Predicative presentation of stably locally compact locales

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Every stably compact locale is an adjoint retract of the ideals of a distributive lattice.

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*Every stably (**locally**) compact locale is a dual frame adjoint retract of the ideals of a (**0-bounded**) distributive lattice.*

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Theorem (This talk)

*Every stably (**locally**) compact locale is a dual frame adjoint retract of the ideals of a (**0-bounded**) distributive lattice.*

☛ The latter seems to capture the duality of stably (locally) compact locales better.

Preliminary

Definition

- A subset U of a poset $(P, \leq)$ is **directed** if it is inhabited and any two elements of U have an upper bound in U.
- A **dcpo** is a poset $(D, \leq)$ in which every directed subset $U \subseteq D$ has a least upper bound, denoted by

$$\bigvee^{\uparrow} U := \text{the directed join of } U.$$

- For elements x, y of a dcpo D, we say that y is **way-below** x, denoted $y \ll x$, if for every directed subset $U \subseteq D$, we have

$$x \leq \bigvee^{\uparrow} U \ \rightarrow\ \exists z \in U\,(y \leq z)\,.$$

- A **continuous domain** is a dcpo D in which every element is a directed join of elements way-below it, i.e., for each $x \in D$, the set $\mathord{\downarrow\!\!\downarrow} x \overset{\text{def}}{=} \{y \in D \mid y \ll x\}$ is directed and $x = \bigvee^{\uparrow} \mathord{\downarrow\!\!\downarrow} x$.

Definition

- A **preframe** is a dcpo $(P, \leq)$ with a meet semilattice structure $(P, 1, \wedge)$ where finite meets distribute over directed joins:

$$x \wedge \bigvee^{\uparrow} U = \bigvee_{y \in U}^{\uparrow} (x \wedge y).$$

- A **frame** is a complete lattice $(X, 1, \wedge, \bigvee)$ where finite meets distribute over all joins: $x \wedge \bigvee U = \bigvee_{y \in U} (x \wedge y)$.
- A frame is **locally compact** if it is a continuous domain.
- A locally compact frame is **stably locally compact** if

$$x \ll y \,\&\, x \ll z \;\rightarrow\; x \ll y \wedge z.$$

Definition

- Let P and Q be preframes. A function $f : P \to Q$ is a **preframe homomorphism** if f preserves finite meets and **directed joins** (i.e., Scott continuous meet semilattice homomorphism).
- Let X and Y be frames. A function $f : X \to Y$ is a **frame homomorphism** if f preserves finite meets and **joins**.

Scott open filters

Let X be a frame.

Definition

A **Scott open filter** on X is a filter $\alpha \subseteq X$ such that

$$\bigvee^{\uparrow} U \in \alpha \;\rightarrow\; \exists y \in U \,(y \in \alpha)\,.$$

$\mathbf{SOF}(X)$**:** the collection of Scott open filters on X.

Lemma

There exists a bijective correspondence between Scott open filters on X and preframe homomorphisms from X to $\mathsf{Pow}(1)$*.*

☛ We identify $\mathbf{SOF}(X)$ with $\mathrm{PreFrm}(X, \mathsf{Pow}(1))$.

Lemma

1. $\mathbf{SOF}(X)$ *forms a preframe.*
2. *Every (pre-)frame homomorphism $f\colon X \to Y$ between frames X and Y induces a preframe homomorphism*
   $f^{\mathsf{d}}\colon \mathbf{SOF}(Y) \to \mathbf{SOF}(X) \quad h\colon Y \to \mathsf{Pow}(1) \mapsto h \circ f.$

Dual frame homomorphisms

Proposition

If X is stably locally compact, then $\mathbf{SOF}(X)$ has binary joins

$$\alpha \vee \beta \stackrel{\text{def}}{=} \uparrow\{a \wedge b \mid a \in \alpha, b \in \beta\}$$
$$= \{x \in X \mid \exists a \in \alpha \exists b \in \beta\, (a \wedge b \leq x)\}$$

which distribute over finite meets (i.e., it is ***almost*** *a frame).*

Definition

A frame homomorphism $f\colon X \to Y$ between stably locally compact frames X and Y is said to be **dual** if $f^{\mathsf{d}}\colon \mathbf{SOF}(Y) \to \mathbf{SOF}(X)$ preserves binary joins.

