More on the Intuitionistic Borel Hierarchy

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In classical theory, the following result is well known:

The set of all eventually zero sequences is $\mathbf{\Sigma}^0_2$-complete.

- i.e., $FIN = \{x \in \mathbb{N}^{\mathbb{N}} : \exists n \forall m > n.\ x(m) = 0\}$ is $\mathbf{\Sigma}^0_2$-complete.

Here, for $A, B \subseteq \mathbb{N}^{\mathbb{N}}$:

- A is reducible to B (written $A \leq B$) iff

$$\exists \text{ continuous } \varphi\ \forall x\ (x \in A \iff \varphi(x) \in B)$$

- A is $\mathbf{\Gamma}$-complete if A is a $\leq$-greatest element among $\mathbf{\Gamma}$ sets.

The same result holds if "continuous" is changed to "computable".

$\mathbf{\Sigma^0_2}$-completeness of FIN is "*trivial*" to those of us familiar with classical theory, but it is not necessarily true in intuitionistic mathematics.

Theorem (Veldman 2008)

In a certain intuitionistic system,
$FIN = \{x \in \mathbb{N}^{\mathbb{N}} : \exists n \forall m > n.\ x(m) = 0\}$ is not $\mathbf{\Sigma^0_2}$-complete.

- One of the key intuitionistic principles assumed by Veldman is Brouwer's continuity principle.
- The details of Veldman's assumptions are not explained here.

Veldman also showed other results that contradict those in the classical theory.

Theorem (Veldman 2022)

In a certain intuitionistic system, the set of (the codes of) trees which are ill-founded w.r.t. the Kleene-Brouwer ordering is not $\mathbf{\Sigma^1_1}$-complete.

Objective

- We clarify that Veldman's results are valuable not only in intuitionistic mathematics, but also in classical math.
  - ▷ Veldman's results can be understood as results about "Levin reducibility" in classical math.
- Further refine Veldman's results using techniques in classical math.
  - ▷ Veldman's insights provide a new refinement to the classical arithmetical/Borel hierarchy.

What is "Levin reducibility"?

- Introduced by Leonid Levin in 1973.
- Levin's 1973 paper is a monumental paper in complexity theory that showed NP-completeness of SAT, but what Levin was really dealing with was "Levin reducibility" between search problems.

- A search problem is a binary relation $R \subseteq \Sigma^* \times \Sigma^*$.
- Any y satisfying $R(x,y)$ is called a witness for $x \in |R|$, where $|R| = \{x : \exists y R(x,y)\}$.

Definition (Levin 1973)

For a complexity class C and search problems A and B, A is C-Levin reducible to B if there exist C-functions φ, ℓ, r such that for any $x, y, z \in \Sigma^*$ the following holds:

1. $x \in |A|$ if and only if $\varphi(x) \in |B|$.
2. If y is a witness for $x \in |A|$ then $r(x,y)$ is a witness for $\varphi(x) \in |B|$.
3. If z is a witness for $\varphi(x) \in |B|$ then $\ell(x,z)$ is a witness for $x \in |A|$.

- φ is a reduction for $|A| \leq |B|$
- r is a realizer for "$x \in |A| \implies \varphi(x) \in |B|$"
- ℓ is a realizer for "$x \in |A| \impliedby \varphi(x) \in |B|$"

The “standard model” of intuitionistic math that satisfies Veldman’s assumptions would be those based on Kleene’s second algebra K_2.

Thee main “algebras” $(\mathbb{A}, \mathbb{A}_{eff}, *)$:

- Kleene’s first algebra K_1
  - The algebra of computability on natural numbers.
  - $\mathbb{A} = \mathbb{A}_{eff} = \mathbb{N}$ and $e * x = \varphi_e(x)$
  - where φ_e is the eth partial computable function on $\mathbb{N}$.
- Kleene’s second algebra K_2
  - The algebra of continuity on infinite strings.
  - $\mathbb{A} = \mathbb{A}_{eff} = \mathbb{N}^{\mathbb{N}}$, and $e * x = \psi_e(x)$
  - where ψ_e is the partial continuous function on $\mathbb{N}^{\mathbb{N}}$ coded by e.
- Kleene-Vesley algebra KV
  - The algebra of computability on infinite strings.
  - $\mathbb{A} = \mathbb{N}^{\mathbb{N}}$, $\mathbb{A}_{eff}$ = computable strings, and $e * x = \psi_e(x)$

Let $(\mathbb{A}, \mathbb{A}_{eff}, *)$ be a relative pca, i.e, K_1, K_2, KV or so.

