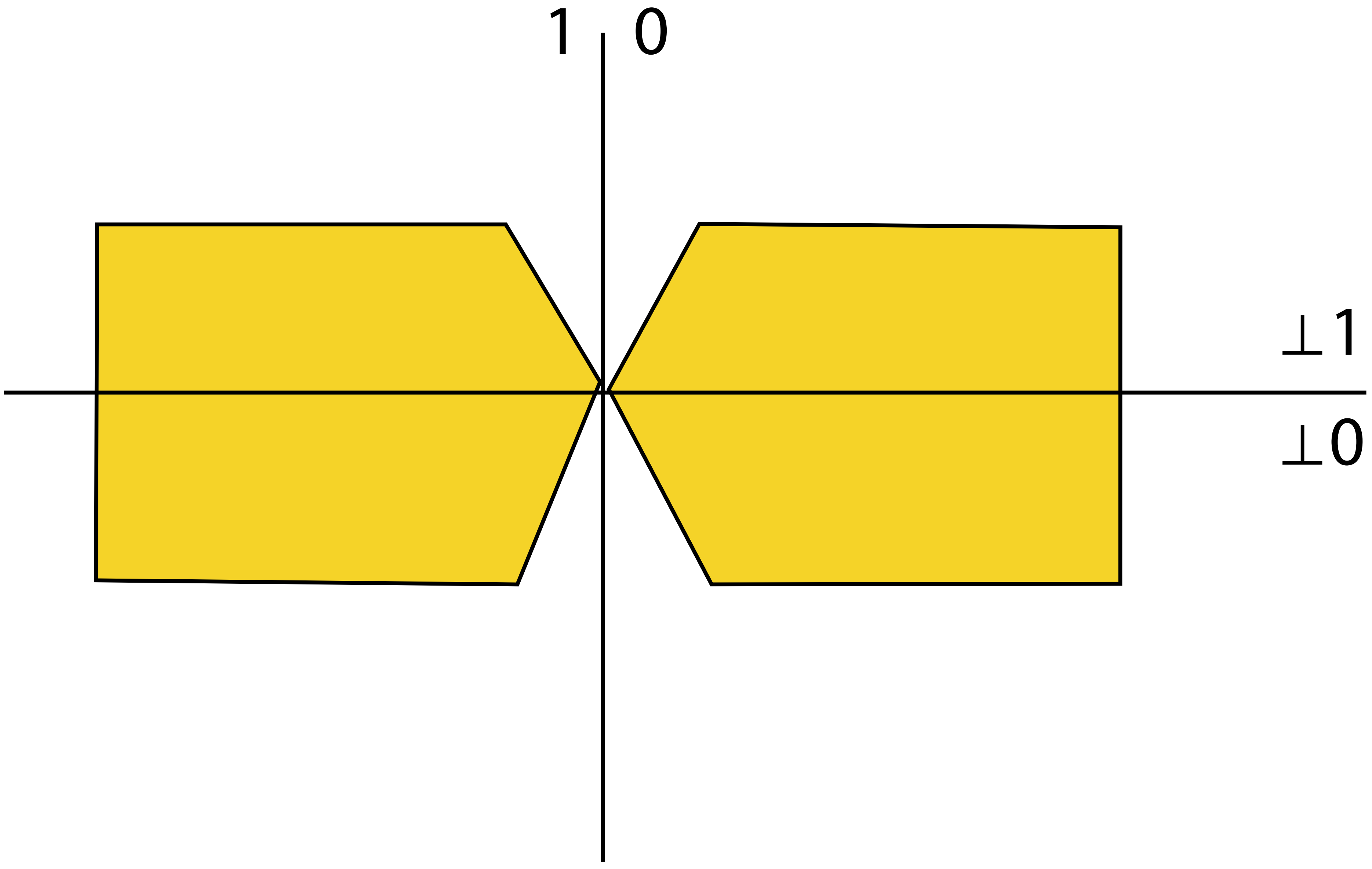
- Let X be a Hausdorff space and S be a dyadic subbase of X.
- Theorem. S is strongly proper if and only if $D_S = \hat{D}_S$. That is, $\overline{S}(p) \neq \emptyset$ implies $S(p) \neq \emptyset$ for $p \in \{0,1,\delta\}^*$.
- Ex. $\varphi_S(x) = \perp\perp p$, then $00p$, $01p$, $10p$, $11p$, $0\perp p$, $1\perp p$, $\perp 0p$,



