Fibred sets within a predicative and constructive variant of Hyland's effective topos

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The two levels are linked by a **setoid model** of emTT in mTT.

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Recall that Feferman's $\widehat{\mathsf{ID}_1}$ is classical but strictly predicative, while Aczel's CZF is constructive and predicative.

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1. $\mathcal{C}_r$: the category of definable classes of natural numbers and (restrictions of) computable functions between them (represented by numerals).
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So we started from a square of embeddings over $\mathcal{C}_r$

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The top right corner can be filled by shifting to fibrations and considering the codomain fibration of $\mathbf{pEff}$, however **how can we fill the top left corner?**

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These conditions, expressed here informally, can be expressed in precise categorical terms.

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These objects and arrows, together with the compositions and identities inherited from $\mathbf{Col}^r(\mathsf{A})$, define a category $\mathbf{pEff}^c_{col}(\mathsf{A}, \mathsf{R})$.

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If one assumes the axiom of countable choice, we get that $\mathbf{pEff}^c_{col}(\mathsf{A},\mathsf{R})$ is equivalent to $\mathbf{pEff}^c/(\mathsf{A},\mathsf{R})$.

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Theorem

$\mathbf{pEff}^c_{set}(A, R)$ *is a* **locally cartesian closed list-arithmetic pretopos** *and its embedding in* $\mathbf{pEff}^c_{col}(A, R)$ *preserves all the structure.*

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The right notion of dependent proposition in this framework should keep track of some computational information:

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Moreover $(\mathsf{P}, \rho) \leq (\mathsf{P}', \rho')$ means $\mathsf{P} \leq \mathsf{P}'$ in $\mathbf{Prop}^r(\mathsf{A})$.

Theorem

Dependent propositions with the order just defined form a Heyting prealgebra $\mathbf{pEff}^c_{prop}(\mathsf{A}, \mathsf{R})$*. Moreover,* $\mathbf{pEff}^c_{prop}(\mathsf{A}, \mathsf{R})$ *can be embedded in* $\mathbf{pEff}^c_{col}(\mathsf{A}, \mathsf{R})$ *by the assignment*

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Similarly, one defines the Heyting prealgebra $\mathbf{pEff}^c_{prop_s}(\mathsf{A},\mathsf{R})$ of dependent small propositions (using $\mathbf{Prop}^r_s$ instead of $\mathbf{Prop}^r$) and shows that it embeds in $\mathbf{pEff}^c_{set}(\mathsf{A},\mathsf{R})$.

The picture over (A, R)

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\mathbf{pEff}^c_{set}(\mathsf{A}, \mathsf{R}) \arrow[r, hook] & \mathbf{pEff}^c_{col}(\mathsf{A}, \mathsf{R}) \\
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If we consider the formalization in $CZF + uREA + RDC$, $\mathbf{pEff}^c$ models MF + inductive formal topologies + coinductive topological definitions thanks to (Maietti, M., Rathjen 2022) and all coinductive definitions thanks to (PhD thesis Sabelli 2023)

Thanks for your attention!