- An represented space is a pair of a set X and a partial surjection $\delta :\subseteq \mathbb{A} \to X$.
  - ▷ If $\delta(p) = x$ then p is called a name of $x \in X$.
- A function $f : X \to Y$ is realizable if there exists $a \in A_{eff}$ such that if p is a name of $x \in X$ then $a * p$ is a name of $f(x) \in Y$

A represented space is also known as a modest set.

- Fact: The category of represented spaces and realizable functions is a locally cartesian closed category with NNO, whose internal logic corresponds to the realizability interpretation.
- The standard model of intuitionistic mathematics satisfying Veldman's assumptions would be the category $Rep(K_2)$ of K_2-represented spaces (or the realizability topos $RT(K_2)$ over K_2).

In the category of represented spaces:

- A formula is interpreted as something like a "witness-search problem (or a realizer-search problem)"

Example: The type $\mathbb{N}^{\mathbb{N}}$ formula "$\varphi(x) \equiv \exists n \forall m \geq n.\ x(m) = 0$" is interpreted as a subobject $FIN \rightarrowtail \mathbb{N}^{\mathbb{N}}$ such that

- the underlying set is $\{x \in \mathbb{N}^{\mathbb{N}} : \exists n \forall m \geq n.\ x(m) = 0\}$
- a name of $x \in FIN$ is a pair of $\langle x, n \rangle$, where n is an existential witness.

Fact: Every subobject of X has a representative of the following form:

- an underlying set A is a subset of X
- a name of $x \in A$ is the pair of a name p of $x \in X$ and some $q \in \mathbb{A}$. This q is considered as a "witness".

Roughly speaking:

- A subobject is a subset with witnesses.
- A regular subobject is a subset without witnesses.

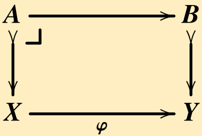
Recall: for $A, B \subseteq \mathbb{N}^{\mathbb{N}}$, A is reducible to B (written $A \leq B$) iff

$$\exists \text{ continuous } \varphi \ \forall x \ (x \in A \iff \varphi(x) \in B)$$

That is, $A = \varphi^{-1}[B]$.

Its categorical version would be something like:

Def: Let X, Y be objects in a category C having pullbacks.
A mono $A \overset{\alpha}{\rightarrowtail} X$ is reducible to $B \overset{\beta}{\rightarrowtail} Y$ if $A \overset{\alpha}{\rightarrowtail} X$ is a pullback of $B \overset{\beta}{\rightarrowtail} Y$ along some morphism $\varphi : X \to Y$.



When this notion is interpreted in the category of represented spaces, we obtain (computable/continuous) Levin reducibility.

A subobject $A \rightarrowtail X$ $\approx$ a subset with witnesses:

- an underlying set A is a subset of X
- a name for $x \in A$ is a pair of a name $x \in X$ and a witness.

For subobjects $A, B \rightarrowtail X$, A is Levin reducible to B if there exist a morphism $\varphi \colon X \to X$ and realizable functions ℓ, r such that for any x, y, z the following holds:

1. $x \in A$ if and only if $\varphi(x) \in B$.
2. If $\dot{x}$ is a name of $x \in X$ and $\dot{y}$ is a name of witness for $x \in A$ then $r(\dot{x}, \dot{y})$ is a witness for $\varphi(x) \in B$.
3. If $\dot{x}$ is a name of $x \in X$ and $\dot{z}$ is a name of $\varphi(x) \in B$ then $\ell(\dot{x}, \dot{z})$ is a witness for $x \in A$.

- φ is a reduction for $A \leq B$ (on underlying sets)
- r is a realizer for "$x \in A \implies \varphi(x) \in B$"
- ℓ is a realizer for "$x \in A \impliedby \varphi(x) \in B$"

Theorem (Veldman 2008)

In a certain intuitionistic system,
$FIN = \{x \in \mathbb{N}^{\mathbb{N}} : \exists n \forall m > n.\, x(m) = 0\}$ is not Σ^0_2-complete.

This is because:

- The witness-search problem for FIN is not Levin-complete among the witness-search problems for Σ^0_2 formulas even in classical mathematics.

Theorem (Veldman 2022)

In a certain intuitionistic system, the set $IF(KB)$ of trees which are ill-founded w.r.t. the Kleene-Brouwer ordering is not Σ^1_1-complete.

This is because:

- The witness-search problem for $IF(KB)$ is not Levin-complete among the witness-search problems for Σ^1_1 formulas even in classical mathematics.

- Our observation shows that Veldman's seemingly strange results can also be understood by classical mathematicians as results regarding Levin reducibility for witness-search problems.
- However, simply interpreting previous results in a different context is of course not very interesting.
  - It is interesting when the interpretation leads to a truly new discovery.
- One of our new discoveries is that the "three layers" of $\mathbf{\Sigma}^0_2$ formulas w.r.t Levin reducibility (in classical mathematics).

