# Inductive and Coinductive predicates in the Minimalist Foundation

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# The Minimalist Foundation – *Why?*

- A **minimal** setting to formalise **constructive** mathematics
- A **common ground** compatible with other foundations
- A suitable framework for developing **Formal Topology**

  In particular, it does not validate the

  **Axiom of Unique Choice**, which allows to extract functions from functional relations;

  formally $\forall x \in A \exists ! y \in B \,.\, R(x, y) \Rightarrow \exists f : A \rightarrow B \forall x \in A \,.\, R(x, f(x))$

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***Toward a minimalist foundation for constructive mathematics***
M.E. Maietti and G. Sambin, Oxford Univ. Press, Oxford, 2005

***A minimalist two-level foundation for constructive mathematics***
M.E. Maietti, Ann. Pure Appl. Logic 160, 2009

# **The Minimalist Foundation** – *How?*

Key slogan:

**Minimalist in Assumptions, Maximal in Distinctions**

# *A two-levels type theory:*

**MF**

**eMF**

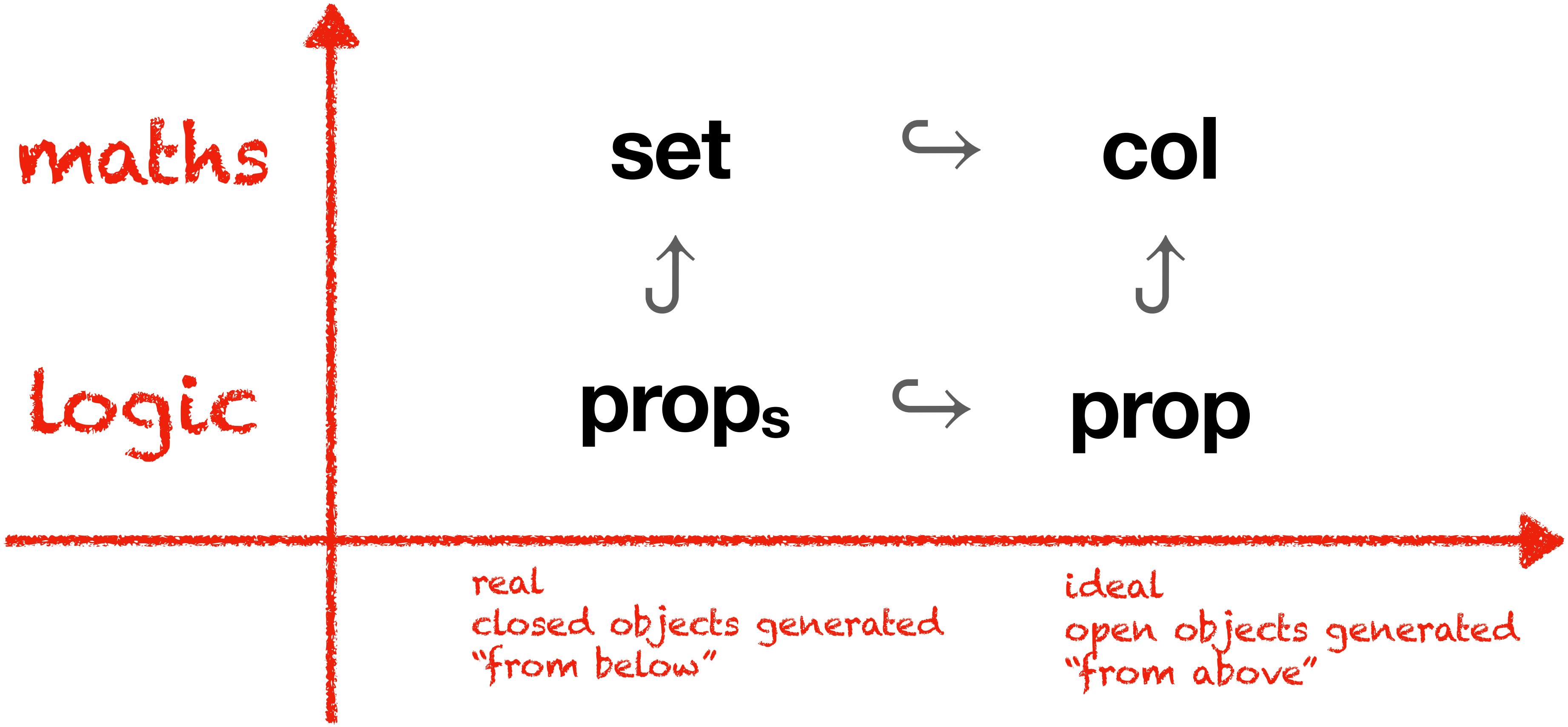
The intended place to do maths:
- function extensionality
- proof-irrelevance
- quotients
- powers

