Metric spaces in computable structure theory

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"Algebraic" computable structure theory

A structure $\mathcal{A}$ is *computably presented* **if its domain is** $\mathbb{N}$ and all the functions and relations on $\mathcal{A}$ are computable.

This works very well for linear orders, groups, rings, Boolean algebras, etc.

Metric structures, though, require us to adapt to a continuous setting. *How do we do this?*

Metric structures

Definition

A *metric structure* is a quintuple $\mathcal{M} = (U, d, \mathcal{O}, \mathcal{F}, \mathcal{C})$ such that (U, d) is a complete metric space and

1. For each $T \in \mathcal{O}$, there is a positive integer n so that T is a uniformly continuous n-ary operation on U.
2. For each $f \in \mathcal{F}$, there is a positive integer n so that f is a uniformly continuous n-ary functional on U; i.e., $f : U^n \to \mathbb{F}$ and is uniformly continuous.
3. $\mathcal{C} \subseteq U$.

A presentation of a metric structure $\mathcal{M}$ is a pair $(\mathcal{M}, (p_n))$ such that the p_ns (the *distinguished points*) generate $\mathcal{M}$, and the *rational points* of a presentation are the points in the subspace generated by the distinguished points.

Quick observation

We could view algebraic structures as special cases of metric structures with the discrete metric.

Metric signatures

Definition

A *metric signature* $\mathcal{S}$ is a quintuple $(\overline{\mathcal{O}}, \overline{\mathcal{F}}, \overline{\mathcal{C}}, \eta, \Delta)$ where

1. $\overline{\mathcal{O}}, \overline{\mathcal{F}}, \overline{\mathcal{C}}$ are pairwise disjoint sets of symbols,
2. $\eta : \overline{\mathcal{O}} \cup \overline{\mathcal{F}} \cup \overline{\mathcal{C}} \to \mathbb{N}$, η is positive on $\overline{\mathcal{O}} \cup \overline{\mathcal{F}}$, and $\eta(\bar{c}) = 0$ for each $\bar{c} \in \overline{\mathcal{C}}$, and
3. $\Delta : (\overline{\mathcal{O}} \cup \overline{\mathcal{F}}) \times \mathbb{N} \to \mathbb{N}$.

Presentations of signatures

A presentation ν of a metric signature $\mathcal{S}$ is a map from $\mathbb{N}$ onto the symbols of $\mathcal{S}$ and is computable if

- its inverse is computable,
- $\eta \circ \nu$ is computable, and
- $\Delta_{\nu(n)}$ is computable uniformly in $n \in \nu^{-1}[\overline{\mathcal{O}} \cup \overline{\mathcal{F}}]$.

Computability of presentations of structures

Definition

Suppose $\mathcal{M}^\#$ is a presentation of a metric structure $\mathcal{M}$ that has a computable signature. We say $\mathcal{M}^\#$ is *computable* if it satisfies the following conditions.

1. The metric of $\mathcal{M}$ is computable on the rational points of $\mathcal{M}^\#$: if d denotes the metric of $\mathcal{M}$, then there is an algorithm that, given any two rational points p_1, p_2 of $\mathcal{M}^\#$ and a $k \in \mathbb{N}$, computes a rational number q so that $|q - d(p_1, p_2)| < 2^{-k}$.
2. For every n-ary functional F of $\mathcal{M}^\#$ and all rational points $p_1, \ldots, p_n$ of $\mathcal{M}^\#$, $F(p_1, \ldots, p_n)$ is computable uniformly in $F, p_1, \ldots, p_n$: there is an algorithm that, given $F, p_1, \ldots, p_n$ and $k \in \mathbb{N}$ as input, produces a rational scalar q so that $|F(p_1, \ldots, p_n) - q| < 2^{-k}$.

Computability of maps

Theorem

Suppose $\mathcal{M}_0^\#$ and $\mathcal{M}_1^\#$ are presentations of metric structures with computable signatures, and let $\Phi : |\mathcal{M}_0^\#| \to |\mathcal{M}_1^\#|$. Then Φ is a computable map of $\mathcal{M}_0^\#$ into $\mathcal{M}_1^\#$ if both of the following hold.

1. *Φ is computable on the rational points of $\mathcal{M}_0^\#$. That is, for every rational point p of $\mathcal{M}_0^\#$, $\Phi(p)$ is a computable point of $\mathcal{M}_1^\#$ uniformly in p.*
2. *There is a computable modulus of continuity for Φ.*

Recall: a function $g : \mathbb{N} \to \mathbb{N}$ is a modulus of continuity for f if $d(f(p_1, \ldots, p_n), f(q_1, \ldots, q_n)) < 2^{-k}$ whenever $\max_j d(p_j, q_j) \leq 2^{-g(k)}$.

Isomorphisms

Definition

Suppose $\mathcal{M}_0$ and $\mathcal{M}_1$ are interpretations of a metric signature $\mathcal{S}$, and let $F : |\mathcal{M}_0| \to |\mathcal{M}_1|$. We say F is an *isomorphism* if it is homeomorphic and satisfies the following.

