CCC 2023

Computability, Continuity, Constructivity -
from Logic to Algorithms

Abstracts

Kyoto, 25-29 September 2023

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CCC 2023 – Programme

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1 Invited talks and tutorials

METRIC SPACES IN COMPUTABLE STRUCTURE THEORY

JOHANNA FRANKLIN

Computable structure theory has historically focused on countable algebraic structures. In this talk, I will discuss a framework for considering metric spaces in the context of computable structure theory. As an application, I will consider the question of lowness for isometric isomorphism for metric structures: which Turing degrees are so weak that, if they can compute an isometric isomorphism between two metric structures, there is already a computable isometric isomorphism between them?

Computable Polish groups

Alexander G. Melnikov

This tutorial/survey will be based on a series of papers jointly written with Nies, Montalban, Ng, Lupini, Harrison-Trainor, and Koh. The subject holds significant technical depth. It has interesting connections to other areas of computability theory, most notably computable structure theory and effective topology. Some of these applications are (perhaps) unexpected.

Recent results in constructive reverse mathematics

Takako Nemoto

Constructive reverse mathematics aims to characterize mathematical theorems with some combination of a fragment of choice axioms and logical principles. In this talk, we consider several variations of Kőnig's lemma and related principles and observe some examples of the following phenomena.

– Logical principle raises up the consistency strength of systems.
– Induction principles and choice principles act similarly.

Theoretical and practical aspects of computer arithmetic

Siegfried M. Rump

The ubiquitous IEEE-754 floating-point standard is not the only possibility to perform approximate computations. We first show that much more general arithmetic models allow better error estimates for compound operations. To that end IEEE-754 floating-point arithmetic is just a very special case. We next discuss possibilities to compute mathematically correct error bounds for the solution of numerical problems, such as the solution of nonlinear systems or ordinary differential equations. We show how to lift error estimates for single operations to error estimates for the approximate solution of a numerical problem. In particular the benefit of affine and Taylor arithmetic compared to naive interval arithmetic is discussed.

Step functions in the Weihrauch lattice

Linda Westrick

For any real A in the unit interval, let s_A denote the step function that jumps at A. We consider the partial order of the functions s_A under Weihrauch reducibility, and start to paint the picture of how these functions are arranged. Among other results, we find a collection of left-c.e. reals indexed by $\sigma \in \omega^{<\omega}$ such that the Weihrauch reducibility of their step functions coincides with the extension relation on their indices. *Joint work with Arno Pauly.*

2 Contributed talks

Almost everywhere randomness

Laurent Bienvenu, Valentino Delle Rose, Tomasz Steifer

July 31, 2023

In algorithmic randomness, the celebrated van Lambalgen theorem asserts that a pair of infinite binary sequences, or reals, (X, Y) is Martin-Löf random if and only if X is Martin-Löf random and Y is Martin-Löf random relative to X. This theorem is known to fail for other randomness notions such as Schnorr randomness, computable randomness, Demuth randomness, etc.

A corollary of van Lambalgen theorem is that if X is Martin-Löf random, then for almost all Y, X is Martin-Löf random relative to Y. Does this weaker corollary hold for other randomness notions? For example, for computable randomness, this question is equivalent to the following: is there a computably random sequence X (i.e., a sequence such that no deterministic computable martingale succeeds on X) and a *probabilistic* martingale M such that with positive probability, M defeats X? This question was implicitly left open in a paper of Buss and Minnes [2].

In this talk, based on the paper [1], we show that, surprisingly, there does exist such a sequence X. The proof uses a so-called fireworks argument. Furthermore, this result shows that one can define a natural stronger version of computable randomness: we say that X is almost everywhere (or 'a.e.') computably random if for almost all Y, X is computably random relative to Y (due to our main result, this notion is strictly stronger than computable randomness). We then prove that the separation between a.e. computable randomness and computable randomness happens exactly in the almost everywhere dominating Turing degrees.

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Characterisations of polynomial-time and -space complexity classes over the reals

Manon BLANC and Olivier BOURNEZ

Many recent works study how analogue models work, compared to classical digital ones ([6]). By "analogue" models of computation, we mean computing over continuous quantities, while "digital" models work on discrete structures, like bits. This led to a broader use of Ordinary Differential Equation (ODE) in computability theory. An interesting side-effect was the resolution of several open problems in computability and computable analysis, such as the existence of a universal ODE in [5] or various hardness problems related to dynamical systems in [9]. When talking about ODEs, especially in computer science, it is necessary to clarify what type of equations we consider: discrete or continuous. For example, in [5], it is shown that Turing machines can be simulated through continuous ODEs. We will use both formalisms.

From this point of view, the field of *implicit complexity* has also been widely studied and developed. It lies between logic and the theory of programming. For example, it allows one to write alternative characterisations of several computability and complexity classes, e.g. based on safe recursion [1], without any reference to the notion of machine. In the context of implicit complexity a standard notation for denoting the smallest class containing functions $f_1,...,f_n$ and operators $op_1,...,op_m$ is $[f_1,...,f_n;op_1,...,op_m]$.

Using these frameworks, we proved, using arguments from computable analysis, in [2] and [3] that we can algebraically characterise **P** and **PSPACE**, using discrete ODEs and more precisely the so-called Linear Length ODEs schemata.

We did so for **P**. We started by characterising the set of real numbers computable in polynomial time and of real sequences computable in polynomial time ([2]):

$$\overline{\mathbb{LDL}^{\bullet}} = [\mathbf{0},\mathbf{1},\pi_i^k,\ell(x),+,-,\times,\overline{\mathrm{cond}}(x),\frac{x}{2};\textit{composition},\textit{linear length ODE},\textit{ELim}],$$

where π_i^k is the projection of the i^e coordinate of a k-vector, $\ell(x)$ is the size of the binary representation of an integer x, $\overline{\mathrm{cond}}(x)$ valuing 1 for $x > \frac{3}{4}$ and 0 for $x < \frac{1}{4}$, and *ELim* stands for "Effective Limit", that enable us to compute limits and make approximations. Thus we have $\overline{\mathbb{LDL}^{\bullet}} \cap \mathbb{R}^{\mathbb{N}} = \mathbf{P} \cap \mathbb{R}^{\mathbb{N}}$. Then, we moved to the characterisation of functions over the reals. But we cannot apply the same techniques for the computable functions in $\mathbb{R}^{\mathbb{R}}$. The problem comes from the fact that, in our settings, we can no longer guarantee the continuity of the functions and the computations in our contructions.

Thus it was necessary to have a different approach and continuously approximate some discrete functions that are mandatory for the proofs. The following algebra characterises the set of functions over the reals computable in polynomial time in [3]:

$$\overline{\mathbb{LDL}^{\circ}} = [\mathbf{0},\mathbf{1},\pi_i^k,\ell(x),+,-,\tanh,\frac{x}{2},\frac{x}{3};\textit{composition},\textit{linear length ODE},\textit{ELim}],$$

where tanh stands for the hyperbolic tangent. This gives $\mathbb{LDL}^{\circ} \cap \mathbb{R}^{\mathbb{R}} = \mathbf{P} \cap \mathbb{R}^{\mathbb{R}}$. It turns out that we also characterised **PSPACE** using a similar algebra. For that, we introduced the notion of "robust" linear ODE, which means that the Linear ODEs are now assumed numerically stable. And we have the class $\overline{\mathbb{RLD}^{\circ}}$ characterising **PSPACE** ([3]), and $\overline{\mathbb{RLD}^{\circ}} \cap \mathbb{R}^{\mathbb{R}} = \mathbf{PSPACE} \cap \mathbb{R}^{\mathbb{R}}$.

A natural extension of the above works is to use continuous ODEs as operators instead of discrete ODEs. To do so, we are inspired by [8], [4], where the use of continuous ODEs for such purpose has been developed. The major issues in that framework are the numerical stability and the management of the approximation errors. In order to link this characterisation to the previous one, we use a a construction due to [7], often called "Branicky trick", to replace discrete settings by continuous ones, thus the need to deal with an error bound. And, to have an equivalence, we need a proper notion of numerical stability, so the ODEs are stable under discretisation. We will present some original results on these aspects.

