Deterministic vs probabilistic gamblers

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joint work with

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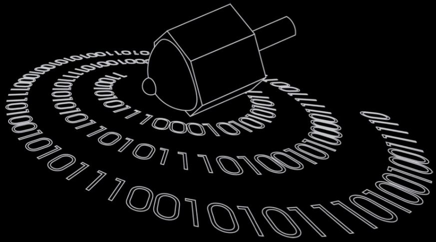
1. Algorithmic randomness



Random infinite sequences

Algorithmic randomness is the theory whose goal is to give a satisfactory meaning to the notion of random individual objects (**in this talk: infinite binary sequences**).

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Not random!

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Looks random!

Kolmogorov complexity

Definition

Let x be a **finite** binary string. We call **Kolmogorov complexity of** x the quantity $K(x)$ defined by

$$K(x) = \text{the shortest computer program (in binary) that generates } x$$

Of course formalizing all this needs some care. A sound formalism can be achieved using Turing machines (and programs=inputs of universal Turing machine).

Modulo this formalization, we can now say that a string x is *random* with randomness deficiency at most c if $K(x) \geq |x| - c$.

Martin-Löf randomness

Definition

An infinite sequence X is Martin-Löf **random** if and only if for some c and for all n,

$$K(X_0 ... X_n) > n - c \quad ^*$$

* for this definition we need to consider a variant of Kolmogorov complexity, the so-called prefix-free Kolmogorov complexity

Unpredictability

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- The player tries to predict the values of these cards in order. Starting with a capital of $\$1$, she bets at each move some amount of her money on the value of the next card.
- The player wins the infinite game if her capital tends to infinity throughout the game.

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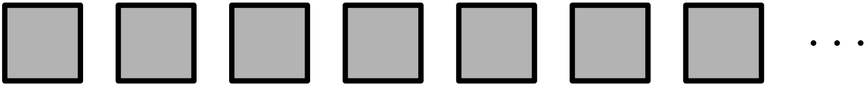

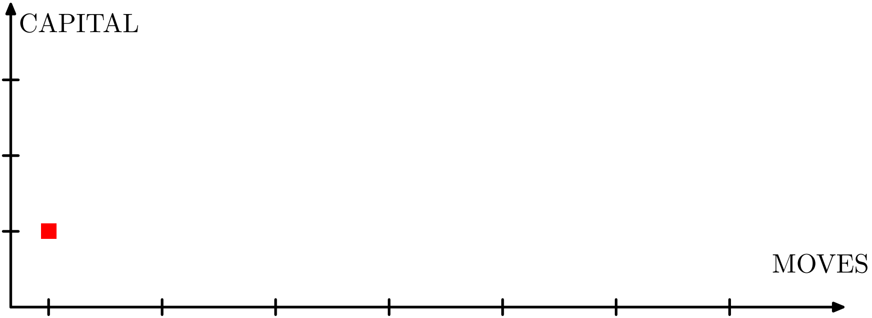

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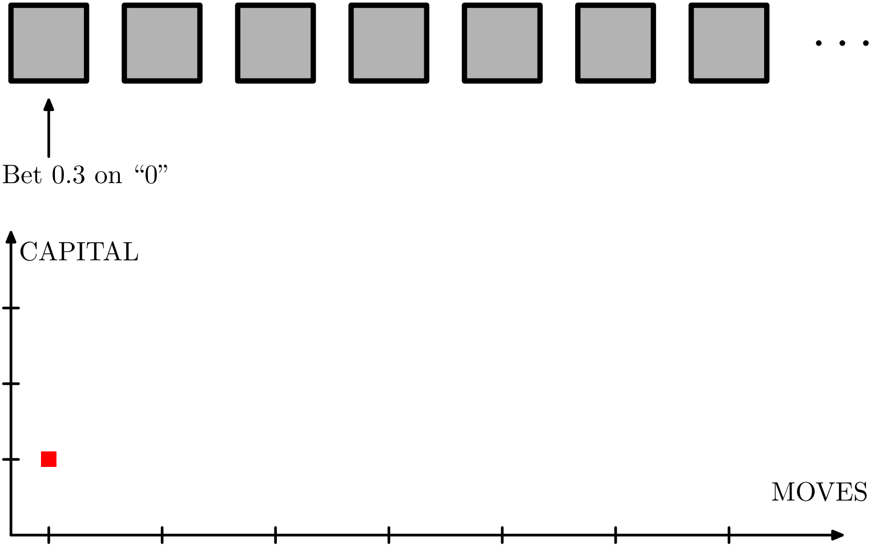