Categories of stably locally compact frames

SLCF: Category of stably locally compact frames and **frame homomorphisms**.

$\mathrm{SLCF_D}$: Category of stably locally compact frames and **dual frame homomorphisms**.

Splitting of idempotents

Definition

Let $\mathbb{C}$ be a category.

- An **idempotent** is a morphism $f\colon A \to A$ such that $f \circ f = f$.
- An idempotent $f\colon A \to A$ **splits** if there exist morphisms $r\colon A \to B$ and $s\colon B \to A$ such that $s \circ r = f$ and $r \circ s = \mathrm{id}_B$.

$$\begin{array}{ccc} A & \xrightarrow{\quad f \quad} & A \\ & {\scriptstyle r}\searrow \quad \nearrow{\scriptstyle s} & \\ & B & \end{array}$$

Proposition

In SLCF *and* $\mathrm{SLCF_D}$*, every idempotent splits.*

Proof.

If $f\colon X \to X$ is an idempotent in SLCF (for example), f factors through $X_f = f[X]$ as

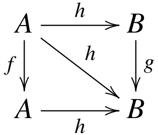
$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & X \\ & {\scriptstyle f\restriction_X}\searrow \quad \nearrow{\scriptstyle i} & \\ & X_f & \end{array} .$$

Karoubi envelop

Definition

Let $\mathbb{C}$ be a category. The **Karoubi envelop** (or splitting of idempotents) of $\mathbb{C}$ is a category $\mathbf{Karoubi}(\mathbb{C})$ where

- objects are **idempotents** in $\mathbb{C}$,
- morphisms $h\colon (f\colon A \to A) \to (g\colon B \to B)$ are morphisms $h\colon A \to B$ in $\mathbb{C}$ such that $g \circ h = h = h \circ f$:



Proposition

Let $\mathbb{C}$ be a category in which every idempotent splits (e.g. SLCF*,* $\mathsf{SLCF_D}$*). If $\mathbb{D}$ is a full subcategory of $\mathbb{C}$ such that every object in $\mathbb{C}$ is a retract of an object of $\mathbb{D}$, $\mathbb{C}$ is equivalent to $\mathbf{Karoubi}(\mathbb{D})$.*

☛ Find a suitable $\mathbb{D}$ for SLCF and $\mathsf{SLCF_D}$.

0-bounded distributive lattices

Definition

- A **0-bounded distributive lattice** (0-bounded distributive lattice) is a distributive lattice with a least element 0.
- An **ideal** of a 0-bounded distributive lattice $(S, 0, \vee, \wedge)$ is a subset $I \subseteq S$ which is downset and closed under finite joints.
- The **ideal completion** $\mathrm{Idl}(S)$ of a 0-bounded distributive lattice S is the collection of ideals of S.

Lemma

For any 0-bounded distributive lattice $(S, 0, \vee, \wedge)$, the ideal completion $\mathrm{Idl}(S)$ is a stably locally compact frame.

Proof.

The frame structure of $\mathrm{Idl}(S)$ is given by

$$0 \stackrel{\text{def}}{=} \emptyset, \qquad I \vee J \stackrel{\text{def}}{=} \downarrow\{a \vee b \mid a \in I \;\&\; b \in J\},$$

$$\bigvee_{k \in I}^{\uparrow} I_k \stackrel{\text{def}}{=} \bigcup_{k \in K} I_k, \qquad I \wedge J \stackrel{\text{def}}{=} \{a \wedge b \mid a \in I, b \in J\}.$$

Retraction

Theorem

Every stably locally compact frame is a dual frame retract of the ideal completion of a 0-bounded distributive lattice.

Proof.

Let X be a stably locally compact frame.

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First try (AVM 2023)

Consider X as a distributive lattice as in Johnstone (1982), and define $f\colon X \to \mathrm{Idl}(X)$ by $f(x) = \twoheaddownarrow x$.

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Second try (Verona 2023)

Consider $\mathbf{SOF}(X)^{\mathsf{op}}$, which is almost a frame (lacking the top). Let

$$S = (\mathbf{SOF}(X)^{\mathsf{op}}, 1, \wedge, \vee).$$

There exists a frame homomorphism $f\colon X \to \mathrm{Idl}(S)$ defined by $f(x) \overset{\text{def}}{=} \downarrow \{\uparrow a \mid a \ll x\}$.

