Totalization of ODEs

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Introduction

Let $[a, b] \subset \mathbb{R}$, $E \subset \mathbb{R}^n$, $f : E \to E$ and $y_0 \in E$. We consider IVPs on $[a, b]$ of the type:

$$\begin{cases} y' = f(y(t)) \\ y(a) = y_0 \end{cases}$$

- Dynamic is given by ordinary differential equations with a unique solution
- $y : [a, b] \to E \subseteq \mathbb{R}^n$ is the unique solution.

Analysis

If f is continuous, obtain y as limit of a sequence of continuous functions

Computability

if f is continuous, compute y with the Ten thousand monkeys approach [CG09]

Motivation

Relaxing continuity for f

Question 1: analysis

Can we obtain y?

- Is it always the case that we can obtain y analytically?
- Which method should be used?

Question 2: definability

What is the set theoretical complexity of y relative to f?

- Can we rank these IVPs depending on how hard is to obtain the solution y?
- Boldface and lightface analysis

Antidifferentiation

Antidifferentiation is a particular type of ODE solving when the derivative is known explicitely

- Let $F : [a, b] \subseteq \mathbb{R} \to \mathbb{R}$ be a function differentiable on $[a, b]$
- Let $f : [a, b] \subseteq \mathbb{R} \to \mathbb{R}$ be such that $F'(x) = f(x)$ for all $x \in [a, b]$

Antidifferentiation

Can we obtain F from f? How?

- Lebesgue integration is not general enough
- Investigate antidifferentiation for non-Lebesgue integrable derivatives
- Denjoy totalization 1912, [Den12]

Denjoy totalization

- Condition on f: f is a derivative

Definition 1 (Nonsummable points of f)

Let E be a closed set $E \subseteq [a, b]$ and f a Lebesgue measurable function. A point $x \in E$ is a *nonsummable point of f on E* if f is not Lebesgue integrable in every $I \in E$, I an open interval containing x.

Theorem 2

If F is a differentiable function on $[a, b]$ and $E \subseteq [a, b]$ is closed, then the nonsummable points of f on E form a closed nowhere dense set[1] *in E*

[1] A subset A of a topological space X is nowhere dense in X if the closure of A has empty interior

Transfinite process

- Let $E_1 = \{x \in [a, b]$ such that x is a nonsummable point of f on $[a, b]\}$, and let $\{(a_i, b_i)\}_i$ be its contiguous intervals
- Obtain $F(d) - F(c)$ for all $[c, d] \subseteq [a, b]$ such that $[c, d] \cap E_1 = \emptyset$
- Since F is continuous, take limits to obtain $F(b_i) - F(a_i)$ for all i.

Inductive step

Let $E_2 = \{x \in E_1$ such that x is a nonsummable point of f on $E_1\}$

- We know by the same theorem that E_2 is nowhere dense in E_1
- Repeat the above for E_2
- Proceed by transfinite induction, taking intersections at limit ordinals

- If $E_{\alpha+1} \subseteq E_\alpha$ with $E_{\alpha+1}$ nowhere dense in E_α, $\Rightarrow E_{\alpha+1} \subset E_\alpha$
- The process converges due to Cantor-Baire stationary principle [EV10] $\Rightarrow E_\alpha = \emptyset$ for some $\alpha < \omega_1$
- We obtain $F(d) - F(c)$ for all $[c, d] \subseteq [a, b] \Rightarrow$ Totalization leads to the antiderivative in the most general setting

Theorem 3 ([DK91])

The operation of antidifferentiation is not Borel. [2]

Theorem 4 ([Wes20])

The set $\{f \in M(I) : f$ *is Denjoy integrable*$\}$ *is* Π^1_1*-complete.*

- Ajtai, $F \in \Delta^1_1(f)$

[2] More precisely, there is no Borel set $B \subseteq C[a, b]^{\mathbb{N}}$ such that for f derivative $f \in B \iff F(b) - F(a) > 0$.

Back to ordinary differential equations

$$\begin{cases} y' = f(y(t)) \\ y(a) = y_0 \end{cases}$$

- We find a set of conditions on f that allows to apply a techinque inspired by the totalization method
- We isolate the problematic parts of E via transfinite recursion
- On the remaining parts we apply a search method inspired by the Ten thousand monkeys approach [CG09]

Nonsummability points $\longrightarrow$ Discontinuity points

Lebesgue integration $\longrightarrow$ Search method

Isolating the discontinuities

Definition 5 (sequence of f-removed sets on E)

Let $\{E_\alpha\}_{\alpha<\omega_1}$ be a transfinite sequence of sets and $\{f_\alpha\}_{\alpha<\omega_1}$ a transfinite sequence of functions $f_\alpha = f\restriction_{E_\alpha}: E_\alpha \to \mathbb{R}^r$ defined as following:

- Let $E_0 = E$
- For all $\alpha = \beta + 1$ let $E_\alpha = \{x \in E_\beta : f_\beta \text{ is discontinuous in } x\}$
- For all α limit ordinal, let E_α be $E_\alpha = \cap_\beta E_\beta$ with $\beta < \alpha$

Connections with:

- [Her96]: continuous computational trees
- [dB14]: realizability of jump operators

Search method at step α

Definition 6 ((α)Monkeys approach)