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On the δ-decidability of decision problems for neural network questions

Olivier Bournez[1] Johanne Cohen[2] Valentin Dardilhac[3]

[1] LIX, Ecole Polytechnique, IP Paris, France
[2] Université Paris-Saclay, CNRS, LISN, 91405, Orsay, France.
[3] ENS Paris-Saclay, CNRS, LISN, Université Paris Saclay

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A extended abstract with more details is available on:
`https://www.lix.polytechnique.fr/~bournez/load/tmp/ccc-2023-more-details.pdf`

The current success of deep learning applications renewed interest in various models of computation over the reals and their associated complexity classes. This phenomenon mirrors what happened during the second wave of neural networks, which led to the development and funding of several research projects related to computational models over the reals such as the Blum Shub Smale model [3, 6, 9]. Notably, the complexity class $\exists\mathbb{R}$, popularized in [12], corresponds to the constant-free Boolean part of the class $\mathrm{NP}_{\mathbb{R}}$ from [3, 9]. The currently popular real RAM model [4] can be viewed as an extension or formalization of models such as the real Turing machine model considered in [6, 9].

One reason for $\exists\mathbb{R}$'s popularity is its natural appearance in discussions about the complexity of decision questions for neural networks. Formally, let $\mathrm{Th}_{\exists}(\mathbb{R})$ be the set of all true sentences over $\mathbb{R}$ of the form $\exists x_1, \ldots, x_n \in \mathbb{R} : \phi(x_1, \ldots, x_n)$, where ϕ is a quantifier-free Boolean formula of equalities and inequalities of polynomials with integer coefficients. Then, $\exists\mathbb{R}$ is defined as the closure of $\mathrm{Th}_{\exists}(\mathbb{R})$ under polynomial-time many-one reductions [12]. Indeed, the training of neural networks has then been shown to be $\exists\mathbb{R}$ complete in [1]. Later, the authors of [2] proved the hardness of training for fully connected two-layer neural networks, even in the basic case of fully connected two-layer ReLU neural networks with exactly two input and output dimensions. In the context of verification of neural networks, it is established that reachability is NP-complete even for the simplest neural networks using the ReLU activation function [11]. This result has been further extended in [14] to prove that verification of neural networks is $\exists\mathbb{R}$ complete for every non-linear polynomial activation function.

However, in all these statements, completeness is established at the price of assuming that activation functions are either the $\mathrm{ReLU(x)} = \max(0, \mathrm{x})$ function or, more generally, piecewise algebraic functions. This assumption allows them to demonstrate that the model can be effectively simulated by the real RAM model of computation [4, 3, 9]. Otherwise, it is not clear that we have $\exists\mathbb{R}$-membership as the function is not supported by the real RAM model of computation [4, 3, 9]. Notably, works like [10] state that computations involving analytic functions may be undecidable.

The question of what happens when the activation function is not piecewise algebraic, such as the widely used sigmoid function $\sigma(x) = \frac{1}{1+e^{-x}}$, or tanh, is however very relevant. Let us define the complexity class $\exists\mathbb{R}_{\mathcal{F}}$ as the collection of decision problems that have a polynomial-time many-one reduction to closed formulae of existential real arithmetic with additional functions from $\mathcal{F}$. Here, $\mathcal{F}$ is a set of unary real-valued functions intended to capture the allowed activation functions. In that more general case, the neural network training problem $\mathrm{NN}_{\mathcal{F}}$-TRAINING is $\exists\mathbb{R}_{\mathcal{F}}$-complete, as proved in [7]. Specifically, NN_{σ}-TRAINING is $\exists\mathbb{R}_{\exp}$-complete [7]. However, the question of whether $\exists\mathbb{R}_{\exp}$ is decidable is a well-known open problem, initially posed by Tarksi, and is known to be related to some open questions in number theory [13]. In other words, while we have some completeness results, the hardness with respect to classical complexity, and even the decidability of neural network training or verification remain open questions.

Our first contribution is to present this unified framework that encompasses the previous results. This helps to better understand the connections between various models and complexity classes. A second main contribution is to demonstrate that the potential undecidability arises from the fact that exact tests are assumed in these models and classes. To address this, we consider a more robust notion of decision, called δ-decidability, proposed by [5]. We prove this leads to the definition of more natural and provably tractable robust complexity classes.

Namely, δ-decision corresponds to the following: Fix a collection $\mathcal{F}$ of computable functions over the reals. The associated logic might be undecidable, in the general case: there might be no algorithm that decides whether a first-order sentence φ over the corresponding signature is true. However, using arguments from computable analysis, the authors of [5] prove that there always exists an algorithm that, given any first-order sentence φ, containing only bounded quantifiers, and a positive rational number δ, decides either "φ is true", or "a δ-strengthening of φ is false". The proof from [5] consists in transforming the question into the question the sign of a computable function, expressed using functions from $\mathcal{F}$, and max and min functions. Then, using arguments from computable analysis, one can determine whether such a function is positive or negative, if we authorize an arbitrary answer (possibly incorrect) when close to 0.

In relation with this approach, we introduce the concept of the non-binary decision problem: the idea is that an algorithm may answer incorrectly for some inputs, i.e. may be incorrect in some "grey" zone. As an application, we establish that while the (exact) complexity of training neural networks remains open (in particular $\mathrm{NN}_{\tanh} - \mathrm{TRAINING}$), its (bounded) non-binary decision lies between NP and PSPACE. Our results follow from the previous arguments and an improved complexity analysis. The analysis of the approach in [5] is primarily based on results from Ko in [8] about the complexity of minimisation. We improve their analysis, focusing on the case of Lipschitz functions. This is the case of the tanh or the sigmoid σ functions, proving that the obtained algorithm is polynomial with oracle in NP.

In particular, as algorithms over neural networks are often implemented using non-exact arithmetic and bounded coefficients, we believe that the approach of δ-decision instead of decision is very natural and highly relevant. Fundamentally, our approach connects previous statements, primarily derived using the real RAM model, with essential questions and concepts in the realm of computable analysis.

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FIBRED SETS WITHIN A PREDICATIVE AND CONSTRUCTIVE VARIANT OF HYLAND'S EFFECTIVE TOPOS

CIPRIANO JUNIOR CIOFFO, MARIA EMILIA MAIETTI AND SAMUELE MASCHIO

In [MM21] the authors construct a strictly predicative variant of Hyland's effective topos based on Feferman's predicative theory of non-iterative fixpoints $\widehat{\mathbf{ID}_1}$. This category **pEff** is obtained by applying the elementary quotient completion in [MR13] to a first-order hyperdoctrine $\mathbf{Prop}^r$ (representing Kleene realizability logic) defined over the category $\mathcal{C}^r$ of definable classes of $\widehat{\mathbf{ID}_1}$ and recursive functions between them. Actually, over the category $\mathcal{C}^r$ one can define four indexed categories corresponding to the four kinds of dependent types in the intensional level of the Minimalist Foundation (**MF**) [Mai09].

$$\begin{tikzcd}
\mathbf{Set}^r \arrow[r, hook] & \mathbf{Col}^r \\
\overline{\mathbf{Prop}^r_s} \arrow[u, hook] \arrow[r, hook] & \overline{\mathbf{Prop}^r} \arrow[u, hook]
\end{tikzcd}$$

The indexed categories $\overline{\mathbf{Prop}^r}$ and $\overline{\mathbf{Prop}^r_s}$ are defined as posetal reflections of **Col** and **Set**, respectively. The indexed category **Set** instead is obtained as an indexed subcategory of the indexed category of dependent collections **Col** by means of a fix-point representing a type theoretic universe of sets.

The work done in [MM21] produces a base category **pEff** and two first-order hyperdoctrines $\mathbf{pEff}_{prop}$ and $\mathbf{pEff}_{props}$ by using elementary quotient completion (here in fact coinciding with ex/lex completion). However, while a notion of dependent collection could be defined by means of the codomain fibration, a notion of dependent set was still missing there.

In this talk we enrich the fibration $\mathbf{pEff}_{col}$ of dependent collections over **pEff** with a categorical structure of dependent sets needed to interpret the whole extensional level of the Minimalist Foundation extended with Church's thesis within the predicative version of Hyland's effective topos in [MM21] using the interpretation given in [IMMS18].

We conclude by observing that, by applying the same technique, we can also build a *constructive version* of Hyland's Effective topos formalizable within $\mathbf{CZF} + \mathbf{REA}$ and capable of interpreting **MF** extended with inductive and coinductive definitions using the realizability interpretation in [MMR21, MMR22].

Such predicative and constructive versions of Hyland's effective topos can be used as universes where to extract programs from proofs in **MF** extended with inductive and coinductive definitions.