Σ^0_2 subobject $\approx$ Σ^0_2 subset with existential witnesses

Classification of Σ^0_2 formulas $\exists n \forall m \varphi(n,m)$:

- (Unique Witness) "$\exists n \forall m \varphi(n,m) \leftrightarrow \exists! n \forall m \varphi(n,m)$"
- (Increasing Witness) "$k \leq n$ and $\forall m \varphi(k,m) \to \forall m \varphi(n,m)$"

Definition:

- A u.w. Σ^0_2 subobject is a subobject defined by a Σ^0_2 formula satisfying (Unique Witness).
- A i.w. Σ^0_2 subobject is a subobject defined by a Σ^0_2 formula satisfying (Increasing Witness).

Example:

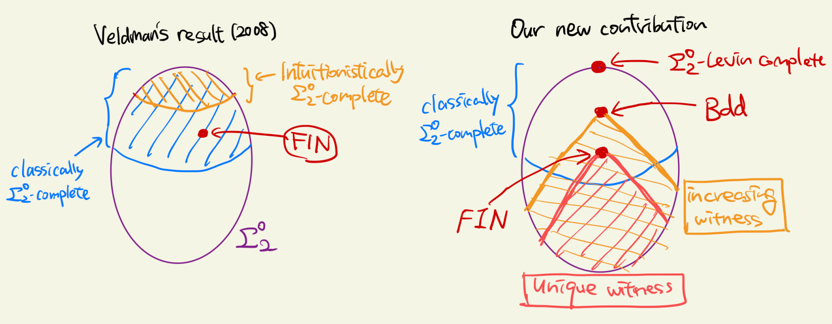
- $FIN = \{x \in \mathbb{N}^{\mathbb{N}} : \exists n \forall m \geq n.\ x(m) = 0\}$ is a u.w. Σ^0_2 subobj. of $\mathbb{N}^{\mathbb{N}}$.
- $Bdd = \{x \in \mathbb{N}^{\mathbb{N}} : \exists n \forall m.\ x(m) < n\}$ is an i.w. Σ^0_2 subobject of $\mathbb{N}^{\mathbb{N}}$.
- Every u.w. Σ^0_2 subobject is an i.w. Σ^0_2 subobject.

Classification of Σ^0_2 formulas $\exists n \forall m \varphi(n,m)$:

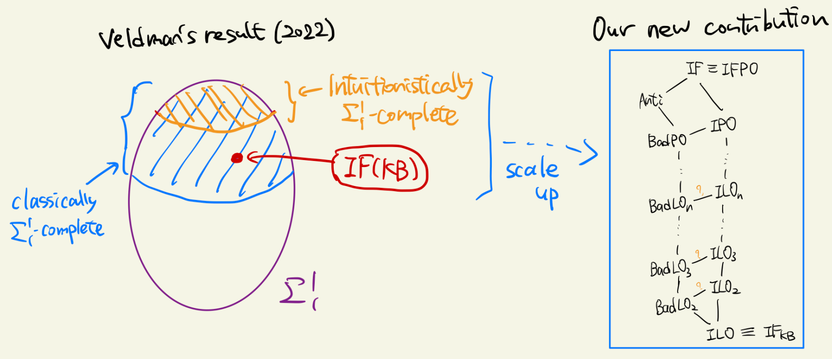
- (Unique Witness) "$\exists n \forall m \varphi(n,m) \leftrightarrow \exists! n \forall m \varphi(n,m)$"
- (Increasing Witness) "$k \leq n$ and $\forall m \varphi(k,m) \to \forall m \varphi(n,m)$"

Theorem

- FIN is Levin complete among u.w. Σ^0_2 subobjects.
- Bdd is Levin complete among i.w. Σ^0_2 subobjects.
- There is a Levin complete subobject among all Σ^0_2 subobjects.
- They have different Levin reducibility degrees from each other.



- ***FIN*** is Levin complete among u.w. $\mathbf{\Sigma^0_2}$ subobjects.
- ***Bdd*** is Levin complete among i.w. $\mathbf{\Sigma^0_2}$ subobjects.
- There is a Levin complete subobject among all $\mathbf{\Sigma^0_2}$ subobjects.


Veldman's result (2022)
Intuitionistically
Σ¹₁-complete
IF(KB)
classically
Σ¹₁-complete
Σ¹₁
scale
up
Our new contribution
IF ≡ IFPO
Anti
IPO
BadPO
BadLOₙ
ILOₙ
BadLO₃
ILO₃
BadLO₂
ILO₂
ILO ≡ IF_KB