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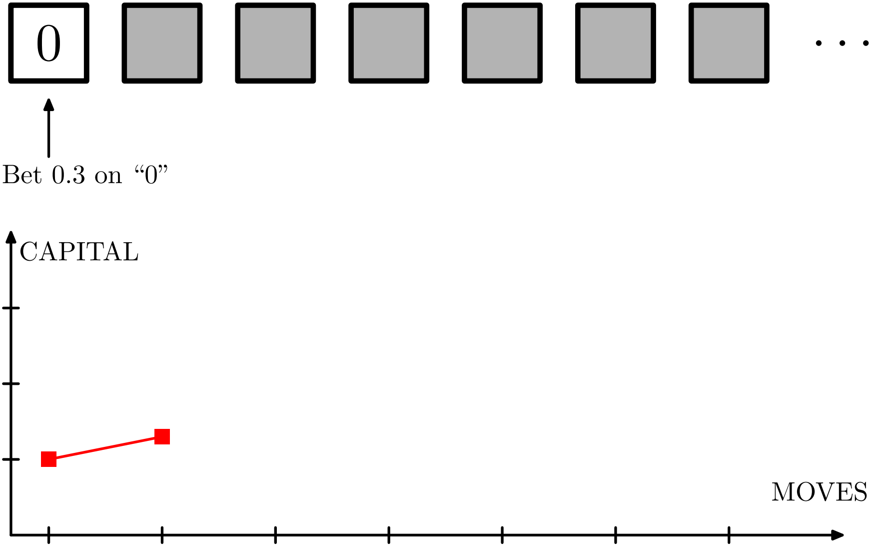

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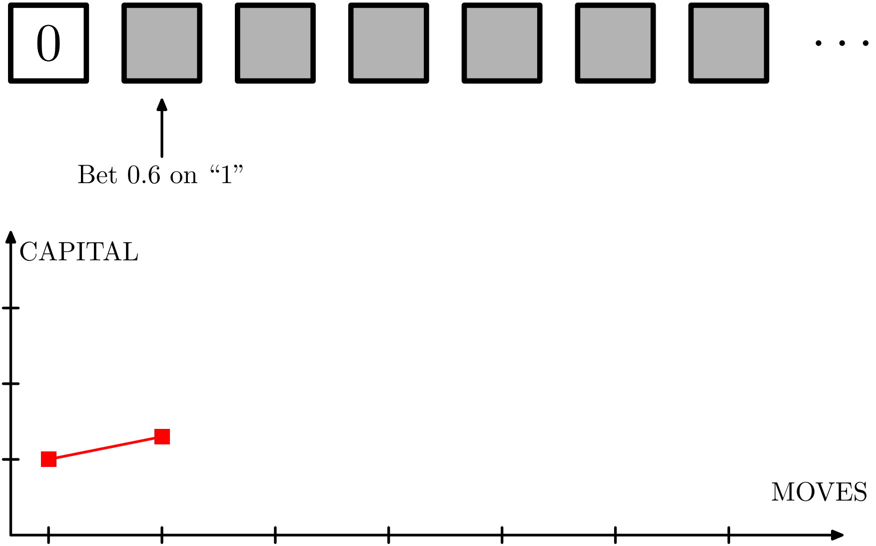

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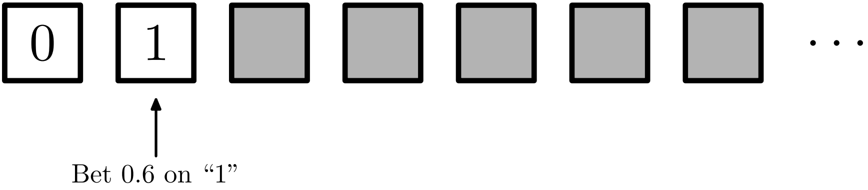

