Kuratowski's problems in constructive topology

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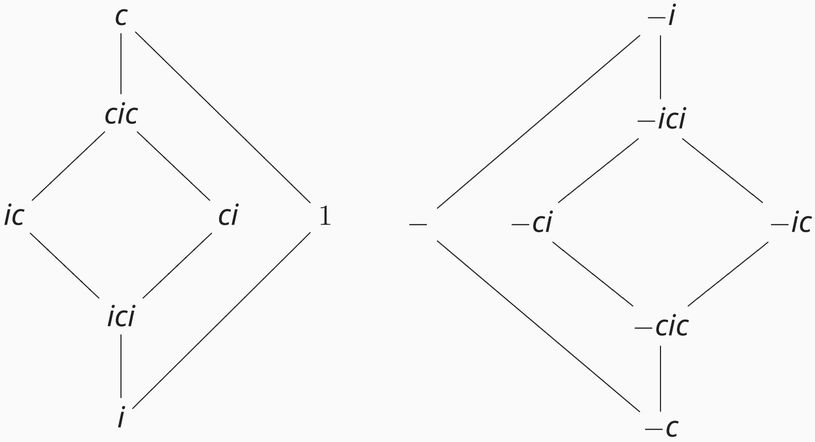
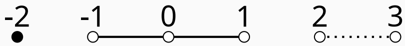
Classical Kuratowski's problem: the closure-complement problem ($i = -c-$)



A 14-set of reals

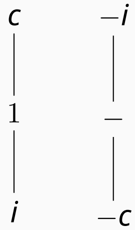


$$\{-2\} \cup (-1,0) \cup (0,1) \cup \big((2,3) \cap \mathbb{Q} \big)$$

Kuratowski's monoids: the case of a Boolean space

Boolean space = the opens are a cBa

A space is Boolean iff $\boxed{c = ic}$ iff $\boxed{ci = i}$ iff ...



What happens CONSTRUCTIVELY (no LEM)?

$i = -c-$ does not make sense

(when $c = 1$ we get $i = --$ which is an interior operator iff LEM)

so we keep i as primitive and

we have 3 possibilities for c

- treat c as primitive too (and no pseudocomplement)
- put $c = -i-$ (the ”negative” definition of closure)
- define c in terms of adherent points (”positive”)

The interior-closure problem

For an interior i

$(ix \leq iy \text{ iff } ix \leq y)$

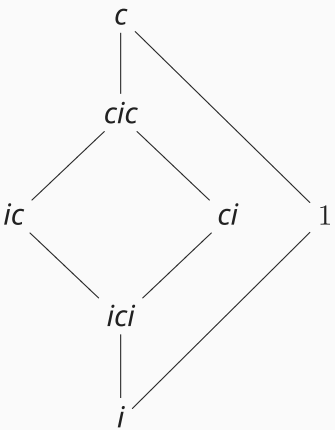
and a closure c

$(cx \leq cy \text{ iff } x \leq cy)$

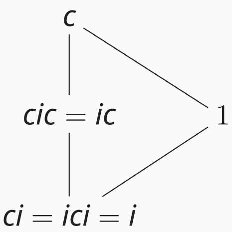
on an arbitrary poset:

Constrcutive and
very general:

- no link between i and c
- no topology



Example: if $\boxed{ci = i}$ (a version of Booleanness), then



(because $ci = i$ implies $ci = ici = i$, and $ci = ici$ is equivalent to $cic = ic$)

The present framework is a bit too general for studying Kuratowski's monoids...

There are 26 additional inequalities which could be imposed:

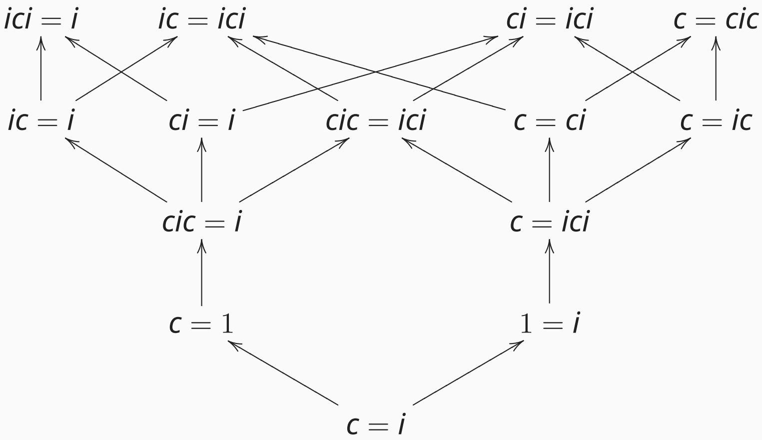
$c = i$, $c = ici$, $c = ic$, $c = ci$, $c = cic$, $c = 1$,
$cic = i$, $cic = ici$, $cic = ic$, $cic = ci$, $cic \leq 1$, $1 \leq cic$,
$ic = i$, $ic = ici$, $ic \leq ci$, $ic \leq 1$, $1 \leq ic$,
$ci = i$, $ci = ici$, $ci \leq ic$, $ci \leq 1$, $1 \leq ci$,
$ici = i$, $ici \leq 1$, $1 \leq ici$,
$1 = i$.