↓ **setoid interpretation**

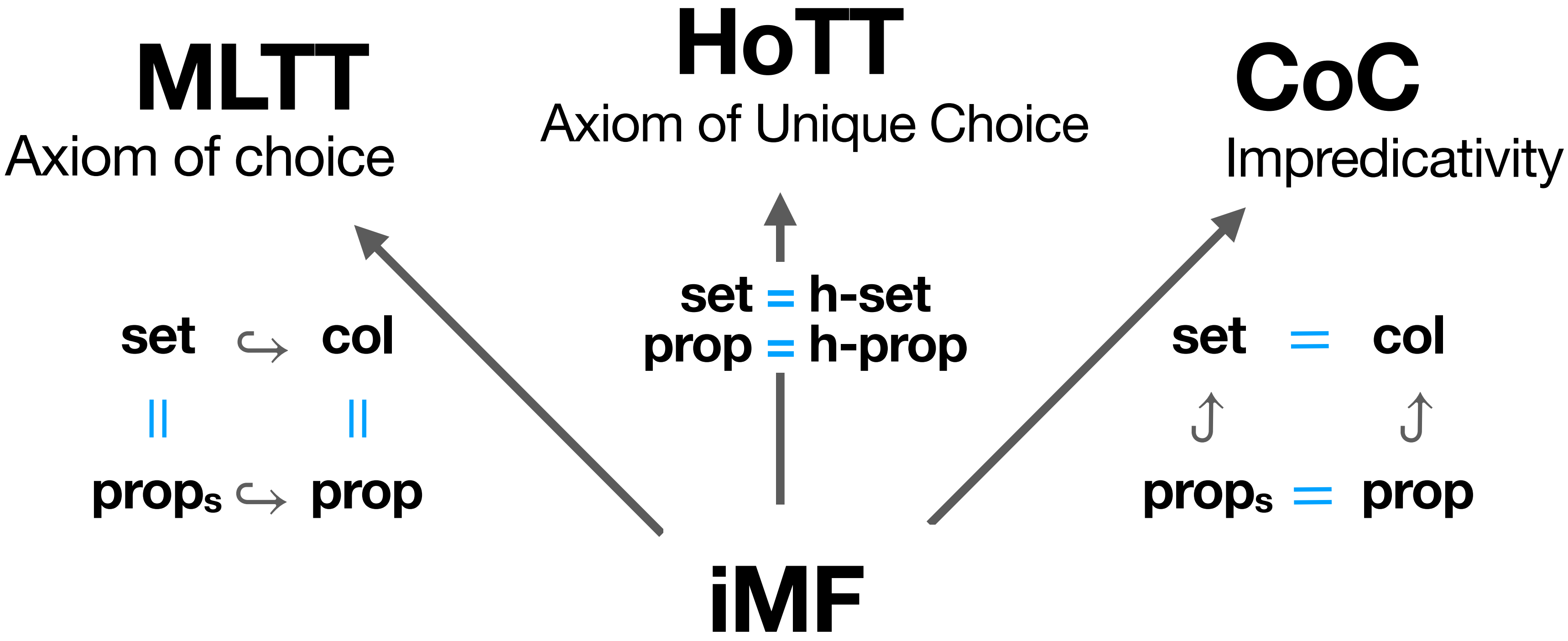
**iMF**

Where the computational content lies:
- proofs-as-programs
- compatible with the formal Church thesis
- enjoys a realizability model