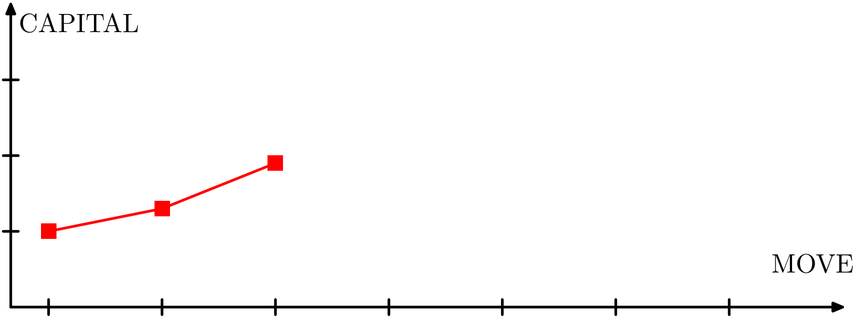

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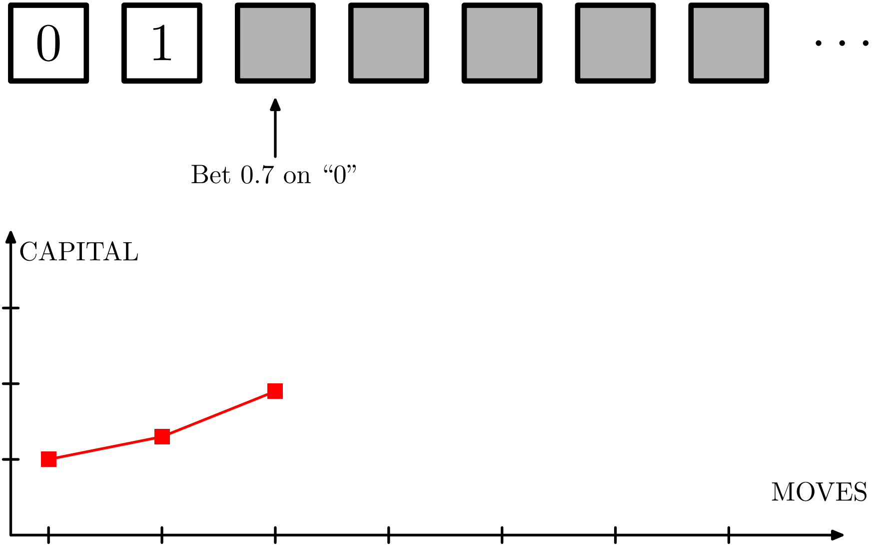

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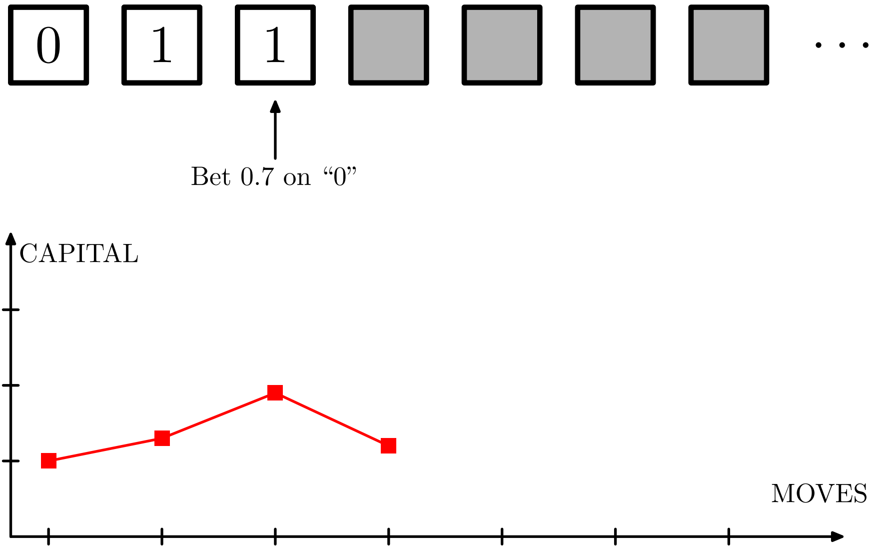