Although some are equivalent...

$cic \leq 1$	$cic = i$	
$1 \leq cic$	$c = cic$	
$ic \leq ci$	$cic = ci$	$ic = ici$
$ic \leq 1$	$ic = i$	
$1 \leq ic$	$c = ic$	
$ci \leq ic$	$cic = ic$	$ci = ici$
$ci \leq 1$	$ci = i$	
$1 \leq ci$	$c = ci$	
$ici \leq 1$	$ici = i$	
$1 \leq ici$	$c = ici$	

(items in the same row are equivalent).

These are the implications which hold in general:



(some counterexample still missing).

Closure via adherent points

In a topological space
$p \in c(Y)$ if every (basic) open neighbourhood of p overlaps Y

if $i(X)$ and $c(Y)$ overlap each other, then in fact $i(X)$ and Y overlap each other

In order to get a general algebraic version of this,
we need a poset with overlap...

Overlap relations

Poset with overlap $(L, \leq, \between)$
= poset $(L, \leq)$ + binary relation $\between$ on L s.t.

$x \between y \Rightarrow y \between x$ (symmetry)

$(x \between y)$ & $(y \leq z) \Rightarrow (x \between z)$ (monotonicity)

$\forall z\ (x \between z \Rightarrow y \between z) \Rightarrow x \leq y$ (density)

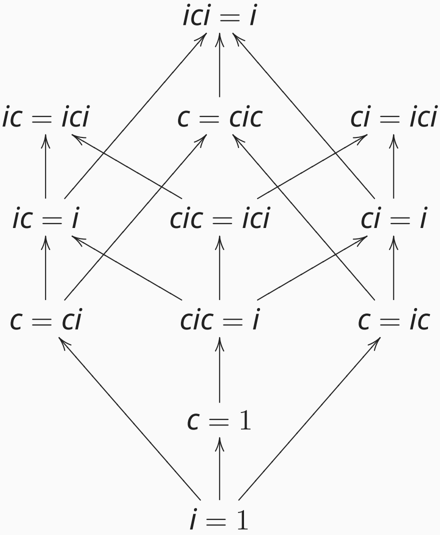
A "positive" link between interior and closure

Compatibility:

$$ix \between cy \;\Rightarrow\; ix \between y$$

In a poset with $\between$
if i and c are compatible then:

- $i = 1 \;\Rightarrow\; c = 1$
- ...
- $c = ic \;\Rightarrow\; ci = i$
- $c = ci \;\Rightarrow\; ic = i$
- ....

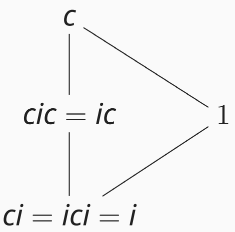


TFAE:

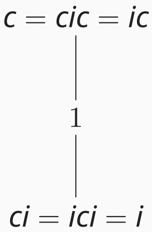
- LEM
- $c = 1 \Rightarrow i = 1$
- $ci = i \Rightarrow c = ic$
- ...

Example: two versions of Booleanness

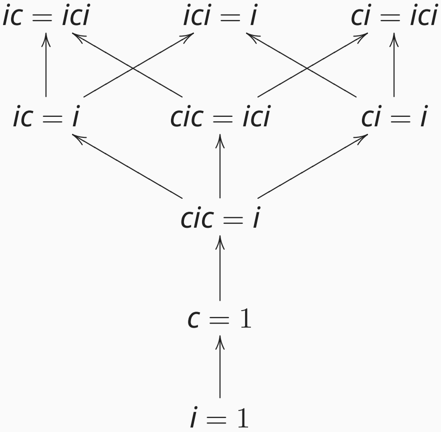
$\boxed{ci = i}$
(every open is closed)



$\boxed{c = ic}$
(every closed is open)



The case of a Kolmogorov space



Example: two versions of Booleanness + T_0

$$(c = ic) + T_0 \iff \underbrace{i = 1}_{(discrete)}$$

$$[(ci = i) + T_0 \iff i = 1] \quad \text{iff} \quad \text{LEM}$$

The interior-pseudocomplement problem

Pseudo-complement operator $-$ on a poset L:

$$x \leq -y \iff y \leq -x$$

($-$ defines an antitone Galois connection on L)

Properties:

$x \leq --x$ $\qquad$ ($1 \leq --$)

$x \leq y \Rightarrow -y \leq -x$ $\qquad$ ($-$ is antitone)

$---x = -x$ $\qquad$ ($--- = -$)

$x \leq --y \Leftrightarrow --x \leq --y$ $\qquad$ ($--$ is a closure operator)

As usual, let $\boxed{i}$ be an interior operator on L.