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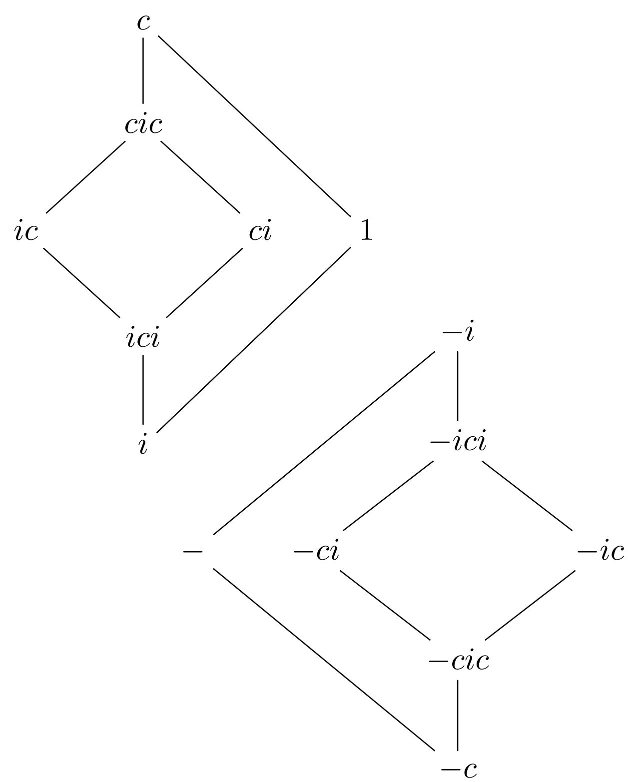
Kuratowski's problems in constructive Topology

Francesco Ciraulo*

A classical result of Kuratowski's states that there are at most 14 different combinations of the operators of interior i, closure c, and complement $-$ in a topological space; they form an ordered monoid (w.r.t. composition and pointwise ordering) whose Hasse diagram is shown below.



Special classes of spaces can be characterized by the fact that two or more of these operators coincide: there are only 6 possible cases [5], that is, 6 quotient of the Kuratowski's monoid above.

*Department of Mathematics, University of Padua

Our purpose is to find a constructive version of this theory, in the sense that we do not assume the Law of Excluded Middle (LEM).

We start with recalling a constructive account of the closure-interior problem (with no reference to the set-theoretic pseudocomplement) that we know from Giovanni Sambin. For the sake of generality, we consider closure and interior operators on an arbitrary poset L. It turns out that the monoid generated by i and c in such a framework is precisely the first piece of the above picture. So, perhaps surprisingly, this fact depends neither on LEM nor on topological notions as strictly understood.

All possible Kuratowski monoids are obtained by imposing some extra condition on those 7 operators (in the form of some inequality between them). So, in order to understand the shapes of such quotient monoids, one needs to understand the implications between the several inequalities. Since we are interested in the topological case, we equip L with an overlap relation [4, 3] in order to be able to express a compatibility condition between i and c (that comes from the fact that, in constructive topology, c is defined via adherent points). There are 26 possible inequalities/equations which, however, are not independent: their number can be reduced to 12 (up to equivalence); their mutual implications are summarized in the following diagram.

$$
\begin{tikzcd}
 & ici = i & \\
ic \leq ci & cic = c \arrow[u] & ci \leq ic \\
ic = i \arrow[u] \arrow[uur] & ci = ic \arrow[ul] \arrow[ur] & ci = i \arrow[u] \arrow[uul] \\
ci = c \arrow[u] \arrow[uur] & cic = i \arrow[ul] \arrow[u] \arrow[ur] & ic = c \arrow[u] \arrow[uul] \\
 & c = 1 \arrow[u] & \\
 & i = 1 \arrow[u] \arrow[uul] \arrow[uur] &
\end{tikzcd}
$$

Some Brouwerian counterexamples [3, 2] show that many of these implications are strict (while few other questions are still open).

Considering also the pseudocomplement in the picture greatly increases the number of possible combinations [1]. To simplify the matter we study the case in which $c = -i-$. Contrary to the classical case, we get 31 possible combinations (instead of 14) which cannot be reduced, apparently.

Finally, we investigate what the Kuratowski's problem looks like in a pointfree (and constructive) framework, that is, within the (constructive) theory of locales. Given the usual notions of closed and open sublocales, it is clear that every sublocale of a given locale has both an interior and a closure. Moreover, it is well known that the sublocales of a given locale form a co-frame; in particular, every sublocale has a co-pseudocomplement. Also, open and closed sublocales are complemented, the complement of a closed sublocale being open, and the complement of an open sublocale being closed. So the Kuratowki's problem for sublocales is related, although in a dual way, to the interior-pseudocomplement problem discussed in the previous paragraph. In view of this, we are able to lower the number of possible combinations of interior, closure and co-pseudocomplement to 21. Showing that the picture cannot be further simplified is only partially achieved and some issues still remain open.

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Advances in verified set and function calculi in Coq

Pieter Collins[1], Sewon Park[2], and Holger Thies[2]

[1]Department of Advanced Computing Sciences,
Maastricht University, The Netherlands
[2]Kyoto University, Japan

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In this talk, we present some advances toward the creation of a unified library of verified operations on sets and functions in the Coq proof assistant. The work is based on several different approaches to verified exact real computation using Coq:

- The Incone library [STT21] is a formalization of the realizability interpretation of computable analysis. Computations on higher-order objects like the reals are reduced to computations on basic types like natural or rational numbers in Coq.
- CAERN [KPT22] defines computable real numbers and operations on them abstractly via axioms. The axioms are formulated in a way that resembles modern implementations of exact real computations, in particular the Haskell library AERN. Axiomatically defined types/operations are mapped directly to corresponding types/operations in AERN, allowing to extract efficient high-level Haskell/AERN programs from proofs in the system.
- The verified rigorous numerics library of [Col23] implements interval arithmetic and polynomial models that can be used to represent function types. It is formulated in a generic way, and can be applied to any class supporting rounded arithmetic.

The main extensions include progress towards:

- A compatible set of representations, including descriptions of sets by membership predicates and the use of explicit bases.
- A consistent set of operations, including interval arithmetic on bounds and balls, division, antidifferentiation and sweeping for polynomial function models, and (pre)images of sets under functions.

The theories are constructive, and can be exported to Haskell or OCaml using Coq's code extraction features.

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Discontinuous IVPs with unique solutions

Riccardo Gozzi
École Polytechnique, Paris, France

Olivier Bournez
École Polytechnique, Paris, France

The study of ordinary differential equations (ODEs) and initial value problems (IVPs) with discontinuous right-hand terms has many applications to a wide range of problems in mechanics, electrical engineering and theory of authomatic control. Broadly speaking, discontinuous ODEs of the form $y'(t) = f(t, y)$ can be divided into two main categories [5]: one in which f is continuous in y for almost all t and one in which f is discontinuous on an arbitrary subset of its domain. In the first case, existence and unicity for solutions of the IVPs can be discussed under specific requirements on f, such as the Carathéodory conditions [1]. In the second case, the most common approach is to study the dynamic using differential inclusions of the form $y'(t) \in F(t, y)$ by identifying the correct definition of F on the set of discontinuity points. In both cases, the solution, when unique, is an absolutely continuous function y such that $y'(t) = f(t, y)$ is defined almost everywhere in an interval I.

We choose to analyze a different scenario: discontinuous IVPs for which the solution is necessarily unique and the equation $y'(t) = f(t, y)$ is defined everywhere on I. In other words, we assume existence and unicity and we focus on finding an analytical procedure to obtain such solution from f and the initial condition. In this sense, the point of view is similar to the one in [2], where it is shown that when y is unique, then it is computable if the IVP is. Nonetheless, unicity when f is discontinuous might imply noncomputability of y even when the set of discontinuity points is trivial.

We first demonstrate the difficulty by means of an example: we construct a bidimensional IVP such that, despite having computable initial condition and f computable everywhere except a straight line, has a solution that assumes a noncomputable value at a fixed integer time. This demonstrates the capability of these dynamical systems of generating highly complicated solutions even when the structure of the discontinuity points in the domain is particularly simple. Therefore, our goal is to characterize the definability of y, generalizing the result obtained in [2] for computability to a wider class of IVPs with unique solutions.

Our approach is close in spirit to that of A. Denjoy, who provided with his totalization method an extension to the Lebesgue integral in order to generalize the operation of antidifferentiation to a wider class of derivatives [3]. This perspective fits within a wider research field that explores set theoretical descriptions of the complexity of operations such as differentiation and integration. For example, [8] makes use of the notion of differentiable rank from [6] to present a precise lightface classification of differentiable functions based on how complex their derivative can be. In [4],[7] instead the authors conduct similar treatments for antidifferentiation.

