Recent results in constructive reverse mathematics

Takako Nemoto

Tohoku University

CCC 2023, Kyoto

Table of contents

What is reverse mathematics?

Something often in usual mathematics

- From the assumption A, the conclusion B is derived. (Weaken A as much as possible)
- Sometimes, the conclusion B also derive the assumption A (A and B are equivalent)

- If we take a set comprehension axiom as A, then we can classify usual theorems in mathematics by set comprehension axioms.

Friedman-Simpson's reverse math

- Reverse mathematics with classical logic, using systems of second order arithmetic
- "The reverse mathematics" (?)

Language of 2nd order arithmetic and base systems

Language $L_2(\exp)$

- Constants $0, 1$
- Binary functions $+$, $\cdot$, $\exp$
- Binary relations $<$, $\in$
- 1st order variables x, y, z,...
- 2nd order variables X, Y, Z,...
- 1st order equality $=$

The systems RCA_0^* and RCA_0

Basic arithmetic

Successor	$n+1 \neq 0, \quad n+1 = m+1 \to n = m,$
Addition	$n+0 = n, \quad n+(m+1) = (n+m)+1,$
Multiplication	$n \cdot 0 = 0, \quad n \cdot (m+1) = n \cdot m + n,$
Order	$\neg m < 0, \quad m < n+1 \leftrightarrow m \leq n,$
Exponentiation	$\exp(n,0) = 1, \quad \exp(n,m+1) = \exp(n,m) \cdot n.$

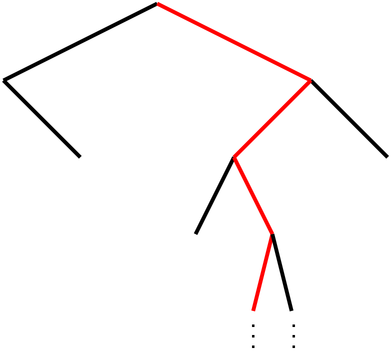
Σ^0_0-IND $\;A(0) \wedge \forall n(A(n) \to A(n+1)) \to \forall n A(n)$, for $A \in \Sigma^0_0$.

Δ^0_1-CA $\;\forall n(A(n) \leftrightarrow B(n)) \to \exists X \forall n(A(n) \leftrightarrow n \in X)$,
for $A \in \Sigma^0_1$ and $B \in \Pi^0_1$.

The famous base system RCA_0 can be defined by $\mathsf{RCA}_0^* + \Sigma^0_1$-IND.

Weak König's lemma

- A binary *tree* T is a subset of $\{0,1\}^*$ closed under initial segments.
- A binary tree T is *infinite* if $\forall n \exists s \in T(|s| = n)$, where $|s|$ is the length of a finite tree s.
- A *path* of a binary tree T is a function α s.t. $\forall n(\overline{\alpha}n \in T)$, where $\overline{\alpha}n$ is a finite sequence $\langle \alpha(0), \ldots, \alpha(n-1) \rangle$.
- *Weak König's Lemma (*WKL*)*: "Every infinite binary tree has a path"



Some results from Friedman-Simpson reverse math

TFAE over RCA_0 ([8])

- Weak König's lemma: Every infinite binary tree has a path.
- Heine-Borel's covering theorem
- Every continuous function on $[0,1]$ is uniformly continuous.
- Every continuous function on $[0,1]$ has infimum.
- Every continuous function on $[0,1]$ has a point attaining the infimum.
- Every continuous function on $[0,1]$ is Riemann integrable.
- Gödel's completeness theorem
- Every countable ring contains a prime ideal.
- Brouwer's fixed point theorem
- Peano's existence theorem for solution of ODE.
- Separable Hahn-Banach theorem
- Π^0_1 axiom of choice

$\mathsf{WKL}_0 = \mathsf{RCA}_0 +$Weak König's lemma

Intuitionistic logic and constructive reverse math

Usual mathematics

Based on classical logic

Constructive mathematics

Based on intuitionistic logic

Constructive reverse mathematics

A mathematical theorem are characterized with a combination of

- choice principle (asserting the existence of a function)
- logical principles

which are necessary and sufficient to prove it.

Base theory EL_0^*

Language L_{EL}

- Constant 0
- Function symbols for all elementary functions S, f,.....
- Application symbol AP
- Abstraction operator λ
- bdd. $\boldsymbol{\mu}$ operator $\boldsymbol{\mu}$
- 1st order $=$
- 1st order variables x, y, z,...
- 2nd order variables α, β, γ...