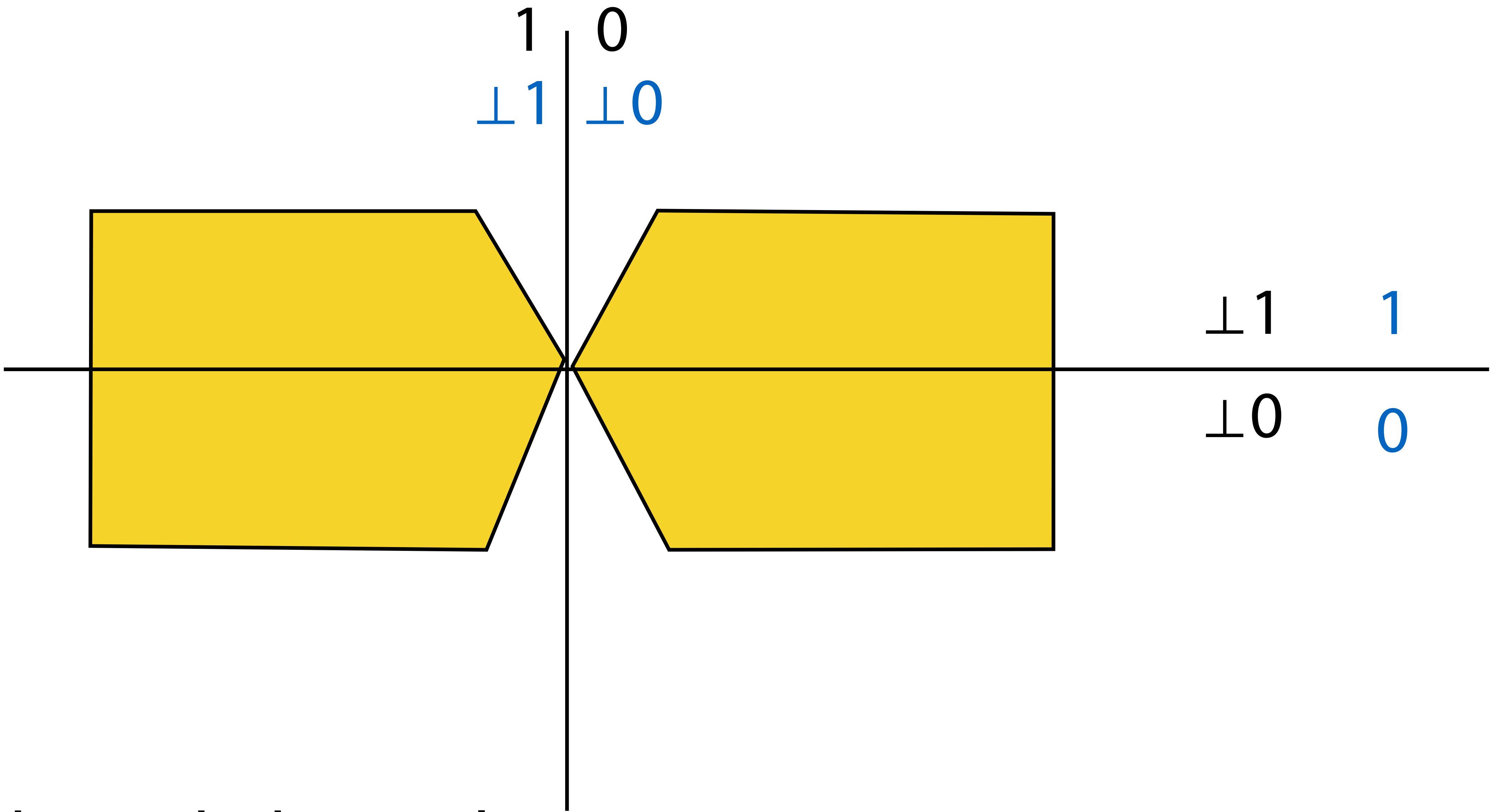


- Recall that $K_S = \{ p \in \mathbb{T}^\omega \mid S(p^\delta) \neq \emptyset\}$.
- $\hat{K}_S = \{ p \in \mathbb{T}^\omega \mid \overline{S}(p^\delta) \neq \emptyset\}$.

Example of a not strongly proper dyadic subbase.



Example of a not strongly proper dyadic subbase.



D_S is bounded complete.

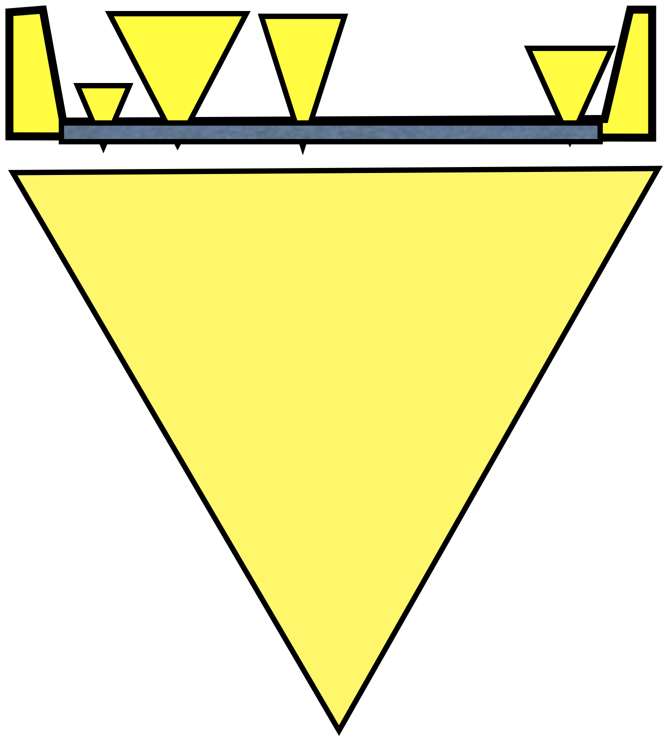
$D_{S'}$ is not bounded complete.

Bounded complete Domain.

- For a proper dyadic subbase S, D_S is not bounded complete in general. (Therefore we defined a bounded complete domain $\hat{D}_S$. in addition to D_S.)
- For a strongly proper dyadic subbase S, D_S is bounded complete. (Because $D_S = \hat{D}_S$.)
- Moreover, D_S is bounded complete not depending on the ordering of the components of S. (Because strongly properness is independent on it.)
- We also have the converse.
- **Theorem.** Let S be a proper dyadic subbase of a Hausdorff space X. S is strongly proper if and only if for all permutations $\pi : \omega \to \omega$, $D_{S\pi}$ is bounded complete.

Characterization of Regularity via strongly properness.