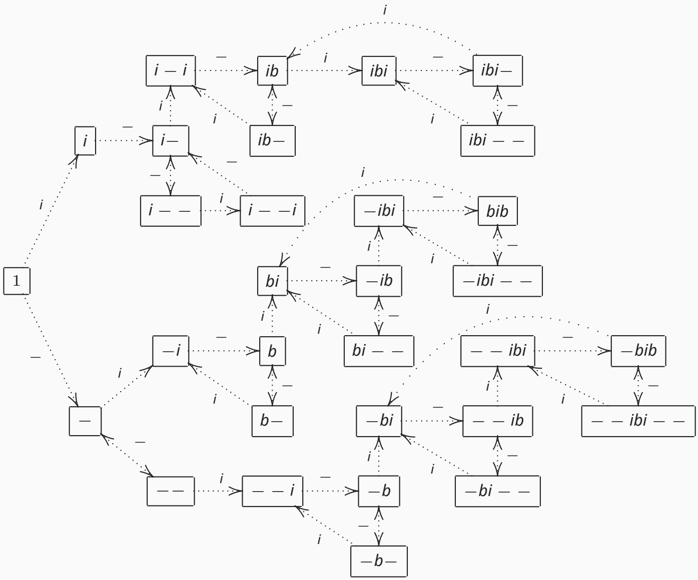
Put $\boxed{b = -i-}$ (closure operator).

31 possible combinations

monotone
1 – –
i *i* – – – – *i* –*b*– *i* – –*i*
b
ib – – *ib*
bi *bi* – –
ibi *ibi* – – – – *ibi* – – *ibi* – –
bib
17

antitone
–
i– –*b*
–*i* *b*–
i – *i* *ib*– –*bi* –*bi* – –
–*ib*
ibi– –*bib*
–*ibi* –*ibi* – –
14

Cayley graph of the monoid generated by *i* and —



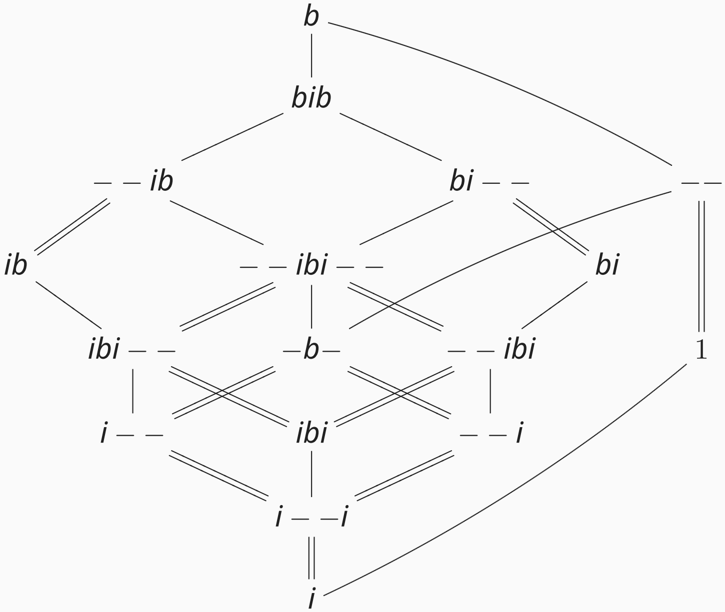
$i - b = i - -i- = i-$

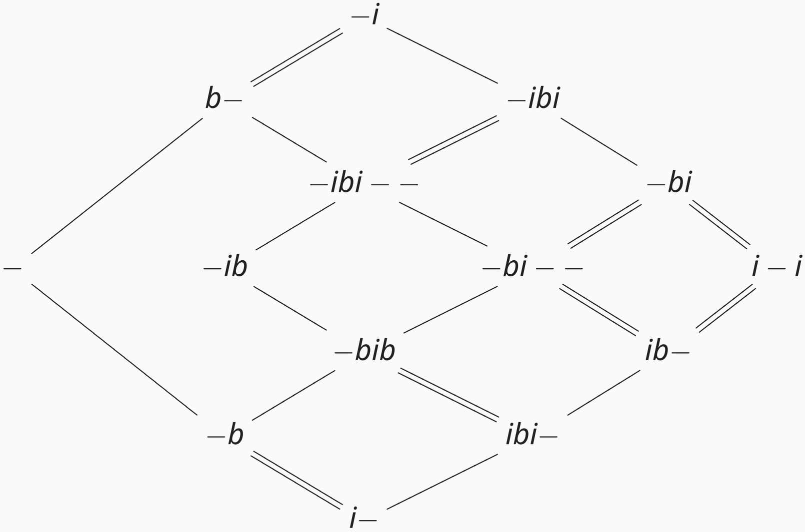
$b - i = -i - -i = -i$

...