System EL_0^*

Successor $\neg S0 = 0$

Defining equations for elementary functions $x{+}0 = x$, $x{+}Sy{=}S(x{+}y)$...

Σ_0^0 induction $A(0) \wedge \forall n(A(n) \to A(n+1)) = \forall n A(n)$, for $A \in \Sigma_0^0$

λ conversion $(\lambda x.t)s = t[x/s]$

Bdd. $\boldsymbol{\mu}$ operator $\boldsymbol{\mu}(t, \varphi, t') =$ "the least $k{\le}t'$ s.t. $\varphi(k) = 0$ if exists, or t'"

$\mathrm{QF\text{-}AC}^{00}$ $\forall x \exists y A(x, y) \to \exists \alpha \forall x A(x, \alpha(x))$, for $A(x, y) \in \Pi_0^0$

EL_0 can be defined by $\mathrm{EL}_0^* + \Sigma_1^0\text{-IND}$

RCA_0 and EL_0

RCA_0

- Classical logic
- Set based language
- Allowing primitive recursion

EL_0

- Intuitionistic logic
- Function based language
  - It yields $A \vee \neg A$ for Σ^0_0 formulae
- Allowing primitive recursion

Conservation results ([7])

For any Π^0_2 sentence A in $L_2(\exp)$,

- $\mathsf{RCA}^*_0 \vdash A$ yields $\mathsf{EL}^*_0 \vdash A$
- $\mathsf{RCA}_0 \vdash A$ yields $\mathsf{EL}_0 \vdash A$

Characterizing WKL

Theorem (Essentially by [1])

The following are equivalent over EL_0^*

1. WKL
2. $\Pi^0_1\text{-}\mathrm{AC}^\vee + \Sigma^0_1\text{-}\mathrm{DML}$
    - $\Pi^0_1\text{-}\mathrm{AC}^\vee$:
      $\forall x(A(x) \vee B(x)) \to \exists \alpha \forall x((\alpha(x) = 0 \to A(x)) \wedge (\alpha(x) \neq 0 \to B(x)))$
      for $A, B \in \Pi^0_1$
    - $\Sigma^0_1\text{-}\mathrm{DML}$: $\neg(A \wedge B) \to (\neg A \vee \neg B)$ for $A, B \in \Sigma^0_1$

De Morgan's law

- $\neg(A \lor B) \leftrightarrow (\neg A \land \neg B)$
  - $\surd$ $\neg(A \lor B) \to \neg A \land \neg B$
  - $\surd$ $\neg A \land \neg B \to \neg(A \lor B)$
- $\neg(A \land B) \leftrightarrow (\neg A \lor \neg B)$
  - $\neg(A \land B) \to \neg A \lor \neg B$
  - $\surd$ $\neg A \lor \neg B \to \neg(A \land B)$

De Morgan's law

- $\neg(A \vee B) \leftrightarrow (\neg A \wedge \neg B)$
  - $\surd$ $\neg(A \vee B) \to \neg A \wedge \neg B$
  - $\surd$ $\neg A \wedge \neg B \to \neg(A \vee B)$
- $\neg(A \wedge B) \leftrightarrow (\neg A \vee \neg B)$
  - $\neg(A \wedge B) \to \neg A \vee \neg B \leftarrow$ does not hold in EL_0^*
  - $\surd$ $\neg A \vee \neg B \to \neg(A \wedge B)$

De Morgan's law

- $\neg(A \vee B) \leftrightarrow (\neg A \wedge \neg B)$
  - $\surd$ $\neg(A \vee B) \to \neg A \wedge \neg B$
  - $\surd$ $\neg A \wedge \neg B \to \neg(A \vee B)$
- $\neg(A \wedge B) \leftrightarrow (\neg A \vee \neg B)$
  - $\neg(A \wedge B) \to \neg A \vee \neg B \leftarrow$ does not hold in EL_0^*
  - $\surd$ $\neg A \vee \neg B \to \neg(A \wedge B)$