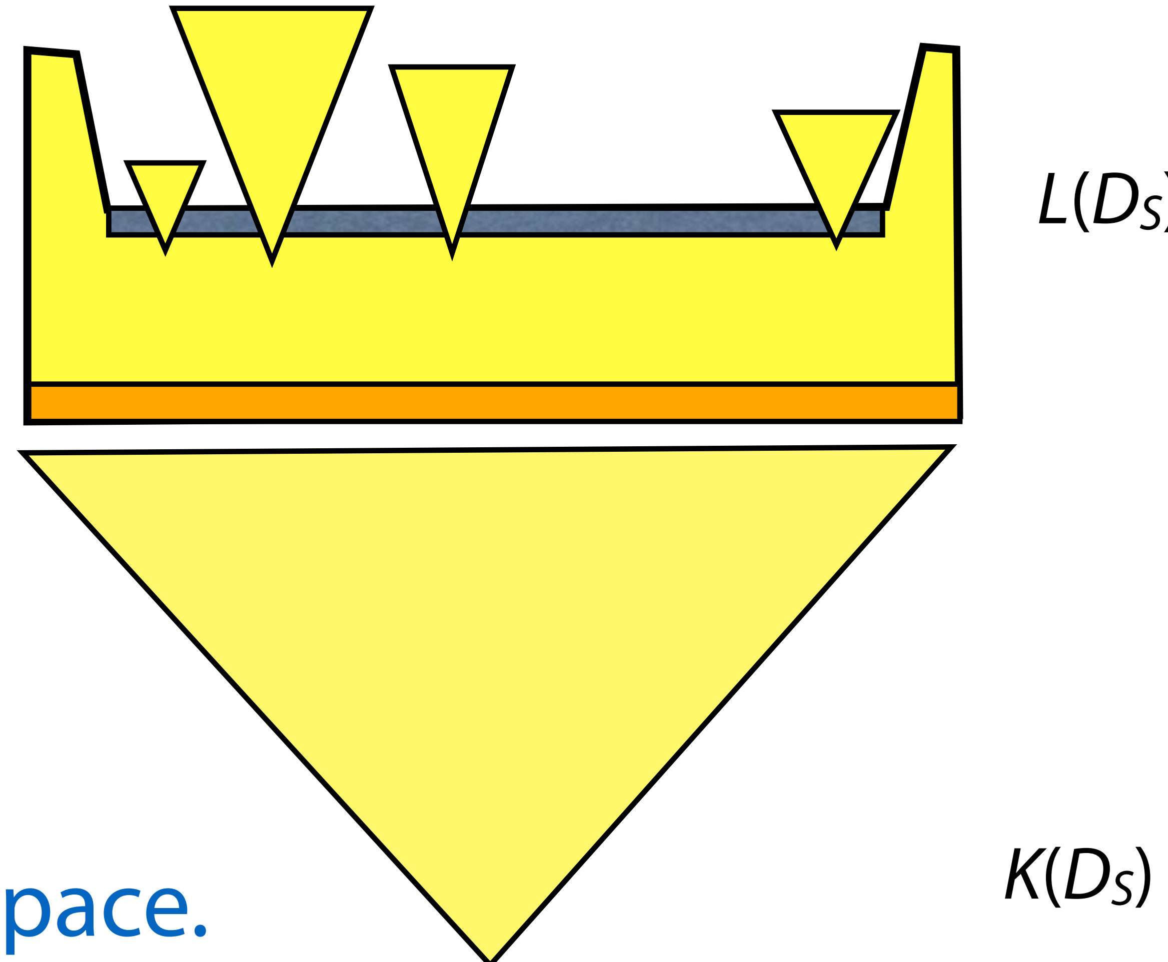
- **Theorem 5.** Suppose that S is a proper dyadic subbase. If X is regular, then $p \uparrow \varphi(x)$ in $\mathbb{T}^\omega$ implies $\varphi(x) \sqsubseteq p$ for $p \in L(D)$.
- The converse also holds for a strongly proper dyadic subbase.
- **Theorem .** Suppose that S is a proper dyadic subbase of X. X is regular if and only if $p \uparrow \varphi(x)$ in $\mathbb{T}^\omega$ implies $\varphi(x) \sqsubseteq p$ for $p \in L(D)$.



Strongly independent dyadic subbase

[Tsukamoto, T]

Theorem 4. min(L(D_S)) exists for a strongly adhesive space X.



Example of a Hausdorff Adhesive space.

For this example, min(L(D_S)) exists.

- Is there a space X and a dyadic subbase S for which min(L(D_S)) does not exist?
- More strongly, is there (X, S) for which $D_S = \mathbb{T}^\omega$?