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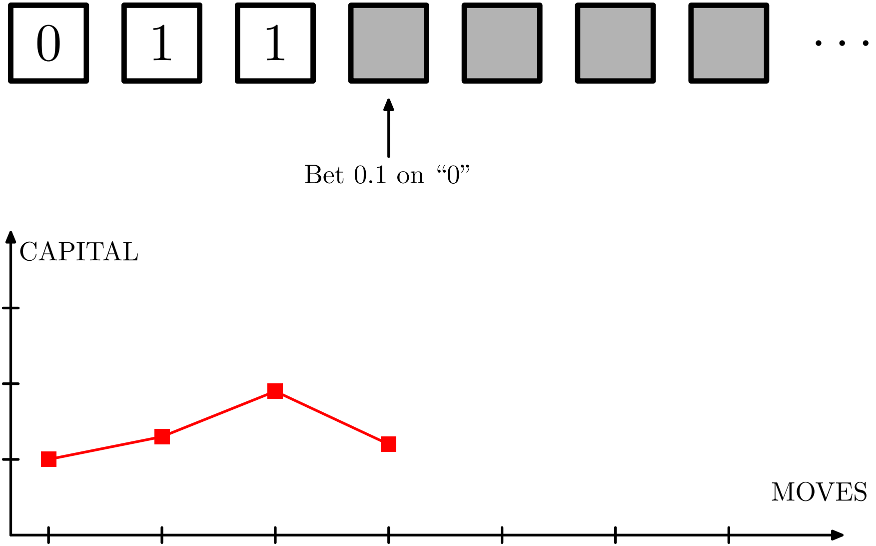

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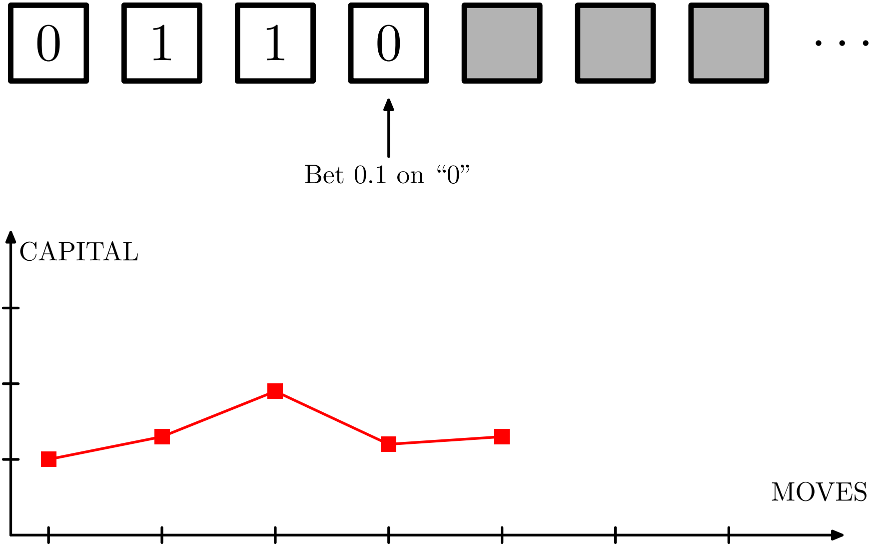