More formally, the dynamical systems we are considering are IVPs of the form: given an interval $[a, b]$, a closed domain $E \subset \mathbb{R}^r$ for some $r \in \mathbb{N}$, a point $y_0 \in E$ and a function

$f : E \to E$ we have:

$$\begin{cases} y'(t) = f(y(t)) \\ y(a) = y_0 \end{cases} \tag{1}$$

for some $y : [a, b] \to \mathbb{R}^r$ with $y([a, b]) \subset E$. Under the assumption of unicity of y we show that two conditions on the right-hand term f are sufficient for obtaining the solution via transfinite recursion up to a countable ordinal. In order to do so we make use of a construction inspired by the Cantor-Bendixson analysis where the set derivative operator is replaced by the action of excluding the set of discontinuity points of f. More precisely, we define:

Definition 1 *Consider a closed domain $E \subset \mathbb{R}^r$ for some $r \in \mathbb{N}$ and a function $f : E \to \mathbb{R}^r$. Let $\{E_\alpha\}_{\alpha<\omega_1}$ and $\{f_\alpha\}_{\alpha<\omega_1}$ be transfinite sequences such that $f_\alpha = f \restriction_{E_\alpha}: E_\alpha \to \mathbb{R}^r$ defined as following: let $E_0 = E$; for all $\alpha = \beta + 1$ let $E_\alpha = \{x \in E_\beta : f_\beta$ is discontinuous in $x\}$; for all α limit ordinal let E_α be $E_\alpha = \cap_\beta E_\beta$ with $\beta < \alpha$. We call $\{E_\alpha\}_{\alpha<\omega_1}$ the sequence of f-removed sets on E.*

Studying the structure of the sequence of f-removed sets on E allows us to prove that the two following conditions on f are sufficient for obtaining the solution. The conditions are: 1) f is a function of class Baire one 2) For all closed $K \subseteq E$ the set of discontinuity points of $f \restriction_K$ is a closed set. This result is obtained with our main theorem:

Theorem 2 *Consider a closed interval, a closed domain $E \subset \mathbb{R}^r$ for some $r \in \mathbb{N}$ and a function $f : E \to E$ such that, given an initial condition, the IVP of the form of Equation 1 with right-hand term f has a unique solution on the interval. If f is a function of class Baire one such that for every closed $K \subseteq E$ the set of discontinuity points of $f \restriction_K$ is closed, then we can obtain the solution analytically via transfinite recursion up to an ordinal α such that $\alpha < \omega_1$.*

This result expresses the analogy between this context and the totalization method in [3], leaving open the possibility of defining a rank for the IVPs related to constructible ordinals in order to populate a hierarchy similar to the one presented in [8] for differentiable functions.

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A number that has an elementary contractor and no elementary sum approximation

Keita Hiroshima (Kyoto University)*

A function is called *elementary* [4, Section VIII.7] if it can be computed in time bounded by a tower of powers of constant height. A *trace function* for a irrational number α is a function $T\colon \mathbb{Q} \to \mathbb{Q}$ such that $|T(q) - \alpha| < |q - \alpha|$ for all $q \in \mathbb{Q}$. A *contractor* $F\colon \mathbb{Q} \to \mathbb{Q}$ for α is a trace function for α such that $|F(q) - F(q')| < |q - q'|$ for all q and $q' \in \mathbb{Q}$ with $q \neq q'$. The *sum approximation* for α is the strictly increasing function $S\colon \mathbb{N} \to \mathbb{N}$ such that $\alpha = a + \sum_{i=0}^{\infty} 2^{-(S(i)+1)}$ for some integer a.

Trace functions are primitive recursively [4, Section VIII.8] equivalent to contractors. That is, an irrational number, that has a primitive recursive trace function, has a primitive recursive contractor [1, Section 9]. Kristiansen [3, Theorem 5.4] showed that the sum approximation of a number with a primitive recursive trace function is primitive recursive. The author and Kawamura [2, Part 1 of Theorem 1] showed that the same implication fails for the subclass of elementary functions.

In this talk, we show that there exists an irrational number that has an elementary contractor has no elementary sum approximation, as a claim that strengthens the previous result.

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Predicative presentations of stably locally compact locales

Tatsuji Kawai

Kochi university

We give a lattice theoretic presentation of stably locally compact locales, the class of locales which includes locally compact regular locales as its subclass. Here, a stably locally compact locale is presented by a strong quasi-proximity lattice, a quasi-bounded (or 0-bounded) distributive lattice without 1 equipped with a certain idempotent relation on it. The structure generalises the corresponding (bounded) distributive lattice presentation of stably compact locales by strong proximity lattices due to Jung and Sünderhauf [2].

The characterisation of (non-strong) quasi-proximity lattice can be obtained by a straightforward generalisation of that of proximity lattice (cf. van Gool [3]). On the other hand, the characterisation of the strong variant depends on the dually of stably locally compact locales, and in particular, on the observation that every stably locally compact locale is an adjoint retract of a dually continuous maps on the ideal completion of a 0-bounded distributive lattice.[1] This observation is implicit in the characterisation of strong proximity lattice [2] but does not seem to be explicitly noted before. The fact naturally leads to a characterisation of stably locally compact locales as the Karoubi envelop of the ideal completions of 0-bounded distributive lattices and dually continuous maps, which only requires finitary reducts of the complete lattice structure.

In the setting of strong quasi-proximity lattices, we construct a coreflection from the category of locally compact regular locales and perfect maps to that of stably locally compact locales and perfect maps. The coreflection can be seen as a generalisation of the construction of non-negative real numbers as Dedekind sections. It also gives a predicative manifestation of the topos-valid but impredicative construction of a patch topology of a stably locally compact locale as the frame of Scott continuous nuclei by Martín Escardó [1]

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[1] Here, dually continuous map means a frame map between stably locally compact locales whose inverse image function is a morphism of preframes of Scott open filters on the locales which also preserves finite joins.

MORE ON THE INTUITIONISTIC BOREL HIERARCHY

TAKAYUKI KIHARA

In this talk, we study the notion of many-one reducibility for *sets with witnesses* and reanalyze the arithmetical/Borel/projective hierarchy under this new reducibility notion. In computational complexity theory, the notion of polytime many-one reducibility has been extended to sets with witnesses (a.k.a. search problems or function problems) by Levin [1] already in 1970s, but strangely enough, it seems that its computable analogue has never been studied.

We observe that the same definition as computable Levin reducibility is restored as many-one reducibility in the category of represented spaces. This perspective unexpectedly connects the notion of Levin reducibility with the study of arithmetical/Borel/projective hierarchy in intuitionistic systems.

According to Veldman [2, 3, 4], the intuitionistic Borel/projective hierarchy behaves differently from the classical hierarchy. For instance, Veldman showed that, under a certain intuitionistic system, the set FIN of all sequences which is eventually zero is not Σ^0_2-complete [2, 3], and the set $\mathsf{IF}_{\mathrm{KB}}$ of all trees which is ill-founded w.r.t. the Kleene-Brouwer ordering is not Σ^1_1-complete [4]. We see that these seemingly strange results can be clearly understood using Levin reducibility.

We divide the collection of Σ^0_2 sets into three layers under computable Levin reducibility, and show that FIN is complete in the bottom layer. We also show that the set of all bounded sequences is complete in the middle layer, and the set of all binary trees with more than one path is complete in the top layer. This means that the classical Σ^0_2-complete sets actually had three different patterns upon closer inspection, which improves on Veldman's result for intuitionistic Σ^0_2-complete sets.

A detailed analysis of Σ^1_1-complete sets is performed in a similar way. The sets of all ill-founded linear orders, all ill-founded partial orders, all partial orders which is not wqo, and those restricted to bounded width are all classically Σ^1_1-complete, but under computable Levin reducibility, they form a very complicated structure. This also improves on Veldman's result for intuitionistic Σ^1_1-complete sets.

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Arithmetical complexity for archimedean order positive fields

Margarita Korovina, Oleg Kudinov
A.P. Ershov Institute of Informatics Systems, Sobolev Inst. of Math
(Novosibirsk, Russia)
rita.korovina@gmail.com

The computability notion based on numberings is well established [4, 2] with a number of prominent results. Nevertheless applied to ordered fields it fails to capture some natural properties and there are many natural structures which are not computable in this sense. In [1] we did show that there are no computable copies neither for the field of primitive recursive real numbers nor for smaller fields of $\mathcal{E}^n$-computable real numbers, where $\mathcal{E}^n$ is a level in Grzegorczyk hierarchy, $n \geq 3$. Another prominent example is given in [3] where a computable real closed field is built, all maximal archimedean subfields of which have no computable presentations. From our point of view, in the case when the language of a structure contains the order and the structure has a negative numbering [5] it makes sense to add the following requirements: computable enumerability of strict ordering and partial computability of the inverse function. Thus, we propose to consider order positive fields which satisfy these requirements. We show that order positive fields have interesting properties in particular, order positive real closed fields are closed under taking the archimedean part. We present a general criterion when an archimedean field is order positive. Using this criterion we prove that the field generated by primitive recursive functions and the fields of $\mathcal{E}^n$-computable real numbers are order positive and also study arithmetical complexity of subclasses of order positive fields such as real closed, algebraic, computable fields.