We call the (α)*Monkeys approach for* (f, y_0) the following method: consider all tuples of the form $(X_{i,\beta,j}, h_{i,\beta,j}, B_{i,\beta,j}, C_{i,\beta,j}, Y_{i,\beta,j})$ for $i = 0, \ldots, l-1$, $\beta < \alpha$, $j = 1, \ldots, m_{i,\beta}$, where $h_{i,\beta,j} \in \mathbb{Q}^+$, $l, m_i \in \mathbb{N}$ and $X_{i,\beta,j}$, $B_{i,\beta,j}$, $C_{i,\beta,j}$ and $Y_{i,\beta,j}$ are open rational boxes in E. A tuple is said to be valid if $y_0 \in \bigcup_{\beta,j} X_{0,\beta,j}$ and for all $i = 0, \ldots, l-1$, $\beta < \alpha$, $j = 1, \ldots, m_{i,\beta}$:

1. Either $(B_{i,\beta,j} = \emptyset)$ or $\left(\overline{B}_{i,\beta,j} \cap E_\beta \neq \emptyset \text{ and } \overline{B}_{i,\beta,j} \cap E_{\beta+1} = \emptyset\right)$
2. $f_\beta\left(\overline{B}_{i,\beta,j}\right) \subset C_{i,\beta,j}$;
3. $X_{i,\beta,j} \cup Y_{i,\beta,j} \subset B_{i,\beta,j}$;
4. $X_{i,\beta,j} + h_{i,\beta,j} C_{i,\beta,j} \subset Y_{i,\beta,j}$;
5. $\bigcup_{\beta,j} Y_{i,\beta,j} \subset \bigcup_{\beta,j} X_{i+1,\beta,j}$;

Conditions on right-hand term f

Solvable functions

A function f is called solvable if it is a function of class Baire one [3] and it is such that for every closed set $K \subseteq E$ the set of discontinuity points of function $f\restriction_K$ is a closed set

Theorem 7 (Bournez and G.)

If f is solvable then there exists an ordinal $\alpha < \omega_1$ such that $E_\beta = \emptyset$ for all $\beta \geq \alpha$.

[3] Let X, Y be two separable, complete metric spaces. A function $f : X \to Y$ is Baire one if there exists a sequence of continuous functions from X to Y, $\{f_m\}_m$, such that $\lim_{m\to\infty} f_m(x) = f(x)$ for all $x \in X$.

Main result

Theorem 8 (Bournez and G.)

Consider a closed interval, a closed domain $E \subset \mathbb{R}^r$ *for some* $r \in \mathbb{N}$ *and a function* $f : E \to E$ *such that, given an intital condition, the IVP with right-hand term* f *has a unique solution on the interval. If* f *is solvable, then we can obtain the solution analytically via transfinite recursion up to an ordinal* α *such that* $\alpha < \omega_1$*.*

- The method is bound to terminate for some countable ordinal due to previous theorem
- The transfinite number of steps corresponds to the first ordinal α such that $E_\alpha = \emptyset$ and represents the complexity of ODEs solving for f on E

Obtaining the solution

- Given the domain E, construct the sequence of f-removed sets on E, obtaining the ordinal $\alpha < \omega_1$ such that $E_\alpha = \emptyset$.
- For all $n \in \mathbb{N}$, build a valid tuple of the (α)Monkeys approach such that $\operatorname{rad}(B_{i,\beta,j}) < \delta_{i,\beta,j}(\frac{1}{2n})$ for all $i = 0, \ldots, l-1$, $\beta < \alpha$, $j = 1, \ldots, m_{i,\beta}$. We can then choose $\operatorname{rad}(C_{i,\beta,j}) < \frac{1}{n}$, since $f_\beta\left(\overline{B}_{i,\beta,j}\right) \subset C_{i,\beta,j}$,
- For each valid tuple, define a sequence of times $\{t_i\}_{i=0,\ldots,l}$ and a function $\eta_n : [a, t_l] \to E$ such that $\eta_n(a) = y_0$ and such that for all $i < l$ we have $\eta_n(t_i) \in X_{i,\beta(i),j(i)}$ and $\eta_n'(t_i) \in C_{i,\beta(i),j(i)}$ for all $i = 0, \ldots, l-1$
- Take the limit of the sequence of functions $\{\eta_n\}_n$ on the interval $[a, b]$

Examples

Do solvable systems exist?

- We can transform discontinuous derivatives into right-hand terms for bidimensional IVPs
- We can construct an example of IVP of this type with $E_\alpha \neq \emptyset$ for all $\alpha < \omega_1$

Are solvable systems useful?

Theorem 9 (Bournez and G.)

Let $h : \mathbb{N} \to \mathbb{N}$ be a total, one-to-one computable function enumerating the halting set and let $\mu = \sum_{i=0}^{\infty} 2^{-h(i)}$. Let $E = [0,5] \times [0,5]$. There exists an IVP with unique solution $y : [0,5] \to E$, rational initial condition and right-hand term computable everywhere on E except a straight-line, such that $y_2(5) = \mu$

Conclusions

- Question 1: there is an analytical method that leads to the solution of these discontinuous systems via transfinite recursion.
- Question 2: in parallel with results from [DK91] and [Wes20] for antidifferentiation we expect to obtain similar complexity results for ODEs solving

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Thank you!