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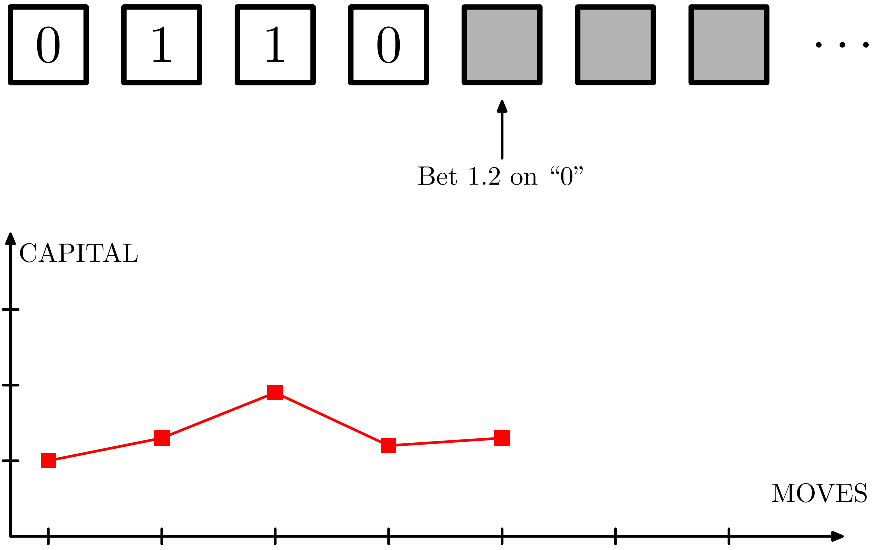

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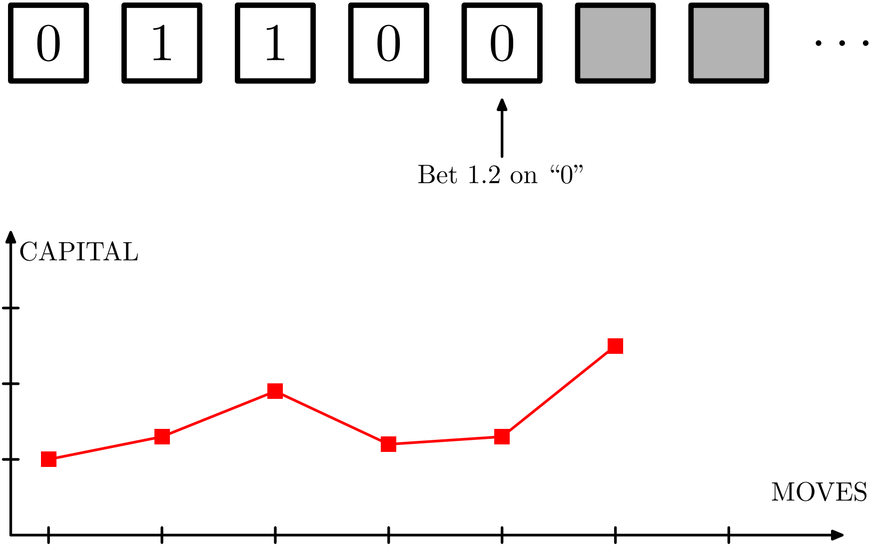

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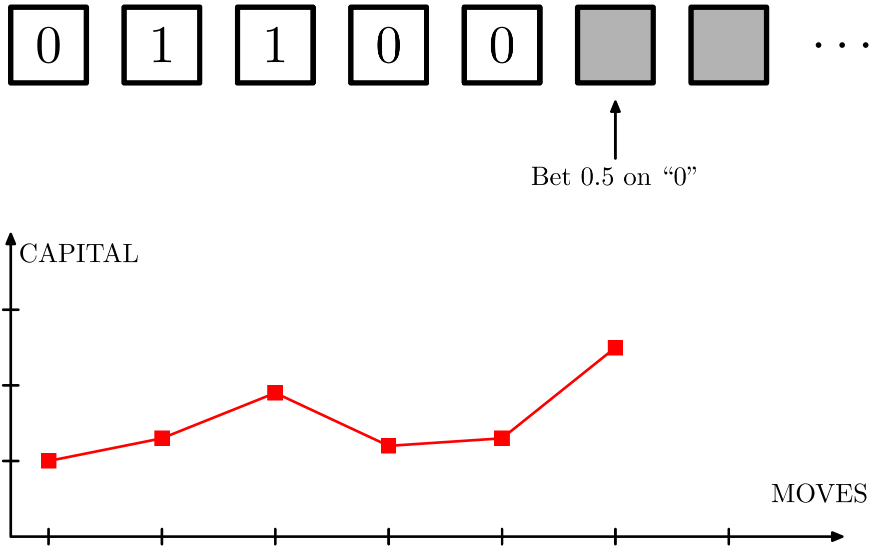