Some generalization

- $\forall x(\neg(\exists i < x)(A(i)) \leftrightarrow (\forall i < x)(\neg A(i)))$
  - $\surd$ $\forall x(\neg(\exists i < x)A(i) \to (\forall i < x)\neg A(i))$
  - $\surd$ $\forall x((\forall i < x)\neg A(i) \to \neg(\exists i < x)A(i))$
- $\forall x(\neg(\forall i < x)A(i) \leftrightarrow (\exists i < x)\neg A(i))$
  - $\forall x(\neg(\forall i < x)A(i) \to (\exists i < x)\neg A(i))$
  - $\surd$ $\forall x((\exists i < x)\neg A(i) \to \neg(\forall i < x)A(i))$

De Morgan's law

- $\neg(A \vee B) \leftrightarrow (\neg A \wedge \neg B)$
  - $\surd$ $\neg(A \vee B) \to \neg A \wedge \neg B$
  - $\surd$ $\neg A \wedge \neg B \to \neg(A \vee B)$
- $\neg(A \wedge B) \leftrightarrow (\neg A \vee \neg B)$
  - $\neg(A \wedge B) \to \neg A \vee \neg B \leftarrow$ does not hold in EL_0^*
  - $\surd$ $\neg A \vee \neg B \to \neg(A \wedge B)$

Some generalization

- $\forall x(\neg(\exists i < x)(A(i)) \leftrightarrow (\forall i < x)(\neg A(i)))$
  - $\surd$ $\forall x(\neg(\exists i < x)A(i) \to (\forall i < x)\neg A(i))$
  - $\surd$ $\forall x((\forall i < x)\neg A(i) \to \neg(\exists i < x)A(i))$
- $\forall x(\neg(\forall i < x)A(i) \leftrightarrow (\exists i < x)\neg A(i))$
  - $\forall x(\neg(\forall i < x)A(i) \to (\exists i < x)\neg A(i)) \leftarrow$ does not hold in EL_0^*
  - $\surd$ $\forall x((\exists i < x)\neg A(i) \to \neg(\forall i < x)A(i))$

Schemata we consider

For a class Γ (Σ^0_k or Π^0_k) of formulae, we consider the following schemata:

- Γ-DML: $\neg(A \wedge B) \to \neg A \vee \neg B$, for $A, B \in \Gamma$
- Γ-GDML: $\forall x(\neg(\forall i < x)A(i) \to (\exists i < x)\neg A(i))$ for $A(i) \in \Gamma$
- Γ-WGDML: $\forall x(\neg(\forall i < x)A(i) \to \neg\neg(\exists i < x)\neg A(i))$ for $A(i) \in \Gamma$

Schemata we consider

For a class Γ (Σ^0_k or Π^0_k) of formulae, we consider the following schemata:

- $\Gamma\text{-DML}$: $\neg(A \wedge B) \to \neg A \vee \neg B$, for $A, B \in \Gamma$
- $\Gamma\text{-GDML}$: $\forall x(\neg(\forall i < x)A(i) \to (\exists i < x)\neg A(i))$ for $A(i) \in \Gamma$
- $\Gamma\text{-WGDML}$: $\forall x(\neg(\forall i < x)A(i) \to \neg\neg(\exists i < x)\neg A(i))$ for $A(i) \in \Gamma$

Easy observation

Over EL^*_0, the following holds:

1. $\Gamma\text{-GDML}$ yields $\Gamma\text{-DML}$
2. $\Gamma\text{-GDML}$ yields $\Gamma\text{-WGDML}$

Schemata we consider

For a class Γ (Σ^0_k or Π^0_k) of formulae, we consider the following schemata:

- $\Gamma\text{-DML}$: $\neg(A \wedge B) \to \neg A \vee \neg B$, for $A, B \in \Gamma$
- $\Gamma\text{-GDML}$: $\forall x(\neg(\forall i < x)A(i) \to (\exists i < x)\neg A(i))$ for $A(i) \in \Gamma$
- $\Gamma\text{-WGDML}$: $\forall x(\neg(\forall i < x)A(i) \to \neg\neg(\exists i < x)\neg A(i))$ for $A(i) \in \Gamma$

Easy observation

Over EL^*_0, the following holds:

1. $\Gamma\text{-GDML}$ yields $\Gamma\text{-DML}$
2. $\Gamma\text{-GDML}$ yields $\Gamma\text{-WGDML}$

Variation of WKL

- WKL: Every infinite binary tree has a path.
- $\mathrm{WKL!!}$: Every infinite binary tree T which has at most one paths, i.e., if there are two paths then they are identical, has a path.
- $\mathrm{dn\text{-}WKL}$: If T is an infinite binary tree, then $\neg\neg$(T has a path)