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There exists a frame homomorphism $f \colon X \to \mathrm{Idl}(S)$ defined by $f(x) \stackrel{\text{def}}{=} \downarrow \{\uparrow a \mid a \ll x\}$. ✘ Its retract is not a frame homomorphism.

Proof continued.

Third try (CCC 2023)

Consider $S_X = (\mathsf{Fin}^+(\downarrow 1), \leq)$ where

$$A \leq B \overset{\text{def}}{\iff} \bigwedge A \leq \bigwedge B \;\&\; \forall C\,(B \ll_U C \to \exists D\,(A \ll_U D \,\&\, \bigwedge D \leq \bigwedge C))$$

where $A \ll_U B \overset{\text{def}}{\iff} \forall b \in B \exists a \in A\,(a \ll b)$.

Define a 0-bounded distributive lattice $S_X = (S_X, 0, \vee, \wedge)$ by

$$0 \overset{\text{def}}{=} \{0\}, \quad A \vee B \overset{\text{def}}{=} \{a \vee b \mid a \in A, b \in B\}, \quad A \wedge B \overset{\text{def}}{=} A \cup B.$$

Lastly, define $f \colon X \to \mathrm{Idl}(S_X)$ by

$$f(x) \overset{\text{def}}{=} \{A \in S_X \mid \exists B\,(A \ll_U B \,\&\, \bigwedge B \leq x)\}.$$

Proof continued.

Third try (CCC 2023)

Consider $S_X = (\mathsf{Fin}^+(\downarrow 1), \leq)$ where

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where $A \ll_U B \overset{\text{def}}{\iff} \forall b \in B \exists a \in A \,(a \ll b)$.
Define a 0-bounded distributive lattice $S_X = (S_X, 0, \vee, \wedge)$ by

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$$f(x) \overset{\text{def}}{=} \{A \in S_X \mid \exists B \,(A \ll_U B \,\&\, \bigwedge B \leq x)\}.$$ ✔ **It works!** □

Definition (Subcategories of stably locally compact frames)

0-DL_{F}**:** A full subcategory of SLCF consisting of ideal completions of 0-bounded distributive lattices.

0-DL_{D}**:** A full subcategory of $\mathsf{SLCF}_{\mathsf{D}}$ consisting of ideal completions of 0-bounded distributive lattices.

Theorem

1. SLCF *is equivalent to* $\mathbf{Karoubi}(0\text{-DL}_{\mathsf{F}})$.
2. $\mathsf{SLCF}_{\mathsf{D}}$ *is equivalent to* $\mathbf{Karoubi}(0\text{-DL}_{\mathsf{D}})$.

Strong quasi-proximity lattices

Proximity relations

First, We look into SLCF $\cong$ $\mathbf{Karoubi}$(0-$\mathsf{DL_F}$).

Aim

Give a predicative characterisation of 0-$\mathsf{DL_F}$.

☛ Then, we obtain a predicative representation of SLCF by the construction $\mathbf{Karoubi}$(0-$\mathsf{DL_F}$).

Definition

Let $(S, 0, \vee, \wedge)$ and $(S', 0', \vee', \wedge')$ be 0-bounded distributive lattices. A **proximity relation** from S to S' is a relation $r \subseteq S \times S'$ such that

1. $r^{-}b \overset{\text{def}}{=} \{a \in S \mid a\ r\ b\}$ is an ideal of S,
2. $ra \overset{\text{def}}{=} \{b \in S' \mid a\ r\ b\}$ is a filter of S',
3. $a\ r\ 0' \ \rightarrow\ a = 0$,
4. $a\ r\ (b \vee' c)\ \rightarrow\ \exists b'\ r\ b \exists c'\ r\ c\ (a \leq b' \vee c')$.

Proximity relations

Proposition

Let $(S, 0, \vee, \wedge)$ *and* $(S', 0', \vee', \wedge')$ *be 0-bounded distributive lattices.*

- *There exists a bijective correspondence between frame homomorphisms* $f\colon \mathrm{Idl}(S') \to \mathrm{Idl}(S)$ *and proximity relations* $r\colon S \to S'$*.*
- *The identity function on* $\mathrm{Idl}(S)$ *corresponds to the underlying order* $\leq$ *on* S*.*
- *The composition of frame homomorphisms contravariantly corresponds to the relational composition of proximity relations.*

$\mathsf{0\text{-}DL_{PX}}$: Category of 0-b. dist. lattices and **proximity relations**.