Strongly independent subbase

- **Definition 3.** A dyadic subbase S of X is independent if $S(p) \neq \emptyset$ for every $p \in \mathbb{T}^*$.
- **Definition 3.** A dyadic subbase S of X is strongly independent if $S(p) \neq \emptyset$ for every $p \in \{0,1,\delta,\perp\}^*$.
- A space X with a strongly independent subbase is adhesive.
  - (For p, q $\in \{0,1,\perp\}^*$, let k = max(len(p), len(q)). $S(\delta^k 0) \neq \emptyset$ and any $x \in S(\delta^k 0)$ is in $\overline{S}(p) \cap \overline{S}(q)$.)
- A proper dyadic subbase S on a space X is strongly independent if and only if (1) S is independent, (2) S is strongly proper, and (3) X is adhesive.
- $D_S = \mathbb{T}^\omega$ for a strongly independent subbase.
- **Question.** Is there a Hausdorff space with a strongly independent subbase?

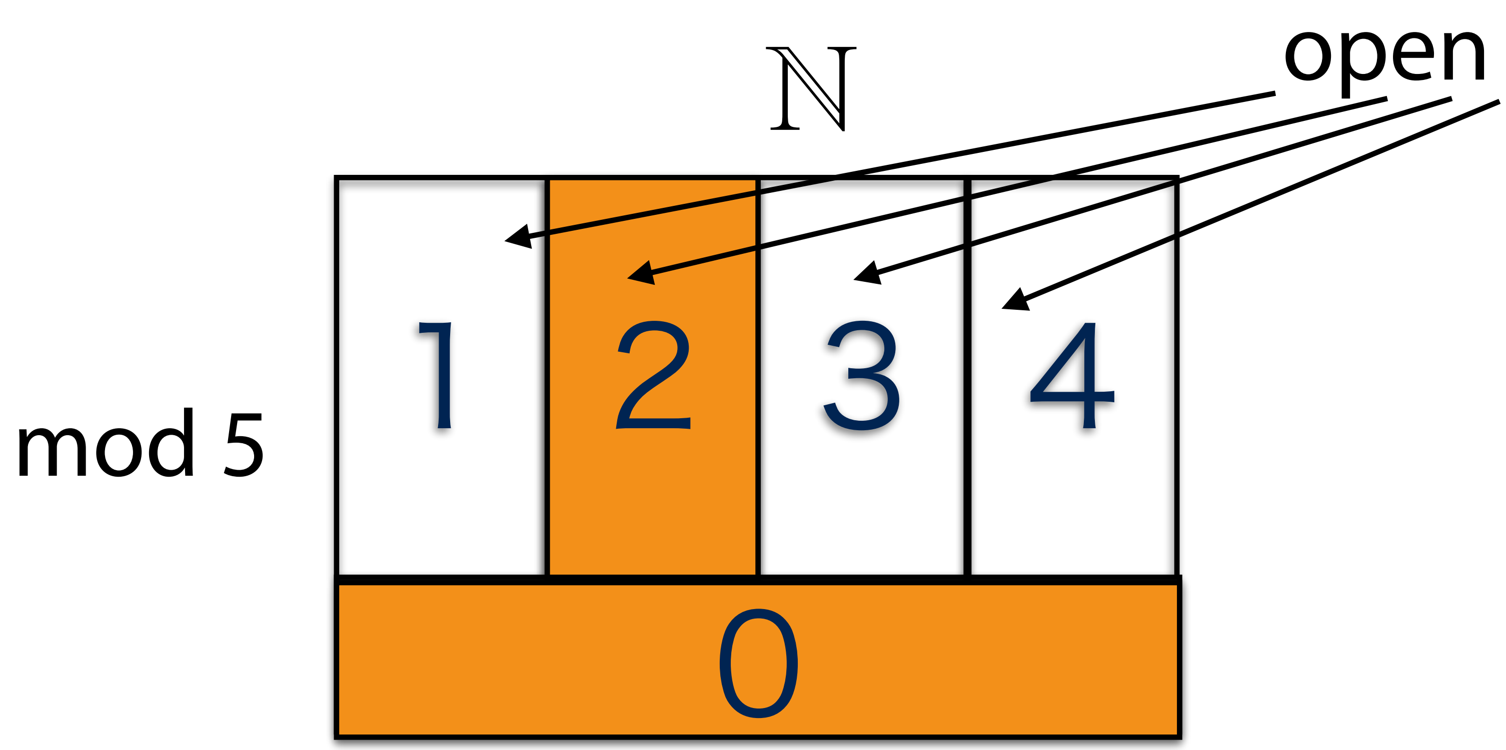
Prime integer topology

- We construct such a space as a modification of the prime integer topology $\mathcal{P}$ on $\mathbb{N} = \{1,2,3,\ldots\}$. [Steen, Seebach 1995]
- The prime integer topology $\mathcal{P}$ is generated by