Schemata we consider

For a class Γ (Σ^0_k or Π^0_k) of formulae, we consider the following schemata:

- $\Gamma\text{-DML}$: $\neg(A \wedge B) \to \neg A \vee \neg B$, for $A, B \in \Gamma$
- $\Gamma\text{-GDML}$: $\forall x(\neg(\forall i < x)A(i) \to (\exists i < x)\neg A(i))$ for $A(i) \in \Gamma$
- $\Gamma\text{-WGDML}$: $\forall x(\neg(\forall i < x)A(i) \to \neg\neg(\exists i < x)\neg A(i))$ for $A(i) \in \Gamma$

Easy observation

Over EL_0^*, the following holds:

1. $\Gamma\text{-GDML}$ yields $\Gamma\text{-DML}$
2. $\Gamma\text{-GDML}$ yields $\Gamma\text{-WGDML}$

Variation of WKL

- WKL: Every infinite binary tree has a path.
- $\mathrm{WKL!!}$: Every infinite binary tree T which has at most one paths, i.e., if there are two paths then they are identical, has a path.
- $\mathrm{dn\text{-}WKL}$: If T is an infinite binary tree, then $\neg\neg(T$ has a path$)$

Observation

$\mathrm{WKL} \Rightarrow \mathrm{WKL!!} \Rightarrow \mathrm{dn\text{-}WKL}$ ([6])

Something I know so far

Let $\Delta(\Gamma)$ be the smallest class containing Γ and closed under $\wedge, \vee, \to, \neg, \forall x < t, \exists x < t$

In the presence of an appropriate induction

- $\mathsf{EL}_0^* + \Delta(\Sigma_k^0)\text{-IND} \vdash \Sigma_k^0\text{-DML} \Leftrightarrow \Sigma_k^0\text{-GDML}$
- $\mathsf{EL}_0^* + \Delta(\Pi_k^0)\text{-IND} \vdash \Pi_k^0\text{-DML} \Leftrightarrow \Pi_k^0\text{-GDML}$
- $\mathsf{EL}_0^* + \Pi_k^0\text{-IND} \vdash \Sigma_k^0\text{-WGDML}.$

Something I know so far

Let $\Delta(\Gamma)$ be the smallest class containing Γ and closed under $\wedge, \vee, \rightarrow, \neg, \forall x < t, \exists x < t$

In the presence of an appropriate induction

- $\mathsf{EL}_0^* + \Delta(\Sigma_k^0)\text{-IND} \vdash \Sigma_k^0\text{-DML} \Leftrightarrow \Sigma_k^0\text{-GDML}$
- $\mathsf{EL}_0^* + \Delta(\Pi_k^0)\text{-IND} \vdash \Pi_k^0\text{-DML} \Leftrightarrow \Pi_k^0\text{-GDML}$
- $\mathsf{EL}_0^* + \Pi_k^0\text{-IND} \vdash \Sigma_k^0\text{-WGDML}$.

In the presence of WKL variants or some choice principles

- $\mathsf{EL}_0^* + \mathrm{WKL} \vdash \Sigma_1^0\text{-GDML}$
- $\mathsf{EL}_0^* + \mathrm{WKL!!} \vdash \Pi_1^0\text{-GDML}$
- $\mathsf{EL}_0^* + \text{dn-WKL} \vdash \Sigma_1^0\text{-WGDML}$

Something I know so far

Let $\Delta(\Gamma)$ be the smallest class containing Γ and closed under $\wedge, \vee, \rightarrow, \neg, \forall x < t, \exists x < t$

In the presence of an appropriate induction

- $\mathsf{EL}_0^* + \Delta(\Sigma_k^0)\text{-IND} \vdash \Sigma_k^0\text{-DML} \Leftrightarrow \Sigma_k^0\text{-GDML}$
- $\mathsf{EL}_0^* + \Delta(\Pi_k^0)\text{-IND} \vdash \Pi_k^0\text{-DML} \Leftrightarrow \Pi_k^0\text{-GDML}$
- $\mathsf{EL}_0^* + \Pi_k^0\text{-IND} \vdash \Sigma_k^0\text{-WGDML}.$