Corollary

- $\mathsf{0\text{-}DL_{PX}}$ *is dually equivalent to* $\mathsf{0\text{-}DL_{F}}$*.*
- $\mathbf{Karoubi}(\mathsf{0\text{-}DL_{PX}})$ *is dually equivalent to* $\mathbf{Karoubi}(\mathsf{0\text{-}DL_{F}})$*.*
- $\mathbf{Karoubi}(\mathsf{0\text{-}DL_{PX}})$ *is equivalent to* $\mathsf{SLCF}^{\mathsf{op}}$*.*

An explicit description of $\mathbf{Karoubi}(\text{0-DL}_{\text{PX}})$.

Definition

A **quasi-proximity lattice** is a structure $(S, 0, \vee, \wedge, \prec)$ where $(S, 0, \vee, \wedge)$ is a 0-bounded distributive lattice and $\prec$ is a relation on S satisfying

1. $\prec \circ \prec = \prec$,
2. $\downarrow_{\prec} a \overset{\text{def}}{=} \{b \in S \mid b \prec a\}$ is an ideal of S,
3. $\uparrow^{\succ} a \overset{\text{def}}{=} \{b \in S \mid b \succ a\}$ is a filter of S,
4. $a \prec 0 \ \rightarrow\ a = 0$,
5. $a \prec (b \vee c) \ \rightarrow\ \exists b' \prec b \exists c' \prec c \, (a \leq (b' \vee c'))$.

Definition

Let $(S, 0, \vee, \wedge, \prec)$ and $(S', 0', \vee', \wedge', \prec')$ be quasi-proximity lattices.
A **proximity relation** from S to S' is a relation $r \subseteq S \times S'$ such that

1. $\prec' \circ\, r = r = r \circ \prec$,
2. $r^{-}b$ is an ideal of S,
3. ra is a filter of S',
4. $a\ r\ 0' \ \rightarrow\ a = 0$,
5. $a\ r\ (b \vee' c) \ \rightarrow\ \exists b'\ r\ b \exists c'\ r\ c\, (a \leq b' \vee c')$.

qPxL : Category of quasi-proximity lattices and proximity relations.

Theorem (Predicative presentation of SLCF)

qPxL $:= \mathbf{Karoubi}(\text{0-DL}_{\mathsf{PX}}) \cong \mathbf{Karoubi}(\text{0-DL}_{\mathsf{F}})^{\mathsf{op}} \cong \mathsf{SLCF}^{\mathsf{op}}$.

Strong quasi-proximity lattices

Using representation $\mathsf{SLCF_D} \cong \mathbf{Karoubi}(\mathsf{0\text{-}DL_D})$, we refine qPxL to obtain a stronger structure.

Definition

Let $(S, 0, \vee, \wedge)$ and $(S', 0', \vee', \wedge')$ be 0-bounded distributive lattices. A **dual proximity relation** from S to S' is a proximity relation $r \subseteq S \times S'$ which satisfies

$$(a \wedge b)\; r\; c \;\rightarrow\; \exists a', b' \in S' \left(a\; r\; a' \;\&\; b\; r\; b' \;\&\; (a' \wedge b') \leq' c\right).$$

Proposition

Let $(S, 0, \vee, \wedge)$ and $(S', 0', \vee', \wedge')$ be 0-bounded distributive lattices. Then, the bijection between frame homomorphisms $f\colon \mathrm{Idl}(S') \to \mathrm{Idl}(S)$ and proximity relations $r\colon S \to S'$ restricts to dual frame homomorphisms and dual proximity relations.

Strong quasi-proximity lattices

0-$\mathsf{DL}_{\mathsf{DPX}}$: Category of 0-bounded distributive lattices and **dual proximity relations**.

Corollary

- 0-$\mathsf{DL}_{\mathsf{DPX}}$ *is dually equivalent to* 0-DL_{D}.
- $\mathbf{Karoubi}(\text{0-}\mathsf{DL}_{\mathsf{DPX}})$ *is dually equivalent to* $\mathbf{Karoubi}(\text{0-}\mathsf{DL}_{\mathsf{D}})$.
- $\mathbf{Karoubi}(\text{0-}\mathsf{DL}_{\mathsf{DPX}})$ *is equivalent to* $\mathsf{SLCF}_{\mathsf{D}}{}^{\mathsf{op}}$.