1. For each n-ary operation symbol $\overline{T}$ of $\mathcal{S}$ and all $p_1, \ldots, p_n \in |\mathcal{M}_0|$, $F(\overline{T}^{\mathcal{M}_0}(p_1, \ldots, p_n)) = \overline{T}^{\mathcal{M}_1}(F(p_1), \ldots, F(p_n))$.
2. For each n-ary functional symbol $\overline{\phi}$ of $\mathcal{S}$ and all $p_1, \ldots, p_n \in |\mathcal{M}_0|$, $F(\overline{\phi}^{\mathcal{M}_0}(p_1, \ldots, p_n)) = \overline{\phi}^{\mathcal{M}_1}(F(p_1), \ldots, F(p_n))$.
3. For each constant symbol $\overline{c}$ of $\mathcal{S}$, $F(\overline{c}^{\mathcal{M}_0}) = \overline{c}^{\mathcal{M}_1}$.

A map $\Phi : |\mathcal{M}_0| \to |\mathcal{M}_1|$ is *isometric* if it preserves distances.

Lowness

A set X is *low* for a relativizable class $\mathcal{C}$ if $\mathcal{C}^X = \mathcal{C}$.

Lowness has been studied in

- degree theory,
- randomness,
- computational learning theory, and
- computable structure theory.

Goal

Add computable metric structure theory to this list.

Lowness for isomorphism

Definition
A Turing degree is *low for isomorphism* if, whenever it can compute an isomorphism between two computably presented structures, there is a computable isomorphism between them.

This class of degrees is not simple to describe.

Warmup: metric spaces

Theorem (F. and McNicholl)

A Turing degree is low for isomorphism if and only if it is low for isometry.

We use Melnikov's technique of representing a graph as a metric space. For a graph $G = (V, E)$, we define

$$d_G(v_0, v_1) = \begin{cases} 0 & v_0 = v_1 \\ 1 & (v_0, v_1) \in E \\ 2 & \text{else} \end{cases}.$$

Let $M(G)$ be the metric space defined using the above metric.

Proof

Suppose **d** is low for isometry and the graph $G_0 = (V_0, E_0)$ is **d**-isomorphic to the graph $G_1 = (V_1, E_1)$.

- Since **d** can compute an isomorphism from G_0 to G_1, **d** can compute an isometry between $M(G_0)$ to $M(G_1)$.
- Since **d** is low for isometry, there is a computable isometry from $M(G_0)$ to $M(G_1)$.
- Since **0** can compute a distance-preserving function from $M(G_0)$ to $M(G_1)$, there is a computable function that maps each pair of vertices in G_0 to another pair of points in G_1 with the same distance between them, that is, another pair of points with the same edge-relation.
- This function will give us a graph isomorphism from G_0 to G_1.

Complications

In our previous result, we were able to encode our metric space as a graph, which is a classic algebraic structure.

How do we get a result like this in the metric structure setting?

Definition

1. Suppose $\mathcal{M}^\#$ is a computable presentation of a metric structure $\mathcal{M}$. We say that $\mathbf{d}$ is *low for $\mathcal{M}^\#$ isometric isomorphism* if every computable presentation of $\mathcal{M}$ that is $\mathbf{d}$-isometrically isomorphic to $\mathcal{M}^\#$ is also computably isometrically isomorphic to $\mathcal{M}^\#$.
2. Suppose $\mathcal{M}$ is a computably presentable metric structure. We say $\mathbf{d}$ is *low for $\mathcal{M}$ isometric isomorphism* if it is low for $\mathcal{M}^\#$ isometric isomorphism whenever $\mathcal{M}^\#$ is a computable presentation of $\mathcal{M}$.
3. Suppose $\mathcal{K}$ is a class of computably presentable metric structures. We say $\mathbf{d}$ is *low for isometric isomorphism of $\mathcal{K}$-structures* if $\mathbf{d}$ is low for $\mathcal{M}$ isometric isomorphism for every $\mathcal{M} \in \mathcal{K}$.
4. We say $\mathbf{d}$ is *low for isometric isomorphism* if it is low for $\mathcal{M}$ isometric isomorphism for every computably presentable structure $\mathcal{M}$.

Lowness for paths

Definition

A real A is *low for paths for Baire space* (or *low for paths for Cantor space*) if every Π^0_1 class $\mathcal{P} \subseteq \mathbb{N}^{\mathbb{N}}$ (or $\mathcal{P} \subseteq 2^{\mathbb{N}}$) with an A-computable element has a computable element.

Lowness for paths

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Question

Are these notions different?

The answer

Theorem (F. and Turetsky)

A sequence A is low for paths for Baire space if and only if it is low for paths for Cantor space.