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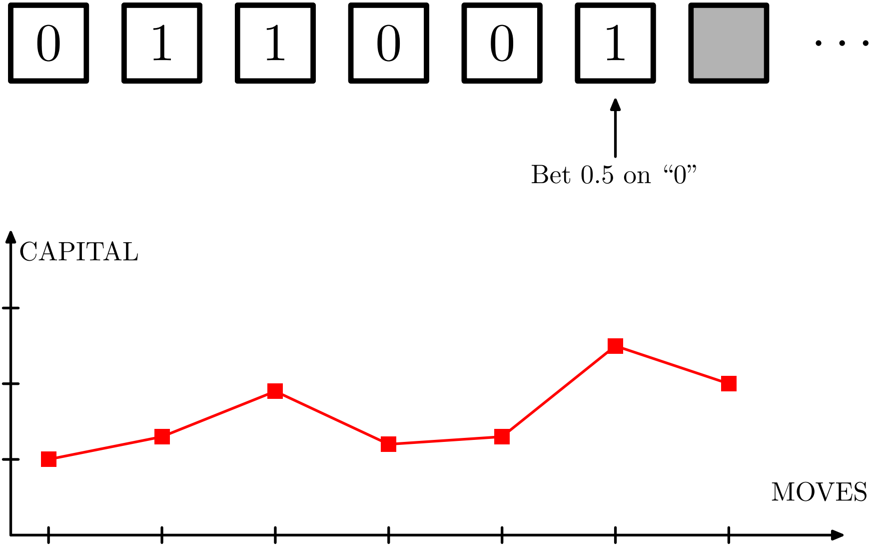

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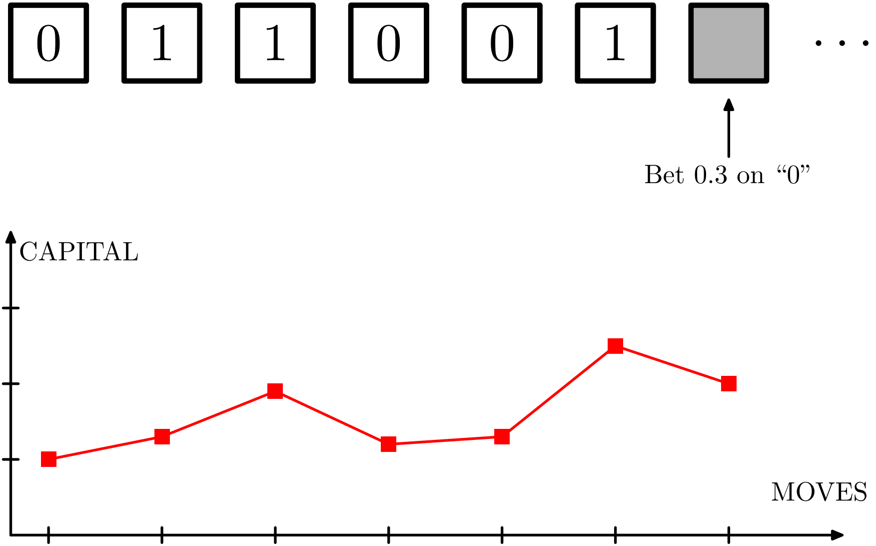