In the presence of WKL variants or some choice principles

- $\mathsf{EL}_0^* + \mathrm{WKL} \vdash \Sigma_1^0\text{-GDML}$
- $\mathsf{EL}_0^* + \mathrm{WKL!!} \vdash \Pi_1^0\text{-GDML}$
- $\mathsf{EL}_0^* + \text{dn-WKL} \vdash \Sigma_1^0\text{-WGDML}$

Σ_k^0 and Π_k^0

- $\mathsf{EL}_0^* + \Sigma_1^0\text{-DML} \vdash \Pi_1^0\text{-DML}$ ([4])
- $\mathsf{EL}_0^* + \Sigma_1^0\text{-GDML} \vdash \Pi_1^0\text{-GDML}$ (Kawai)
- $\mathsf{EL}_0^* + \Sigma_{n+1}^0\text{-GDML} + \Sigma_n^0\text{-DNE} + \Sigma_n^0\text{-IND} \vdash \Pi_{n+1}^0\text{-GDML}$

Facts

- $\mathsf{EL}_0 \nvdash \Pi^0_1\text{-IND}$ ([2]), $\mathsf{EL}_0^* + \Pi^0_1\text{-IND} \nvdash \Sigma^0_1\text{-IND}$ ([9]).
- $\mathsf{EL}_0^* + \Pi^0_1\text{-IND}$ proves $\neg\neg\Sigma^0_1\text{-IND}$ ([9]), i.e., for each $A(x) \in \Sigma^0_1$,
$$\neg\neg A(0) \wedge \forall x(\neg\neg A(x) \to \neg\neg A(x+1)) \to \forall x \neg\neg A(x),$$
- EL_0^* proves, for each $A(i,j) \in \Sigma^0_1$,
$$\forall x((\forall i)\exists j A(i,j) \leftrightarrow \exists y(\forall i < x)(\forall j < y)A(i,j))$$
- $\mathsf{EL}_0^* + \Pi^0_1\text{-IND}$ proves, for $A(i,j) \in \Sigma^0_1$,
$$\forall x((\forall i < x)\neg\neg\exists j A(i,j) \leftrightarrow \neg\neg\exists n(\forall i < x)(\forall j < n)A(i,j))$$

Facts

- $\mathsf{EL}_0 \nvdash \Pi^0_1\text{-IND}$ ([2]), $\mathsf{EL}^*_0 + \Pi^0_1\text{-IND} \nvdash \Sigma^0_1\text{-IND}$ ([9]).
- $\mathsf{EL}^*_0 + \Pi^0_1\text{-IND}$ proves $\neg\neg\Sigma^0_1\text{-IND}$ ([9]), i.e., for each $A(x) \in \Sigma^0_1$,
$$\neg\neg A(0) \wedge \forall x(\neg\neg A(x) \to \neg\neg A(x+1)) \to \forall x \neg\neg A(x),$$
- EL^*_0 proves, for each $A(i,j) \in \Sigma^0_1$,
$$\forall x((\forall i)\exists j A(i,j) \leftrightarrow \exists y(\forall i < x)(\forall j < y)A(i,j))$$
- $\mathsf{EL}^*_0 + \Pi^0_1\text{-IND}$ proves, for $A(i,j) \in \Sigma^0_1$,
$$\forall x((\forall i < x)\neg\neg\exists j A(i,j) \leftrightarrow \neg\neg\exists n(\forall i < x)(\forall j < n)A(i,j))$$

Theorem

$\mathsf{EL}^*_0 + \Pi^0_1\text{-IND}$ proves $\Sigma^0_1\text{-WGDML}$.

(Proof)
$$\begin{aligned}
&\neg(\forall i < x)\exists j B(i,j) \to \neg\neg(\exists i < x)\neg\exists j B(i,j) \\
\iff\ &\neg(\exists i < x)\neg\exists j B(i,j) \to \neg\neg(\forall i < x)\exists j B(i,j) \\
\iff\ &(\forall i < x)\neg\neg\exists j B(i,j) \to \neg\neg(\forall i < x)\exists j B(i,j) \\
\underset{\Pi^0_1\text{-IND}}{\iff}\ &\underbrace{(\forall i < x)\neg\neg\exists j B(i,j) \to \neg\neg\exists y(\forall i < x)(\exists j < y)B(i,j)}_{\Pi^0_1\text{-IND yields this}}
\end{aligned}$$

Induction, DML and GDML

Theorem

$\mathsf{EL}_0^* + \Delta(\Sigma_k^0)\text{-IND} + \Sigma_k^0\text{-DML} \vdash \Sigma_k^0\text{-GDML}$

Induction, DML and GDML

Theorem

$\mathsf{EL}_0^* + \Delta(\Sigma_k^0)\text{-}\mathrm{IND} + \Sigma_k^0\text{-}\mathrm{DML} \vdash \Sigma_k^0\text{-}\mathrm{GDML}$

(Idea for the proof)
Assume $\neg(\forall i < x)\exists j A(i,j)$ for $A(i,j) \in \Pi_{k-1}^0$.