The first clearly implies the second. For the other direction, we need the following:

Lemma (Simpson)

If $\mathcal{P} \subseteq \mathbb{N}^{\mathbb{N}}$ and $\mathcal{Q} \subseteq 2^{\mathbb{N}}$ are nonempty Π^0_1-classes, then there is a Π^0_1-class $\mathcal{R} \subseteq 2^{\mathbb{N}}$ with $\mathcal{R} \equiv_w \mathcal{P} \cup \mathcal{Q}$.

Lowness for paths and lowness for isomorphism

Theorem (F. and Turetsky)

A Turing degree is low for paths if and only if it is low for isomorphism.

...and lowness for isometric isomorphism

Definition

A Turing degree is *low for isometric isomorphism* if for every computably presented metric structure $\mathcal{M}$ and any two of its computable presentations $\mathcal{M}^+$ and $\mathcal{M}^\#$, whenever it can compute an isometric isomorphism between these presentations, there is a computable isometric isomorphism between them.

Theorem (F. and McNicholl)

A Turing degree is low for isomorphism if and only if it is low for isometric isomorphism.

Lemma

Let $\mathcal{M}^\#$ and $\mathcal{M}^+$ be computable presentations of a metric structure with signature $\mathcal{S}$. Then there is a Π^0_1 class $\mathcal{R} \subseteq \mathbb{N}^\mathbb{N}$ so that for every Turing degree $\mathbf{d}$*,* $\mathbf{d}$ *computes a point in $\mathcal{R}$ if and only if* $\mathbf{d}$ *computes an isometric isomorphism Φ of $\mathcal{M}^\#$ onto $\mathcal{M}^+$.*

Let $\Psi = \Phi^{-1}$, and let (x_i) be the rational points of $\mathcal{M}^\#$ and (y_j) the rational points of $\mathcal{M}^+$.

Proof of Lemma

We define $\mathcal{R}$ to be the set of all $(f,g) \in (\mathbb{N}^{\mathbb{N}\times\mathbb{N}})^2$, recoded as elements of $\mathbb{N}^{\mathbb{N}}$, that satisfy the following conditions.

1. f and g give us (indices for) strong Cauchy sequences converging to $\Phi(x_m)$ and $\Psi(y_m)$.
2. Distances between two points, values of n-ary operations T, values of n-ary functionals F, and values of constants c under Φ and Ψ can be computed arbitrarily precisely.

This can all be expressed in a Π^0_1 way, so $\mathcal{R}$ is Π^0_1.

For instance: For every n-ary functional F of $\mathcal{M}$ and all $m, j_1, \ldots, j_n, k_1, \ldots, k_n \in \mathbb{N}$,

$$|F(x_{j_1}, \ldots, x_{j_n}) - F(y_{f(j_1,k_1)}, \ldots, y_{f(j_n,k_n)})| \leq 2^{-m}$$

provided $\Delta_F(m) \leq \min_s k_s + 1$ for s between 1 and n.

Now we show that...

- $\Phi\Psi$ and $\Psi\Phi$ are the identity on the rational points,
- since Φ and Ψ are **d**-computable on the rational points and they have a computable modulus of continuity, they are **d**-computable,
- Φ and Ψ preserve operations, functionals, and constants.

Then if **d** is low for isomorphism and computes an isometric isomorphism from $\mathcal{M}^+$ onto $\mathcal{M}^\#$, it will compute a point of the appropriate $\mathcal{R}$ and thus will be low for isometric isomorphism.

Banach spaces

Let $\mathbb{F}_{\mathbb{Q}} = \mathbb{F} \cap \mathbb{Q}(i)$. We refer to the elements of $\mathbb{F}_{\mathbb{Q}}$ as *rational scalars*.

Let $\mathcal{S}_{\text{Banach}}$ denote the metric signature of Banach spaces, which consists of a binary operation symbol $+$, a unary operation symbol $\cdot_s$ for each rational scalar s, a unary functional symbol $\| \|$, and a constant symbol $\mathbf{0}$.

A Banach space $\mathcal{B}$ can be represented as an interpretation of $\mathcal{S}_{\text{Banach}}$. There is no loss of generality due to the restriction to rational scalars: any map that preserves multiplication by rational scalars also preserves multiplication by scalars.

Presentations of Banach spaces

If $\mathcal{B}^{\#}$ is a presentation of a Banach space $\mathcal{B}$, then the rational points of $\mathcal{B}^{\#}$ are precisely the rational linear combinations of distinguished points of $\mathcal{B}^{\#}$, i.e., vectors that can be expressed in the form $\sum_{j \leq M} \alpha_j v_j$ where $\alpha_j \in \mathbb{F}_{\mathbb{Q}}$ and each v_j is a distinguished point of $\mathcal{B}^{\#}$.

Theorem (F. and McNicholl)

Every Turing degree that is low for isomorphism is also low for isometric isomorphism of Banach spaces.

This follows immediately from the previous theorem.

Why not a bidirectional result?

We don't have a method of effectively encoding members of a sufficiently universal class of countable algebraic structures into Banach spaces.

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Thank you!