$$\{U(p,r) \mid p : \text{prime number},\ 0 < r < p\}.$$

$$U(p,r) := \{n \in \mathbb{N} \mid n \equiv r \pmod p\}.$$

- It is Hausdorff and adhesive (i.e., every pair of nonempty open sets intersect in their closure) (by Chinese Reminder Theorem).



Weakened Prime integer topology

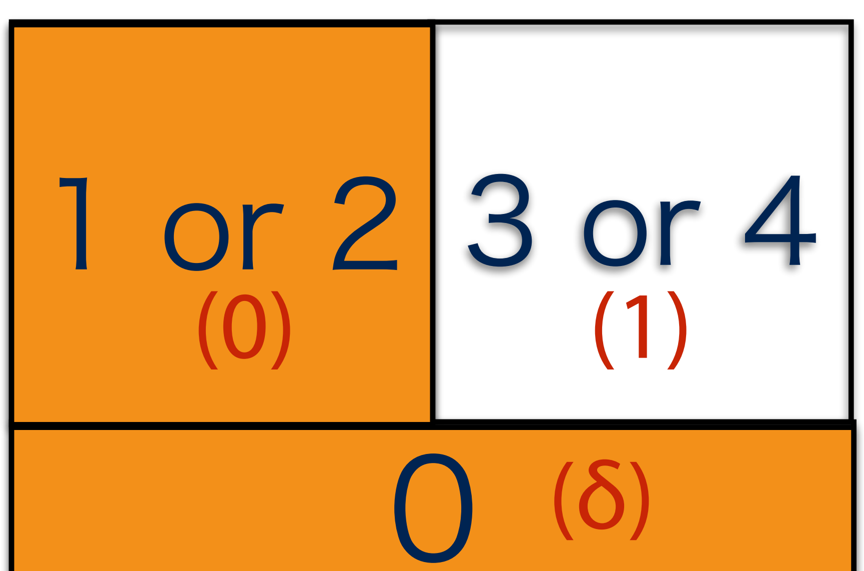
- Let $(p_n) = (3,5,7,11,\dots)$ be the sequence of odd prime numbers.
- We define a topology $\mathcal{P}_2$ on $\mathbb{N}$ generated by

$$S_{n,0} = \{m \mid m \equiv r \pmod{p_n},\ 0 < r < p_n/2\}$$
$$S_{n,1} = \{m \mid m \equiv r \pmod{p_n},\ p_n/2 < r < p_n\}$$

- $\{S_{n,0}, S_{n,1} \mid n = 0,1,2,\dots\}$ is a strongly independent dyadic subbase.
- **Theorem.** $(\mathbb{N}, \mathcal{P}_2)$ is Hausdorff.

  (Use a theorem by Sylvester 1912, Schur 1929, Erdös 1934)
- Even increment function is not continuous on $(\mathbb{N}, \mathcal{P}_2)$.

mod 5 $(= p_1)$



If $n \geq m$, then there exists a number containing a prime divisor greater than m in the sequence $n+1, n+2, \dots, n+m$.

[Erdös 1934]

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