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SOLOVAY REDUCIBILITY AND SIGNED-DIGIT REPRESENTATION

KENSHI MIYABE

Solovay reducibility in the theory of algorithmic randomness [1, Chapter 9] compares two reals in terms of approximability, which is naturally related to computability and randomness. We give some characterizations of Solovay reducibility for weakly computable reals in terms of analysis.

We say that a real $x \in \mathbb{R}$ is weakly computable if there exists a computable sequence $(x_n)_{n\in\omega}$ of rationals such that $x = \lim_{n\to\infty} x_n$ and $\sum_n |x_n - x_{n+1}| < \infty$. Zheng and Rettinger [3] extended Solovay reducibility to weakly computable reals as follows: Let α and β be weakly computable reals. We say that α is Solovay reducible to β if, there exist computable sequences $(a_n)_n$ and $(b_n)_n$ of rationals converging to α, β respectively and $c \in \omega$ such that

$$|\alpha - a_n| \leq c(|\beta - b_n| + 2^{-n})$$

for all n.

We give a characterization of Solovay reducibility for weakly computable reals in terms of the signed-digit representation [2, Chapter 7]. Roughly speaking, $\alpha \leq_S \beta$ if and only if α is computable from β with use bound $h(n) = n+O(1)$ when both of α and β are represented by the signed-digit representation. This should be compared with the fact that computable Lipschitz reducibility is similarly defined with respect to the binary representation and that Solovay reducibility and computable Lipschitz reducibility are incomparable. Solovay reducibility is a continuous concept and has good compatibility with analysis. Also note that it is well-known that the binary expansion is not suitable in the study of computability in analysis while the signed-digit representation is.

Another characterization of Solovay reducibility for weakly computable reals we give is in terms of Lipshcitz functions. Solovay reducibility for left-c.e. reals has a natural characterization in terms of Lipschitz functions. In our previous work, we considered quasi-Solovay reducibility for left-c.e. reals in terms of Hölder continuous functions. Roughly speaking, $\alpha \leq_S \beta$ if and only if there exists a Lipschitz function sandwiched by lower and upper semicomputable functions converging to α as the input goes to β.

We also studied related reducibilities and gave separation among them.

This is joint work with Masahiro Kumabe (The Open University of Japan) and Toshio Suzuki (Tokyo Metropolitan University).

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(K. Miyabe) Meiji University, Japan
Email address: research@kenshi.miyabe.name

Exact Real Arithmetic and the Efficiency of Taylor Models ⋆

Norbert Müller[1]

Abteilung Informatikwissenschaften, Universität Trier, Germany

An essential aspect of software for exact real arithmetic (ERA) is the way how it deals with the composition of functions. The resulting error propagation can be treated manually or automatic. An example for an essentially manual solution would be the explicit computation of the modulus of convergence of a sequence in order to approximate its limit, for example derived from Cauchy's integral formula. A more automatic way would be the use of some kind of interval arithmetic. The latter could be implemented, for example, using naive interval arithmetic or with more elaborate applications of intervals like in Taylor models.

In this short note we want to discuss aspects of this automatic control in a quite special application of ERA: the computation of relatively long trajectories in iterated function systems.

In order to find general applicable results, it is necessary to abstract away from concrete implementations. We will here also just consider the one-dimensional case.

- Suppose $f : \mathbb{R} \to \mathbb{R}$ is computable with $f(D) \subseteq D$ on some bounded $D \subset \mathbb{R}$.
- Consider a computable point $x_0 \in D$ and the trajectory $X := (x_n)_{n\in\mathbb{N}}$ defined by $x_{n+1} = f(x_n)$ for $n \in \mathbb{N}$.

It is easy to see that the trajectory X is uniformly computable, so there is a an algorithm on n and k approximating x_n with an error of at most 2^{-k}.

In the following we will additionally assume that both f and x_0 are computable in time $t : \mathbb{N} \to \mathbb{N}$, where t is 'regular', i.e. t has properties allowing us to simplify terms using t. Additionally, f should be Lipschitz on D, i.e. $|f(x) - f(y)| \leq 2^\ell \cdot |x - y|$ for some $\ell \in \mathbb{Q}$. Then X is uniformly computable in time $\bar{t} : \mathbb{N}^2 \to \mathbb{N}$ where $\bar{t}(n, k) = O(n \cdot t(k + n \cdot \ell))$.

The idea behind the computation of X is that for an approximation of x_n with error 2^{-m} we just need an approximation to x_{n-1} with error $\approx 2^{-m-\ell}$. The disadvantage of this simple 'manual' approach is that we need to know ℓ. Using an algorithm for f based on intervals, we can avoid the explicit knowledge of ℓ and simply repeat an interval based iteration. Depending on the quality of the algorithm, this might lead to a behaviour similar to the complexity bound $\bar{t}$, but it could also be significantly worse. We will have a detailed look at this for the case of the logistic map $L(x) = \mu \cdot x \cdot (1 - x)$ with different control parameters $\mu \in [2, 4]$ and compare naive interval arithmetic to an approach with linear Taylor

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models. We find cases where both approaches necessarily have almost the same behaviour and other cases where the Taylor models show an asymptotically better behaviour.

- Linear Taylor models (easily?) have an error behaviour corresponding to the Lyapunov exponent for IFS, while in the case of naive interval arithmetic we have to take a very careful look on the implementation to achieve this (and perhaps cannot even achieve in higher dimensional examples).
- If we search for upper bounds for the uniform complexity of the iterates in an IFS, we see a difference in the bounds we get depending on whether we have stable orbits or repelling fixed points: Stable orbits should have a uniform complexity of $O(n \cdot t(k))$, while repelling fixpoints lead to $O(n \cdot t(k+n))$. One difference between the two cases is the Lyapunov exponent (negative for stable, positive for repelling).
- In consequence, a real arithmetic based of Taylor models will be able to work with a complexity of $n \cdot t(k)$ more often than version based on naive interval arithmetic, as they often don't show the contraction which would be possible due to the negative Lyapunov exponent.
- The effects can be quite dramatic: For the Logistic map, the computation of many stable orbits can be done in linear time $O(n+k)$ with Taylor models whereas naive interval arithmetic needs more than quadratic time $O(n^2+kn)$.
- However, for repelling fixed points, Taylor models turn out to be not significantly better than naive interval arithmetic.

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ON LEAST REALIZABILITY NOTIONS CORRESPONDING TO SEMI-CLASSICAL AXIOMS IN INTUITIONISTIC ARITHMETIC

SATOSHI NAKATA

The purpose of this talk is to determine "least" realizability notions realizing semi-classical axioms in first-order intuitionistic arithmetic from a topos-theoretic point of view.

Since Kleene defined the first realizability interpretation (*Kleene realizability*) for *Heyting arithmetic* **HA**, many variants have been proposed in the literature: (1) relativization to Turing degree d (d*-realizability*), (2) *Lifschitz realizability*[5], and (3) Kreisel's *modified realizability* [4]. These realizability methods are useful to separate the hierarchy of *semi-classical axioms* over **HA**, such as Σ_n-**DNE** (double negation elimination) and Π_n-**LEM** (law of excluded middle). Akama et al. [1], for instance, show that Σ_n-**DNE** is realizable while Π_n-**LEM** is not under $\emptyset^{(n-1)}$-realizability, meaning that the former does not imply the latter. Similarly, Lifschitz realizability relativized to degree $\emptyset^{(n-1)}$ is used to separate $\Pi_n \vee \Pi_n$-**DNE** and Π_n-**LEM**.

It is known that the realizability notions (1), (2) and (3) are expressed by subtoposes of (extensions of) the effective topos $\mathcal{E}\!f\!f$ in such a way that formula φ is realizable iff φ is true in the corresponding topos [3, 6, 7, 8]. This leads to a topos-theoretic explanation for separation of semi-classical axioms.

Given an elementary topos $\mathcal{E}$, the subtoposes of $\mathcal{E}$ are in one-to-one correspondence with the *local operators* (a.k.a. *Lawvere-Tierney topologies*) in $\mathcal{E}$. A key notion here is what we call *least operator of a formula* φ, that is completely determines the subtoposes of $\mathcal{E}$ which satisfies φ. The same notion appears in Caramello's study of categorical logic [2] and in van Oosten's study of categorical realizability [7]. Above all, van Oosten shows that the local operator $j_{\mathcal{L}if}$ in $\mathcal{E}\!f\!f$ corresponding to the Lifschitz topos $\mathcal{L}if$ is the least operator of an arithmetical formula, which is equivalent to $\Pi_1 \vee \Pi_1$-**DNE** over **HA**. Our emphasis here is the following observation: given two axioms, if their least operators are different, then it automatically follows that they are separable.