Definition (Objects of $\mathbf{Karoubi}(\text{0-}\mathsf{DL}_{\mathsf{DPX}})$)

A **strong quasi-proximity lattice** is a quasi-proximity lattice $(S, 0, \vee, \wedge, \prec)$ which satisfies

$$(a \wedge b) \prec c \ \rightarrow \ \exists a' \succ a \exists b' \succ b \left((a' \wedge b') \leq c\right).$$

SqPxL: Full subcategory of qPxL consisting of strong quasi-proximity lattices.

Theorem

SqPxL *is equivalent to* $\mathsf{SLCF}^{\mathsf{op}}$.

Logical characterisation

Finitary formal topologies

A **finitary formal topology** is a structure $(S, \wedge, \lhd)$, where $(S, \wedge)$ is a semilattice and $\lhd \subseteq S \times \mathrm{Fin}(S)$ is a relation satisfying

$$\frac{a \in A}{a \lhd A} \qquad \frac{a \lhd A \quad A \lhd B}{a \lhd B} \qquad \frac{a \lhd A}{a \wedge b \lhd A} \qquad \frac{a \lhd A \quad a \lhd B}{a \lhd A \wedge B}$$

$$A \lhd B \overset{\text{def}}{\iff} \forall a \in A\, (a \lhd B)\,,$$

$$A \wedge B \overset{\text{def}}{=} \{a \wedge b \mid a \in A, b \in B\}\,.$$

Theorem (Negri (1996). Stone representation)

A finitary formal topology $(S, \wedge, \lhd)$ determines a 0-bounded distributive lattice $\mathrm{L}(S, \wedge, \lhd) = (\mathrm{Fin}(S)/_{\sim}, 0, \vee, \wedge)$ where

$$A \sim B \overset{\text{def}}{=} A \lhd B \,\&\, B \lhd A,$$

$$0 \overset{\text{def}}{=} \emptyset, \qquad A \vee B \overset{\text{def}}{=} A \cup B, \qquad A \wedge B \overset{\text{def}}{=} A \wedge B.$$

Conversely, any 0-bounded distributive lattice $(S, 0, \vee, \wedge)$ can be represented in this way by a finitary formal topology on $(S, \wedge)$:

$$a \lhd_{\vee} A \overset{\text{def}}{=} a \leq \bigvee A.$$

Stably continuous covers

A **stably continuous cover** is a structure $(S, \wedge, \lhd, \prec)$ where $(S, \wedge, \lhd)$ is a finitary formal topology and $\prec$ is a relation on S s.t.

1. $\prec \circ \prec = \prec$,
2. $\downarrow_{\prec} a$ is downward closed,
3. $\uparrow^{\succ} a$ is a filter of S,
4. $a \wedge b \prec c \;\rightarrow\; \exists a' \succ a \exists b' \succ b\, ((a' \wedge b') \leq c)$,
5. $\exists b \in S\, (a \prec b \lhd A) \leftrightarrow \exists B \in \mathsf{Fin}(S)\, (a \lhd B \prec_L A)$,

where $A \prec_L B \overset{\text{def}}{\iff} \forall a \in A \exists b \in B\, (a \prec b)$.

Theorem (Stone representation (continuous version))

A stably continuous cover $\mathcal{S} = (S, \wedge, \lhd, \prec)$ *determines a strong quasi-proximity lattice* $\mathrm{SqPL}(\mathcal{S}) = (\mathrm{L}(S, \wedge, \lhd), \ll)$ *where*

$$A \ll B \overset{\text{def}}{\iff} \exists C \in \mathsf{Fin}(S)\, (A \lhd C \prec_L B)\,.$$

Conversely, any strong quasi-proximity lattice $(S, 0, \vee, \wedge, \prec)$ *can be represented in this way by* $(S, \wedge, \lhd_{\vee}, \prec)$.

Patch topology

Generated finitary formal topologies

Let $(S, \wedge)$ be a semilattice. Given any relation $\lhd_0 \subseteq S \times \mathrm{Fin}(S)$ (called **axioms**), we can inductively generate a finitary formal topology $(S, \wedge, \lhd)$ by the following rules:

$$\frac{a \in A}{a \lhd A} \qquad \frac{a \lhd A}{a \wedge b \lhd A} \qquad \frac{a \lhd_0 A \quad A \wedge b \lhd B}{a \wedge b \lhd B}$$

In this case, $(S, \wedge, \lhd)$ is said to be **generated** by $\lhd_0$.