- $\neg(\forall i < x)\exists j A(i,j)$ implies $\neg(\exists j A(0,j) \wedge \underline{\forall i(0 < i < x \to \exists j A(i,j))})$

Induction, DML and GDML

Theorem

$\mathsf{EL}_0^* + \Delta(\Sigma_k^0)\text{-}\mathrm{IND} + \Sigma_k^0\text{-}\mathrm{DML} \vdash \Sigma_k^0\text{-}\mathrm{GDML}$

(Idea for the proof)
Assume $\neg(\forall i < x)\exists j A(i,j)$ for $A(i,j) \in \Pi_{k-1}^0$.

- $\neg(\forall i < x)\exists j A(i,j)$ implies $\neg(\exists j A(0,j) \wedge \underline{\forall i(0 < i < x \to \exists j A(i,j))})$
- Underlined part is equivalent to some Σ_k^0 formula.

Induction, DML and GDML

Theorem

$\mathsf{EL}_0^* + \Delta(\Sigma_k^0)\text{-}\mathrm{IND} + \Sigma_k^0\text{-}\mathrm{DML} \vdash \Sigma_k^0\text{-}\mathrm{GDML}$

(Idea for the proof)

Assume $\neg(\forall i < x)\exists j A(i,j)$ for $A(i,j) \in \Pi_{k-1}^0$.

- $\neg(\forall i < x)\exists j A(i,j)$ implies $\neg(\exists j A(0,j) \wedge \underline{\forall i(0 < i < x \to \exists j A(i,j))})$
- Underlined part is equivalent to some Σ_k^0 formula.
- Hence, $\Sigma_k^0\text{-}\mathrm{DML}$, we have $\neg\exists j A(0,j)$ or $\neg\forall i(0 < i < x \to \exists j A(i,j))$

Induction, DML and GDML

Theorem

$\mathsf{EL}_0^* + \Delta(\Sigma_k^0)\text{-IND} + \Sigma_k^0\text{-DML} \vdash \Sigma_k^0\text{-GDML}$

(Idea for the proof)
Assume $\neg(\forall i < x)\exists j A(i,j)$ for $A(i,j) \in \Pi_{k-1}^0$.

- $\neg(\forall i < x)\exists j A(i,j)$ implies $\neg(\exists j A(0,j) \wedge \underline{\forall i(0 < i < x \rightarrow \exists j A(i,j))})$
- Underlined part is equivalent to some Σ_k^0 formula.
- Hence, Σ_k^0-DML, we have $\neg\exists j A(0,j)$ or $\neg\forall i(0 < i < x \rightarrow \exists j A(i,j))$
- If the former is the case, repeat this process.

Induction, DML and GDML

Theorem

$\mathsf{EL}_0^* + \Delta(\Sigma_k^0)\text{-IND} + \Sigma_k^0\text{-DML} \vdash \Sigma_k^0\text{-GDML}$

(Idea for the proof)
Assume $\neg(\forall i < x)\exists j A(i,j)$ for $A(i,j) \in \Pi_{k-1}^0$.

- $\neg(\forall i < x)\exists j A(i,j)$ implies $\neg(\exists j A(0,j) \wedge \underline{\forall i(0 < i < x \to \exists j A(i,j))})$
- Underlined part is equivalent to some Σ_k^0 formula.
- Hence, Σ_k^0-DML, we have $\neg\exists j A(0,j)$ or $\neg\forall i(0 < i < x \to \exists j A(i,j))$
- If the former is the case, repeat this process.
- At some $i < x$, we must have $\neg\exists j A(i,j)$.