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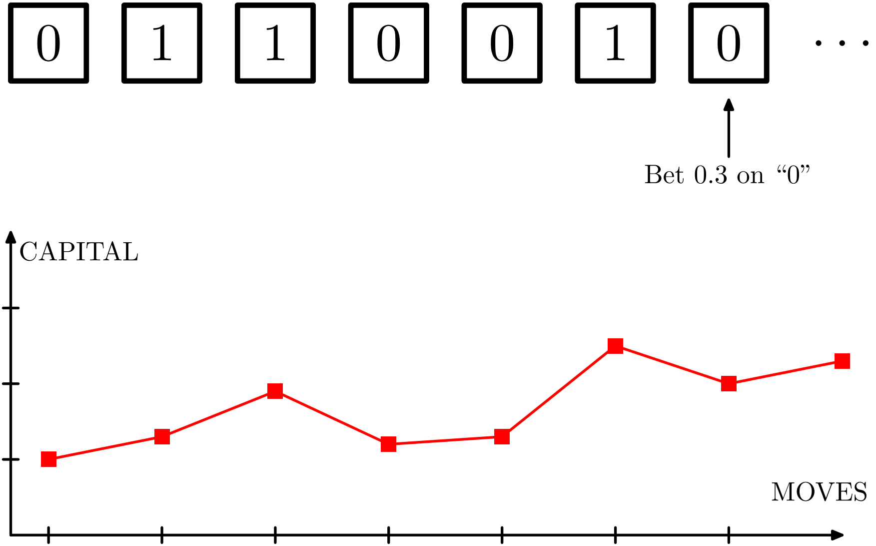

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Nonetheless this is a natural notion worth studying.

2. Randomizing things

Randomized betting strategies

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Do we get a strictly stronger randomness notion?

If so, how does it relate to Martin-Löf randomness?

A restricted model

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Theorem
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But this leaves open the general question, where gamblers can take even more risk by allowing some random choices to trap them in an infinite loop (more risk, more potential reward!).

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The condition that the strategy must be defined with probability 1 ensures that μ is nice, namely, it is a *computable* probability measure.

Thus, the average $\mathbb{E}_\mu(S)$ is a computable strategy, therefore when 'playing' on an unpredictable X, $\mathbb{E}_\mu(S)$ must remain bounded, thus

$$\mu\{S \mid S \text{ remains bounded against } X\} = 1$$

Our main result

If we remove the Buss-Minnes assumption, the situation changes dramatically:

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Take-home message: probabilistic gamblers are strictly better than deterministic gamblers!

A vague idea of the proof

To prove this result, we use the notion of (computability-theoretic) Cohen genericity, namely the class of 1-generics. We say that an object X is 1-generic if for every effectively open set $\mathcal{U}$, X is either in $\mathcal{U}$ or in $Int(\mathcal{U}^c)$.

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1-genericity is usually defined for infinite binary sequences but is a topological notion, so it can be extended to the topological space of strategies. The topology: we view the space of strategies as a (closed) subspace of the space of functions $\{f : 2^{<\omega} \to \mathbb{R}\}$ with the product topology.

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This algorithm induces a (twisted!!) measure ξ on the space of martingales, with $\xi(1GEN) > 0.9999$.

So now we can build an X so close to computable that

$$\xi\left\{S \mid S \text{ defeats } X\right\} > 0.9999$$

... which means that indirectly, we have built a probabilistic martingale which succeeds against X with high probability!

Looking forward

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Theorem
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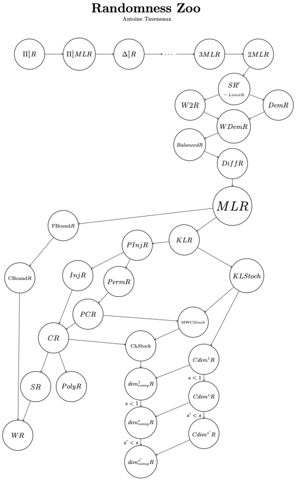
So we have a truly new notion of randomness, which deserves further study.
Also, one now needs to take a new look are the randomness zoo...

Looking forward

... and see which ones have an non-trivial ‘a.e.’ counterpart!



References

[1] Sam Buss, Mia Minnes, ”Probabilistic Algorithmic Randomness”, Journal of Symbolic Logic 78(2), 2013.

[2] Laurent Bienvenu, Valentino Delle Rose, Tomasz Steifer, ”Probabilistic vs deterministic gamblers”, STACS 2022 and arxiv.org/abs/2112.04460

Thank You !