In this talk, we first develop a general theory of such local operators applicable to an arbitrary topos. To achieve this, we need to restrict our treatment of local operators to *dense* ones. This restriction is not essential since all nondegenerate operators are dense in $\mathcal{E}\!f\!f$. In particular, we show the following existence theorem:

Theorem 1. *For every topos $\mathcal{E}$ with natural number object and every $n \geq 0$, Σ_n-**DNE**, Π_n-**LEM**, Σ_n-**LEM** and $\Pi_n \vee \Pi_n$-**DNE** have least dense operators in $\mathcal{E}$.*

We next consider such least dense operators of semi-classical axioms in $\mathcal{E}\!f\!f$ and determine them in terms of (generalized) Turing degrees. Our contribution is to establish a threefold relationship among realizability notions, semi-classical axioms and least dense operators in $\mathcal{E}ff$ as outlined below:

realizability notions	semi-classical axioms	least dense operators in $\mathcal{E}ff$
$\emptyset^{(n-1)}$-realizability	Σ_n-**DNE**	$j_{\emptyset^{(n-1)}}$
$\emptyset^{(n)}$-realizability	Π_n-**LEM**	$j_{\emptyset^{(n)}}$
$\emptyset^{(n)}$-realizability	Σ_n-**LEM**	$j_{\emptyset^{(n)}}$
$\emptyset^{(n-1)}$-Lifschitz	$\Pi_n \vee \Pi_n$-**DNE**	$j_{\mathcal{L}if^{(n-1)}}$

Here $\emptyset^{(n)}$-Lifschitz stands for Lifschitz realizability relativized to n-th Turing jump $\emptyset^{(n)}$ and $j_{\mathcal{L}if^{(n)}}$ for the corresponding local operator in $\mathcal{E}ff$. This correspondence automatically implies the above-mentioned results of [1]. More interestingly, our framework explains that realizability notions they used are "optimal" in the sense of minimality.

Furthermore, we obtain the following negative consequence because least dense operators give us the complete information about separation of semi-classical axioms by subtoposes of $\mathcal{E}ff$.

Theorem 2. *For any $n \geq 0$, Π_n-**LEM**, Σ_n-**LEM** and Σ_{n+1}-**DNE** are never separable by any subtopos of $\mathcal{E}ff$.*

In addition to the separation results explained above, [1] also separates Π_n-**LEM** and Σ_n-**LEM** by using *monotone modified realizability*. As a by-product of Theorem 2, we find that this realizability notion cannot be captured by a subtopos of $\mathcal{E}ff$.

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Classifying topoi in synthetic guarded domain theory

the universal property of multi-clock guarded recursion

Daniele Palombi
20squares & Cybercat Institute
Mannheim, Germany

Jonathan Sterling
Department of Computer Science
University of Cambridge
Cambridge, UK

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Synthetic guarded domain theory

Beginning with Scott and Strachey's groundbreaking investigations in 1969, the scientific study of programming semantics has been guided by the search for a topology of computation — embodied in monoidal closed categories of spaces called ***domains*** whose points can be thought of as the values of datatypes and computer programs [21–24]. The thesis of denotational semantics under Scott and Strachey is that the computational behavior of expressions in a programming language can be studied by characterizing what values they take when interpreted as continuous functions between domains; the advantage of denotational semantics over the direct/operational study of program behavior is that, unlike the latter, it is compositional and amenable to mathematical methods of reduction and abstraction.

The need to reason about increasingly complex programming languages has drawn researchers toward alternative theories of domains based on (complete, bounded, ultra-) ***metric spaces*** [1, 5, 10, 17, 20] and later simplifications and generalizations to categories of sheaves on frames with well-founded bases and the even simpler case of presheaves on well-founded posets [3], allowing for a *synthetic* and topos-theoretic approach to guarded domain theory along the lines of synthetic domain theory [13]. The core idea of ***synthetic guarded domain theory*** (**SGDT**) is to work with a topos $\mathscr{S}$ that is equipped with an endofunctor $\blacktriangleright$, together with a natural transformation $\mathsf{next} : \mathrm{id}_{\mathscr{S}} \longrightarrow \blacktriangleright$ and a "guarded" fixed point operator $-^{\dagger}$ ensuring that for each $f : \blacktriangleright A \longrightarrow A$, there exists a unique $f^{\dagger} : 1_{\mathscr{S}} \longrightarrow A$ such that $f \circ \mathsf{next}_A \circ f^{\dagger} = f^{\dagger}$. Under mild assumptions (*e.g.* left exactness of $\blacktriangleright$), it can be seen that $\blacktriangleright$ extends to an endofunctor on the fundamental fibration $\mathbf{P}_{\mathscr{S}} \longrightarrow \mathscr{S}$ and therefore gives rise to a true connective in the internal dependent type theory of $\mathscr{S}$.

Guarded recursion and coinduction

Unlike both classical and ordinary synthetic domain theory, SGDT is *effective* in the sense that it gives rise to type theories satisfying the *canonicity* property [12]; this means that SGDT is a *programming language* in addition to a semantic universe for denotational semantics. One early application of guarded recursion in this sense was to provide a more ergonomic and compositional method to write programs involving coinductive types or *final coalgebras.*

Consider the type of infinite streams $\mathsf{S}A$ as an example; this type is the final coalgebra for the endofunctor $\mathsf{F}_A X = A \times X$, and because F_A is ω-continuous we may compute $\mathsf{S}A$ as the limit of the ω-chain $\mathsf{F}_A^n 1 \cong A^n$ by Adámek's theorem. A stream producer $\alpha : X \longrightarrow \mathsf{S}A$ must therefore decompose into a cone of finite approximations $\alpha_n : X \longrightarrow A^n$ for all $n \in \omega$; in simpler terms, we must be able to compute any finite approximation of a stream. It is not difficult to imagine programming *partial* functions on streams $\beta : \mathsf{S}A \rightharpoonup \mathsf{S}B$ by general recursion; such a programming style is easily supported in languages like Haskell. But what is the appropriate linguistic construct for defining *total* functions $\beta : \mathsf{S}A \longrightarrow \mathsf{S}B$? Just as in the dual case for *inductive* data, a programming language must verify that recursive calls are justified and reject any recursive calls that would make (for instance) the projections $\beta_n : \mathsf{S}A \longrightarrow B^n$ ill-defined.

One method to ensure that recursive functions on coinductive types are total is to impose a ***syntactic guardedness*** check: every recursive call must be wrapped in a call to a constructor. Syntactic guardedness checks are employed in several type theoretic languages, such as Agda [19], Coq [11], and Idris [8, 9], but they are unfortunately very brittle and not at all conducive to compositional and modular programming with higher-order functions. A *sound* and thus more promising type-based approach to ensuring the guardedness of recursive calls arises from the later modality $\blacktriangleright$, first viewed as a programming construct by Nakano [18].[1] The idea is to approximate the coinductive type $\mathsf{S}A \cong A \times \mathsf{S}A$ by the *guarded recursive* type $\mathsf{S}_{\blacktriangleright}A \cong A \times \blacktriangleright \mathsf{S}_{\blacktriangleright}A$:

$$\mathsf{S}_{\blacktriangleright} : \mathsf{Type} \to \mathsf{Type} \qquad (::) : A \to \blacktriangleright \mathsf{S}_{\blacktriangleright}A \to \mathsf{S}_{\blacktriangleright}A$$

The guarded fixed point operator then allows recursive definitions of functions on guarded streams, with the caveat that recursive calls must appear underneath the later modality. While this semantic / type-based restriction does automatically ensure totality, it is too conservative: we cannot, for instance, define the projection functions $\mathsf{S}_{\blacktriangleright}A \to A^n$. For instance, the following attempted definition is not well-typed:

$$\begin{aligned}
&\mathsf{take} : \mathbb{N} \to \mathsf{S}_{\blacktriangleright}A \to \mathsf{List}\ A \\
&\mathsf{take}\ 0\ u = [] \\
&\mathsf{take}\ (n+1)\ (x :: u) = x :: \mathsf{take}\ n\ u
\end{aligned}$$

Clock-indexed guarded recursion

At the heart of the problem discussed above is the fact that the guarded streams only approximate the coinductive streams. The remarkable suggestion of Atkey and McBride [2] is to *define* real coinductive types in terms of their guarded approximations by adding an additional notion of ***clock*** to the language; with this combination of features, arbitrary functions on coinductive types can be defined using guarded recursion. In the setting of Atkey and McBride, the later modality ensures that functions are well-defined

[1] The calculus of Nakano [18] was subsequently connected to metric domain theory by Birkedal, Schwinghammer, and Støvring [4].

and the clocks allow the later modality to be *removed* in a type-constrained way. The language of Atkey and McBride contains a new sort of clocks k, together with a clock-indexed family of later modalities $\blacktriangleright_k$ as well as a ***clock quantifier*** $\forall k.A[k]$. In the case where A does not depend on the clock variable k, a ***clock irrelevance*** principle is asserted stating that $A = \forall k.A$; finally the canonical map $\lambda x.\Lambda k.\mathsf{next}(x[k]) : \forall k.A[k] \longrightarrow \forall k. \blacktriangleright_k A[k]$ is asserted to have an inverse force.