Patch topology

Let $(S, 0, \vee, \wedge, \prec)$ be a strong quasi-proximity lattice.

Let $\mathrm{P}(S) \stackrel{\text{def}}{=} S \times S$, which is ordered by

$$(a,b) \leq (c,d) \stackrel{\text{def}}{\iff} c \leq a \mathbin{\&} b \leq d,$$

and is equipped with a semilattice structure

$$(a,b) \wedge (c,d) \stackrel{\text{def}}{\iff} (a \vee c, b \wedge d).$$

Let $\prec$ be an idempotent relation on $\mathrm{P}(S)$ defined by

$$(a,b) \prec (c,d) \stackrel{\text{def}}{\iff} c \prec a \mathbin{\&} b \prec d.$$

Let $\lhd_{\mathrm{PT}}$ be a finitary formal topology on $(\mathrm{P}(S), \wedge)$ generated by $\lhd_0$:

$$
\begin{aligned}
(a \wedge b, c) &\lhd_0 \{(a,c),(b,c)\} \\
(a, b \vee c) &\lhd_0 \{(a,b),(a,c)\} \\
(a,b) &\lhd_0 \emptyset && (b \prec a) \\
(c,d) &\lhd_0 \{(c,b),(a,d)\} && (a \prec b)
\end{aligned}
$$

$\mathrm{Patch}(S) = (\mathrm{P}(S), \wedge, \lhd_{\mathrm{PT}}, \prec)$ is a strong stably continuous cover.

Geometric interpretation

Definition

Let $\mathcal{S} = (S, \wedge, \lhd, \prec)$ be a stably continuous cover. A **model** of $\mathcal{S}$ is a filter $\alpha \subseteq S$ of $(S, \wedge)$ such that

1. $a \in \alpha \leftrightarrow \exists b \prec a \, (b \in \alpha)$,
2. $a \lhd B \,\&\, a \in \alpha \;\rightarrow\; \exists b \in B \, (b \in \alpha)$.

In other words, a model of $\mathcal{S}$ is a **rounded prime filter**.

Example

Let $(S, 0, \vee, \wedge, \prec)$ be a strong quasi-proximity lattice.
A model of $\mathrm{Patch}(S)$ can be identified with a pair (L, U) of a rounded prime ideal L and a rounded prime filter U on S such that

- $L \cap U = \emptyset$ (disjoint),
- $a \prec b \;\rightarrow\; a \in L$ or $b \in U$ (located).

Example (Dedekind cuts)

Let $S = (\mathbb{Q}^{\geq 0}, 0, \max, \min, <)$ be the upper half line.
A model of $\mathrm{Patch}(S)$ can be identified with a Dedekind cut.

Universal property of $\mathrm{Patch}(S)$

Definition

A strong quasi-proximity lattice $(S, 0, \vee, \wedge, \prec)$ is **regular** if

$$\forall x, a, b \in S \, [a \prec b \;\rightarrow\; \exists c \in S \, (c \wedge a = 0 \;\&\; x \prec c \vee b)] \, .$$

Lemma

For any strong quasi-proximity lattice S, $\mathrm{Patch}(S)$ determines a regular strong quasi-proximity lattice $\mathrm{SqPL}(\mathrm{Patch}(S))$.

Definition

A proximity relation $r \colon S \to S'$ is a **perfect map** if

$$a \prec' b \;\rightarrow\; \exists c \in r^{-} b \, (r^{-} a \subseteq {\downarrow_{\prec}} c).$$

Theorem

Let $(S, 0, \vee, \wedge, \prec)$ be a strong quasi-proximity lattice. There exists a perfect map $\varepsilon \colon \mathrm{SqPL}(\mathrm{Patch}(S)) \to S$ such that for any perfect map $r \colon S' \to S$ where S' is regular, there exists a unique perfect map $\widetilde{r} \colon S' \to \mathrm{SqPL}(\mathrm{Patch}(S))$ such that

$$\begin{array}{ccc} S' & \xrightarrow{\;\widetilde{r}\;} & \mathrm{SqPL}(\mathrm{Patch}(S)) \\ & {\scriptstyle r}\searrow & \downarrow{\scriptstyle \varepsilon} \\ & & S \end{array} .$$