Induction, DML and GDML

Theorem

$\mathsf{EL}_0^* + \Delta(\Sigma_k^0)\text{-IND} + \Sigma_k^0\text{-DML} \vdash \Sigma_k^0\text{-GDML}$

(Idea for the proof)

Assume $\neg(\forall i < x)\exists j A(i,j)$ for $A(i,j) \in \Pi_{k-1}^0$.

- $\neg(\forall i < x)\exists j A(i,j)$ implies $\neg(\exists j A(0,j) \wedge \underline{\forall i(0 < i < x \to \exists j A(i,j))})$
- Underlined part is equivalent to some Σ_k^0 formula.
- Hence, Σ_k^0-DML, we have $\neg\exists j A(0,j)$ or $\neg\forall i(0 < i < x \to \exists j A(i,j))$
- If the former is the case, repeat this process.
- At some $i < x$, we must have $\neg\exists j A(i,j)$.

Hou to repeat the process? $\Delta(\Sigma_k^0)$-IND!

Bounded comprehension: Using induction, take
$u = \langle u_0, ..., u_{x-1} \rangle \in \{0,1,2\}^*$ s.t.

- $u_k = 0 \to \neg\exists y A(k,y)$,
- $u_k = 1 \to \neg\forall i(k < i < x \to \exists y A(i,y))$, and
- $u_k = 2 \to (\exists i < k) u_i = 0$

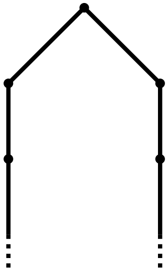
WKL and Σ^0_1-DML (LLPO)

Fact ([1])

$\mathsf{EL}_0^* + \mathrm{WKL} \vdash \Sigma^0_1\text{-DML}$

(Idea for the proof)
Assume $\neg(\exists x A(x) \wedge \exists x B(x))$, where $A(x), B(x) \in \Delta^0_0$.
Consider the following tree T:

$$T = \{u \in \{0,1\}^* : (u_0 = 0 \to \neg A(|u|)) \wedge (u_0 = 1 \to \neg B(|u|))\}$$



Since T must have a branch of any length, T has a path α.
$\alpha(0) = 0$ implies $\neg\exists x A(x)$ and $\alpha(0) = 1$ implies $\neg\exists x B(x)$.

WKL and GDML

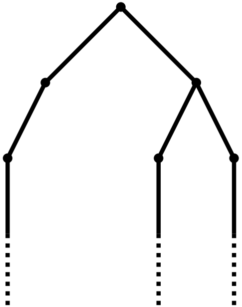
Theorem

$\mathsf{EL}_0^* + \mathrm{WKL} \vdash \Sigma_1^0\text{-GDML}$

(Idea for the proof)
Assume $\neg(\forall i < x)\exists j A(i,j)$ for $A(i,j) \in \Delta_0^0$. Consider the following T:

$$T = \{1^i 0^j : i < x, \neg A(i,j)\}$$

The case of $x = 3$



Since T must have a branch of any length, T has a path α.
Find the least $i < x$ s.t. $\alpha(i) = 1$. Then $\neg\exists j A(i,j)$.

WKL and GDML

Theorem

$\mathsf{EL}_0^* + \mathrm{WKL} \vdash \Sigma_1^0\text{-GDML}$

(Idea for the proof)

- $\neg(\forall i < x)\exists j A(i,j)$ implies $\neg(\exists j A(0,j) \wedge \underline{\forall i(0 < i < x \rightarrow \exists j A(i,j))})$
- Underlined part is equivalent to some Σ_1^0 formula.
- Hence, Σ_1^0-DML, we have $\neg\exists j A(0,j)$ or $\neg\forall i(0 < i < x \rightarrow \exists j A(i,j))$
- If the former is the case, repeat this process.
- At some $i < x$, we must have $\neg\exists j A(i,j)$.

How to repeat the process? Axiom choice!

WKL and GDML

Theorem

$\mathsf{EL}_0^* + \mathrm{WKL} \vdash \Sigma_1^0\text{-GDML}$

(Idea for the proof)

- $\neg(\forall i < x)\exists j A(i,j)$ implies $\neg(\exists j A(0,j) \wedge \underline{\forall i(0 < i < x \to \exists j A(i,j))})$
- Underlined part is equivalent to some Σ_1^0 formula.
- Hence, $\Sigma_1^0\text{-DML}$, we have $\neg\exists j A(0,j)$ or $\neg\forall i(0 < i < x \to \exists j A(i,j))$
- If the former is the case, repeat this process.
- At some $i < x$, we must have $\neg\exists j A(i,j)$.