A clock can be thought of metaphorically as a "time stream"; thus an element of $\forall k. \blacktriangleright_k A$ exhibits an element of $\blacktriangleright_k A$ in all time streams k; thus under this metaphor, the force operation simply instantiates this family at an *earlier* time stream to obtain an element of A. With the clock-indexed later modality in hand, it is now possible to define coinductive streams in terms of their guarded approximations by setting $\mathsf{S}A \coloneqq \forall k.\mathsf{S}_{\blacktriangleright_k} A$; thus we may use guarded recursion to define the take function on coinductive streams:

$$\begin{aligned}
&\mathsf{S}_{\blacktriangleright_k} : \mathsf{Type} \to \mathsf{Type}\\
&(::) : A \to \blacktriangleright_k \mathsf{S}_{\blacktriangleright_k} A \to \mathsf{S}_{\blacktriangleright_k} A
\end{aligned}$$

$$\begin{aligned}
&\mathsf{head} : \mathsf{S}A \to A\\
&\mathsf{head}\ u = \Lambda k.\mathsf{fst}\ (\mathsf{uncons}_k\ u[k])
\end{aligned}$$

$$\begin{aligned}
&\mathsf{tail} : \mathsf{S}A \to \mathsf{S}A\\
&\mathsf{tail}\ u = \mathsf{force}(\Lambda k.\mathsf{snd}\ (\mathsf{uncons}_k\ u[k]))
\end{aligned}$$

$$\begin{aligned}
&\mathsf{take} : \mathbb{N} \to \mathsf{S}A \to \mathsf{List}\ A\\
&\mathsf{take}\ 0\ u = [\,]\\
&\mathsf{take}\ (\mathsf{suc}\ n)\ u = (\mathsf{head}\ u) :: (\mathsf{take}\ n\ (\mathsf{tail}\ u))
\end{aligned}$$

Efforts towards the generalization of Atkey and McBride's clocks to the dependently typed setting have been brought forward by Bizjak and Møgelberg [7] and Sterling and Harper [26].

Main results

Most models of single-clock SGDT are given by presheaf topoi; the models of multi-clock SGDT are also taken in presheaves, but of a different kind than the single-clock version. Thus the work of [2, 6, 7, 26] on multi-clock guarded recurisong raises two questions concerning the relationship between the existing models of single-clock and multi-clock guarded recursion:

(1) Does the passage from single-clock to multi-clock topos models have a universal property?
(2) Can the multi-clock model be rephrased as a special case of the single-clock model of SGDT?

We answer both questions positively. Each multi-clock topos can be seen to be a *partial product* or *bagtopos* [14, 15, 27] for a certain cocartesian fibration of topoi applied at a given model of single-clock guarded recursion as hinted by Sterling [25, §2.2.6]; moreover we show that the model of SGDT in the multi-clock setting is an instance of the single-clock model generalized to the *relative bounded topos theory* over a given elementary topos $\mathscr{S}$, and prove stability properties for models of SGDT under presheaves and localizations. Thus we have contributed a completely modular toolkit for negotiating the two orthogonal axes of variation in multi-clock synthetic guarded domain theory: the properties of the object of clocks, and the properties of each later modality $\blacktriangleright_k$.

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Verifying iRRAM-like implementation of exact real computation *

Sewon Park
Kyoto University, Japan
sewon@kurims.kyoto-u.ac.jp

Holger Thies
Kyoto University, Japan
thies.holger.5c@kyoto-u.ac.jp

By hiding reiterations and approximating computations from users, iRRAM has succeeded in making the reals that users observe and deal with in iRRAM appear to be exact real numbers instead of approximations [Mül00]. To adopt user interactions such as input/output functionalities into this framework, iRRAM uses a dynamically allocated *multi-value cache* to force consistency. When a user asks for some information to be printed, it is only printed once at some iteration, with a promise that the printed information will be consistent throughout the future reiterations. Necessary communications among iterations to force consistency are achieved by the multi-value cache.

Therefore, to verify iRRAM-like iteration-based exact real computation implementations, we must deal with dynamically allocated memory that is carried on through iterations. In this talk, we present our recent progress on verifying basic operations that iRRAM provides, including real arithmetic and multi-valued choices using a compositional verification calculus based on separation logic.

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Inductive and Coinductive predicates in the Minimalist Foundation

Pietro Sabelli

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The Minimalist Foundation [9, 5], for short **MF**, is a dependently typed foundational system for constructive mathematics that serves as a common core between the most relevant foundational theories such as Aczel's constructive set theory [1], Martin-Löf's type theory [10], the internal language of a topos [6], the Calculus of Construction [3], or Homotopy Type Theory [11]. Its development has been strongly driven by the desire to find a suitable foundational system for the constructive treatment of topology, known as Formal Topology [13]. In particular, to implement its powerful (co)inductive methods [4], an extension of **MF** with two new constructors formalising the inductive generation of basic covers and the coinductive generation of positivity relations was presented in [8].

In previous joint work with M. E. Maietti presented at CCC2022, we provided a topological counterpart of well-founded sets in terms of inductive suplattices introduced in Martin-Löf-Sambin's formal topology. Here we dualize this result for general (co)inductive predicates. To this purpose, we extend **MF** with context-independent coinductive predicates and we provide a topological counterpart for them in terms of Martin-Löf-Sambin's positivity relations. Furthermore, we show their equivalence with generalized inductive and coinductive definitions of constructive set theory [12] and M-types in extensional Martin-Löf's type theory and Homotopy Type Theory [2]. Our work let us conclude that the extension of **MF** with inductive suplattices and coinductive positivity relations in [8] (see also [7]) has the full strength of the extension of **MF** with general inductive and coinductive definitions.

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Exploring the abyss in Kleene's computability theory*

Sam Sanders[1]

Department of Philosophy II, RUB Bochum, Germany
sasander@me.com
https://sasander.wixsite.com/academic

Abstract

I will present some results from my ongoing project with Dag Normann on the logical and computational properties of the uncountable, namely a new discovery in Kleene's computability theory based on S1-S9 ([1,2]) and our recent equivalent lambda calculus formulation based on fixed point operators ([5]).

In a nutshell, I will present basic operations stemming from mainstream mathematics that are computable in Kleene's quantifier $\exists^2$, while *slight* mathematical variations are computable in Kleene's quantifier $\exists^3$ but not in weaker oracles ([7]). Since the difference between $\exists^2$ and $\exists^3$ is considerable, we say that these two classes are *separated by an abyss.*

Kleene's computability theory via fixed points

Kleene's S1-S9 computation schemes provide a formal framework for the notion

the object X is computable in the object Y

where X, Y are objects of any finite type. In case X, Y are real numbers, S1-S9-computability reduces to Turing computability, i.e. the former is an extension of the latter. While S1-S8 only formalise a basic form of primitive recursion, the schema S9 is rather ad hoc in nature, hard-coding as it does the recursion theorem for S1-S9-computability theory. By contrast, the recursion theorem for Turing machines is derived from first principles in [8].

In light of the previous, it is a natural question whether there is a (more) conceptually pleasing formulation of S1-S9, which is one of the topics of [5]. In particular, it is shown that S1-S9 can be equivalently captured by the computational model consisting of the following extended lambda calculus:

- Kleene's schemas S1-S8 capturing primitive recursion,

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- λ-abstraction for all finite types,
- the least fixed point operators μ^σ for all finite types σ.

The operator in the final item is such that $\mu^\sigma x.s(x)$ is the least fixed point of the $\sigma \to \sigma$-mapping $\lambda x.s(x)$ which is assumed *monotone* as follows:

$x^\sigma \preceq_\sigma y^\sigma$ means that the graph of y is included in the graph of x and the mapping $\lambda x.s(x)$ is *monotone* if $(\forall x^\sigma, y^\sigma)(x \preceq_\sigma y \to s(x) \preceq_\sigma s(y))$.

We will introduce this new model and briefly sketch its advantages over S1-S9.