$iwi = i - i$ for every w with an odd number of $-$'s

$iwi \in \{i, i - -i, ibi\}$ for every w with an even number of $-$'s


b
bib
− − ib
bi − −
− −
ib
− − ibi − −
bi
ibi − −
−b−
− − ibi
1
i − −
ibi
− − i
i − −i
i



−i
b−
−ibi
−ibi − −
−bi
−
−ib
−bi − −
i − i
−bib
ib−
−b
ibi−
i−

The pointfree Kuratowski's problem

L = poset (co-frame) of all sublocales of a given locale

$\leq$ is the sub-locale relation
(opposite of the pointwise order on nuclei)

$-$ is the co-pseudocomplement on the co-frame L
($-x \leq y \;\Leftrightarrow\; -y \leq x$)

i = the interior in the sense of sublocales
(it is a closure operator on nuclei)

c = the closure in the sense of sublocales
(it is an interior operator on nuclei)

Some facts about open and closed sublocales

Open and closed sublocales are complemented in L:
the complement of an open sublocale is closed and the complement of a closed sublocale is open.

Consequences:

$$- - i = i = -c- = i - - \leq -- \leq c - - = -i- \leq c = - - c$$

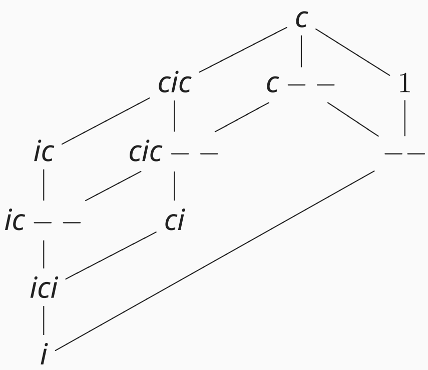
$$-c \leq i- \leq - \leq c- = -i$$

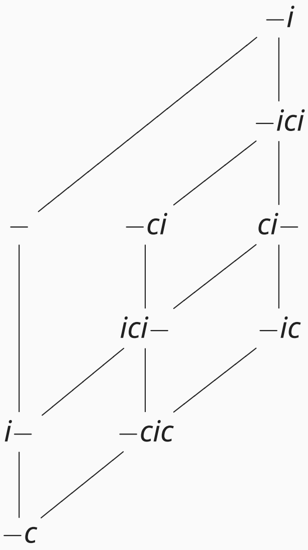
Formally, the Kuratowski's problem for locales is a special case of the interior-pseudocomplement problem above...

Warning: since now $-$ is a co-pseudocomplement, one has to apply the previous result with respect to the opposite order, and so the roles of the interior and closure operators are switched.

As $i = -c-$, we fall under the assumption about b above (where the closure b was supposed to be $-i-$).

Solution: (i) take the previous diagrams, (ii) reverse them, (iii) write c in place of i, (iv) then write i in place of b, and (v) simplify based on the previous facts.


c
cic
c − −
1
ic
cic − −
− −
ic − −
ci
ici
i


−i
−ici
−
−ci
ci−
ici−
−ic
i−
−cic
−c

$c \neq -i-$

X = the opposite of $\omega + 1$ (it's a cHa/frame/locale)
Y = its smallest dense sublocale
(it is defined by the double negation nucleus)
so $cY = X$.
But $-Y = X$ and hence
$-i - Y = \emptyset$ (the smallest sublocale of X).

References

C., Kuratowski's problem in constructive Topology, (submitted)

C., Overlap Algebras as Almost Discrete Locales, Logical Methods in Computer Science (to appear)

C. & Contente, Overlap Algebras: a constructive look at complete Boolean algebras, Logical Methods in Computer Science (2020)

C. & Maietti & Toto, Constructive version of Boolean algebra, Logic Journal of the IGPL (2012)

C. & Sambin, The overlap algebra of regular opens, Journal of Pure and Applied Algebra (2010)

Sambin, Positive Topology. A new practice in constructive mathematics, Clarendon Press, Cambridge (to appear)

Other references?

Al-Hassani & Mahesar & Sacerdoti Coen & Sorge, A term rewriting system for Kuratowski's closure-complement problem, 23rd International Conference on Rewriting Techniques and Applications (2012)

Gardner & MJackson, The Kuratowski closure-complement theorem, New Zealand J. Math. 38 (2008)

He & Zhang, Interior and boundary in a locale, Adv. Math. (China) 29 (2000)

Thank you!