How to repeat the process? Axiom choice!

WKL implies the following choice principle:

- $\Pi_1^0\text{-AC}^\vee$:
  $\forall x(A(x) \vee B(x)) \to \exists\alpha\forall x((\alpha(x) = 0 \to A(x)) \wedge (\alpha(x) \neq 0 \to B(x)))$
  for $A, B \in \Pi_1^0$

By this choice principle, we can take the right direction at once.

Something around

Kőnig's lemma (KL)

Every *finitely branching* infinite tree T, i.e., $\forall u \in T \exists i \forall j (u * \langle j \rangle \in T \to j \le i)$, has a path.

(Weak) Fan Theorem ((W)FT)

Every finitely branching (binary) tree T such that $\forall \alpha \exists n (\overline{\alpha} n \notin T)$ is bounded, i.e., $\exists m \forall u \in T (|u| < m)$.

Some results around KL and FT

- $\mathrm{EL}_0^* + \mathrm{WKL} \vdash \mathrm{WFT}$ (Essentially by Ishihara [5])
- $\mathrm{EL}_0^* + \mathrm{FT} \vdash \Sigma_1^0\text{-IND}$
- $\mathrm{EL}_0^* + \mathrm{KL} + \Sigma_1^0\text{-LEM} \vdash \Sigma_1^0\text{-IND}$ ([7])
- $\mathrm{EL}_0^* + \Sigma_1^0\text{-IND} + \Pi_1^0\text{-AC} \nvdash \mathrm{WKL}, \mathrm{KL}, \Pi_0^1\text{-IND}$ (cf. [7] and [8])
- $\mathrm{EL}_0^* + \Sigma_1^0\text{-IND} + \Pi_1^0\text{-AC} + \Sigma_1^0\text{-LEM} \vdash \mathrm{WKL}, \mathrm{KL}, \Pi_0^1\text{-IND}$ ([7]?)

Some observations

- Constructive reverse mathematics aims to characterize mathematical principles with choice principles (existence of functions), logical principles and sometime induction principles.
- Choice, logical, and induction principles are independent at a glance.
  - Realizability model: Full of choice principle, but weak induction, no logical principle
  - Total recursive function model: Full induction, but weak choice, no logical principle
  - Classical non-standard model: Classical logic, but weak choice, weak induction
- De Morgan's law (DML) is considered as a logical principle.
- But DML is generalized by induction or choice principle.
- How choice, logical and induction principles affect each other?

References

1. J. Berger, H. Ishihara and P.Schuster, *The weak König's lemma, Brouwer's fan theorem and de Morgan's law, and dependence choice*, Reports on mathematical logic 47 (2012), 63–86 DOI:10.4467/20842589RM.12.003.0684
2. S. Buss, *Intuitionistic validity in T-normal Kripke structures*, Ann. Pure Appl. Logic 59, pp. 159-173 (1993)
3. M. Fujiwara and T. Nemoto *On the decomposition of WKL!!*, Phil. Trans. R. Soc. A 381: 20220010. (2023) `https://doi.org/10.1098/rsta.2022.0010`
4. H. Ishihara, *Markov's principle, Church's thesis and Lindelöf's Theorem*, Indagationes Mathematicae 4 (3), pp.321-325 (1993)
5. H. Ishihara, *Weak König ' s Lemma Implies Brouwer ' s Fan Theorem: A Direct Proof*, Notre Dame Journal of Formal Logic 47 (2), pp. 249-252 (2006)
6. J. R. Moschovakis, *Another Unique Weak König ' s Lemma WKL!!*, In: Logic, Construction, Computation, edited by U. Berger, H. Diener, P. Schuster and M. Seisenberger, pp. 343-352, De Gruyter (2012)`https://doi.org/10.1515/9783110324921.343`
7. T. Nemoto and K. Sato, *A marriage of Brouwer's Intuitionism and Hilbert's Finitism I: Arithmetic*, The Journal of Symbolic Logic, 87 (2), pp.437-497 (2022) doi:10.1017/jsl.2018.6
8. S. G. Simpson, *Subsystems of second order arithmetic*, CUP (2009)
9. K. Wehmeier, *Fragments of HA based on Σ_1-induction*, Arch. Math. Logic 37: pp.37–49 (1997)