The abyss in Kleene computability theory

In a nutshell, we identify a number of basic operations (e.g. finding suprema or points of continuity) stemming from mainstream mathematics that are 'close to computable' in that they are computable from $\exists^2$ (or weaker oracles). Perhaps surprisingly, *slight* variations of these basic operations (like extending the function class) are computable in $\exists^3$ but not in weaker oracles. The following examples give the reader an idea of this abyss.

The functional $\exists^2$ *can compute* $\sup_{x\in[p,q]} f(x)$ *for* $f : [0,1] \to \mathbb{R}$ *which is either cadlag, Baire 1, or quasi-continuous.*

The functional $\exists^3$ *can compute* $\sup_{x\in[p,q]} f(x)$ *for* $f : [0,1] \to \mathbb{R}$ *which is either regulated, Baire 2, or cliquish, but no weaker oracle suffices.*

Mathematically speaking, the operations on the first and second line are close as respectively cadlag and regulated, Baire 1 and Baire 2, and quasi-continuous and cliquish, are closely related function classes in real analysis. We discuss similar results for other function classes and operations, like computing points of continuity and computing representations of the set of continuity points.

An important observation concerning the abyss is that the operations computable in $\exists^2$ involve function classes for which $f(x)$ for any $\in [0,1]$ can always be approximated using $f(q)$ for $q \in [0,1] \cap \mathbb{Q}$. The other side of the abyss lacks this approximation device.

Finally, we explain how the logical and computational properties of the uncountability of the reals as follows:

$$\text{there is no injection from } \mathbb{R} \text{ to } \mathbb{N}. \tag{NIN}$$

plays a central role in our project ([3, 4, 6]).

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The Compact Hyperspace Monad: a Constructive Approach*

Dieter Spreen
Department of Mathematics, University of Siegen
57068 Siegen, Germany

1 Extended abstract

As shown by Michael in his seminal 1951 paper [5], the hyperspace $\mathcal{K}(X)$ of non-empty compact subsets of a compact Hausdorff space X is compact Hausdorff again with respect to the Vietoris topology. Moreover, the compact union of compact sets is compact as well. Since the construction $\mathcal{K}$ is functorial, this gives rise to a monad $(\mathcal{K}, \eta, U)$, where η_X maps points $x \in X$ to the singleton $\{x\}$, and U_X compact sets $\mathbb{K} \in \mathcal{K}(\mathcal{K}(X))$ to their union $\bigcup \mathbb{K}$.

In [2] an approach was presented to deal with compact point sets in a constructive way. It allows to extract the computational content from proofs of computationally meaningful statements. The approach is computationally equivalent to Weihrauch's Type-Two Theory of Effectivity [7]. However, contrary to this approach it is purely logical and representation-free. Representations of the computed objects are obtained via a realisability interpretation of the logic, i.e., intuitionistic fixed point logic (IFP) extended by inductive and co-inductive definitions (cf. [3]). Note that although IFP is based on intuitionistic logic a fair amount of classical logic is available: Any true disjunction-free formula can be used as axiom.

The approach in [2] is based on Iterated Functions Systems (IFS), i.e., spaces X coming with a finite set D of continuous self-maps $d\colon X \to X$. Assume that X is a compact metric space and all $d \in D$ are contracting, say with contraction factor $q \in \mathbb{Q}_+$. (X, D) is *covering* if $X = \bigcup \{\, \mathrm{range}(d) \mid d \in D \,\}$. The structure (X, D) is called a *digit space* in this case and the $d \in D$ *digits*. Such spaces can be characterised co-inductively.

Let $\mathbb{C}_X$ be the co-inductively largest subset of X such that

$$x \in \mathbb{C}_X \to \bigvee_{d \in D} (\exists y \in \mathbb{C}_X)\, x = d(y). \tag{1}$$

Then $X = \mathbb{C}_X$. Since X may be a classically defined object, this equality is true only classically. In the constructive approach one works with $\mathbb{C}_X$ instead of X, where in (1) the disjunction has to be interpreted constructively. Thus, given $x_0 \in X$, one effectively obtains $d_0 \in D$ so that for some $x_1 \in X$, $x_0 = d_0(x_1)$. By iterating this procedure, one receives a sequence $(d_i)_{i \in \mathbb{N}}$ of self-maps in D that serves as representation of x_0,

Let $d_1, \ldots, d_r \in D$ be pairwise distinct and define $[d_1, \ldots . d_r]\colon \mathcal{K}(X)^r \to \mathcal{K}(X)$ by

$$[d_1, \ldots, d_r](K_1, \ldots, K_r) \stackrel{\mathrm{Def}}{=} \bigcup_{\nu=1}^{r} \mathcal{K}(d_\nu)(K_\nu),$$

where $\mathcal{K}(d)(K) \stackrel{\mathrm{Def}}{=} d[K]$. Let $\mathcal{K}(D)$ be the finite set of all such maps $[d_1, \ldots, d_r]$. Then $(\mathcal{K}(X), \mathcal{K}(D))$ is an IFS in which the self-maps are no longer required to be unary. We use the notion *extended* IFS in this case [6]. As is well known $\mathcal{K}(X)$ is a compact metric space with respect to the Hausdorff metric. The metric topology is equivalent to the Vietoris topology. If

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$\mathcal{K}(X)^r$ is endowed with the maximum metric the maps $[d_1, \ldots, d_r]$ are also contracting with contraction factor q. Moreover, $(\mathcal{K}(X), \mathcal{K}(D))$ is covering [6]. Therefore,

$$\mathcal{K}(X) = \mathbb{C}_{\mathcal{K}(X)}.$$

One would like to iterate the process described above to obtain co-inductive characterisations of the higher compact hyperspaces $\mathcal{K}^n(X)$ with $n > 1$. Let us consider the case $n = 2$. The digit maps would then be of the form

$$[\vec{d_1}, \ldots, \vec{d_k}]. \tag{2}$$

with

$$\vec{d_i} = [d_1^{(i)}, \ldots, d_{r_i}^{(i)}],$$

for $i = 1, \ldots, k$. Let $\mathcal{K}^2(D)$ be the finite set of all such maps.

The type of the map in Line (2) is

$$\mathcal{K}(\mathcal{K}(X)^{r_1}) \times \cdots \times \mathcal{K}(\mathcal{K}(X)^{r_k}) \to \mathcal{K}^2(X). \tag{3}$$

Thus, these maps are no longer self-maps of $\mathcal{K}^2(X)$ and hence $(\mathcal{K}^2(X), \mathcal{K}^2(D))$ not an extended IFS. For simplicity assume that $r_1 = \cdots = r_k \stackrel{\mathrm{Def}}{=} r$, then the type in Line (3) becomes

$$\mathcal{K}(\mathcal{K}(X)^r)^k \to \mathcal{K}^2(X).$$

In this paper we will inductively construct spaces $\mathcal{K}_i^{(2)}(X)$ and sets of maps $\mathcal{K}_i^{(2)}(D)$ with $\mathcal{K}_0^{(2)}(X) = \mathcal{K}^2(X)$ and $\mathcal{K}_0^{(2)}(D) = \mathcal{K}^2(D)$ such that

$$\mathcal{K}_{i+1}^{(2)}(X) \xrightarrow{\mathcal{K}_i^{(2)}(D)} \mathcal{K}_i^{(2)}(X),$$

where for spaces Y_i and sets F_i of maps $f\colon Y_{i+1} \to Y_i$,

$$Y_{i+1} \xrightarrow{F_i} Y_i$$

means that for each $i \in \mathbb{N}$ and every $y \in Y_i$ there are $f \in F_i$ and $z \in Y_{i+1}$ with $y = f(z)$. Analogously, we will proceed for all $n > 2$.

For each such co-chain $(Y_{i+1} \xrightarrow{F_i} Y_i)_{i\in\mathbb{N}}$ let $\mathfrak{Y} = \sum_{i\in\mathbb{N}} Y_i$ be the topological sum of the family of spaces $(Y_i)_{i\in\mathbb{N}}$ and $\mathfrak{F}$ be the disjoint union of the F_i. Then $(\mathfrak{Y}, \mathfrak{F})$ is an infinite extended IFS (cf. [4]). The maps in $\mathfrak{F}$ operate only locally on the components. If the Y_i are metric spaces, $\mathfrak{Y}$ carries a canonical ∞-metric. IFS of this kind will be the objects of the category underlying the monad.

As shown in [6], Berger's [1] co-inductive inductive characterisation of the uniformly continuous functions on the unit interval can be transferred to the case of extended IFS. A further generalisation to the case of the directed sums just described is presented in the present paper. The function classes obtained from the co-inductive inductive definitions will constitute the hom-sets of the category.

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