# *Four kinds of types:*

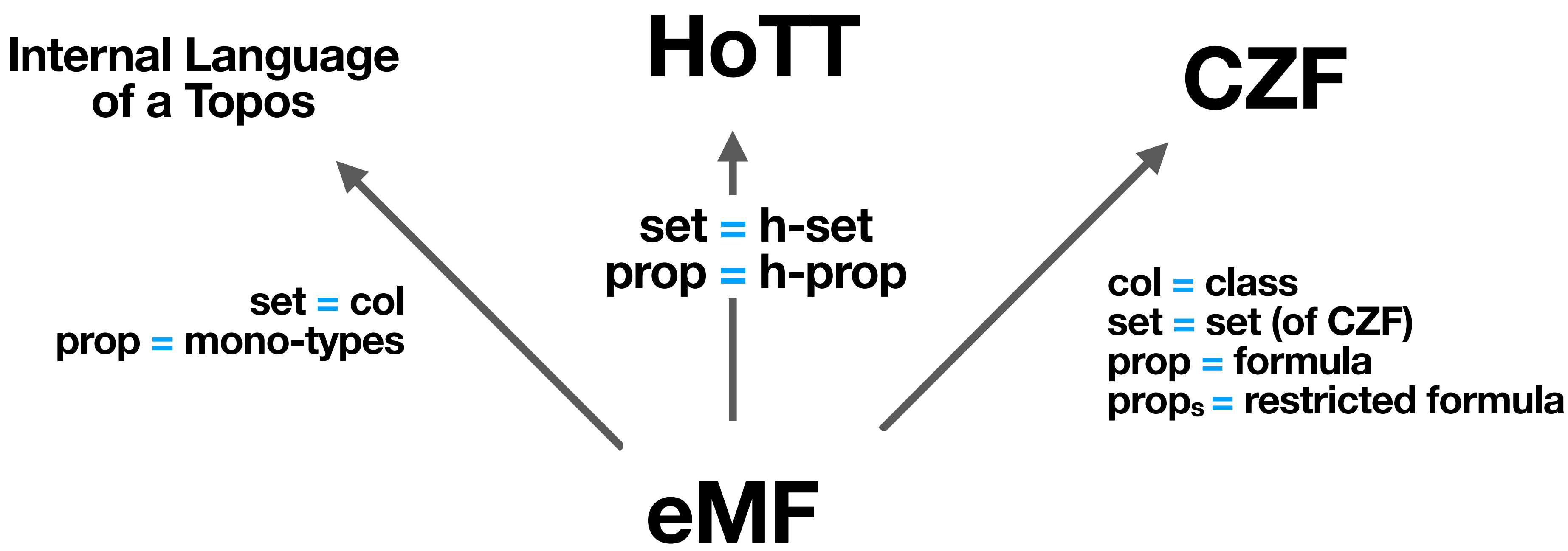


# **Compatibility** – *Intensional level*




***The Compatibility of the Minimalist Foundation with Homotopy Type Theory***
M. Contente, M.E. Maietti

# **Compatibility** – *Extensional level*

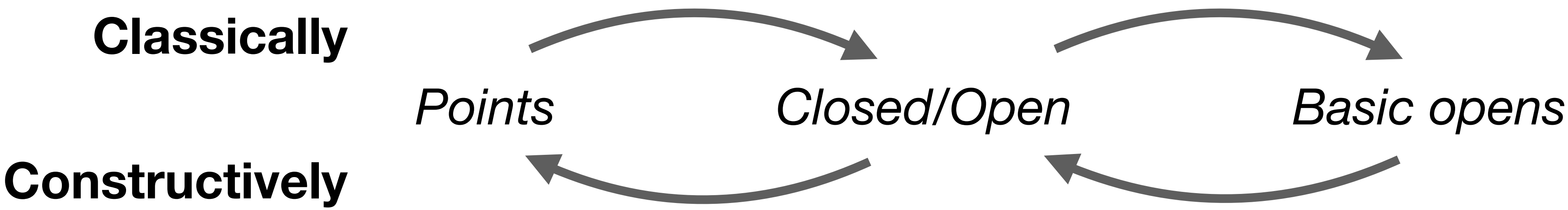



**Note:** all these extensions validates the Axiom of Unique Choice

# Formal Topology

= topology, done *constructively*

Reversing the conceptual order:

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***Intuitionistic formal spaces***
G. Sambin, 1987

The basic data consists of a set of basic opens $A$,
together with two relations $\lhd$ and $\ltimes$ between *elements* and *subsets* of $A$

**Basic Cover**

$a \lhd U$ *such that*

$$\text{rf}\frac{a \,\varepsilon\, U}{a \lhd U} \qquad \text{tr}\frac{a \lhd U \quad \forall x \in U\,.\,(x \,\varepsilon\, U \Rightarrow x \lhd V)}{a \lhd V}$$

*Spatial intuition:* the basic open $a$ is covered by the union of basic opens in the subset $U$

**Positivity Relation**

$a \ltimes U$ *such that*

$$\text{corf}\frac{a \ltimes U}{a \,\varepsilon\, U} \qquad \text{cotr}\frac{a \ltimes U \quad \forall b \in A\,.\,(b \ltimes U \Rightarrow b \,\varepsilon\, V)}{a \ltimes V}$$

*Spatial intuition:* there exists a point in the basic open $a$ whose basic neighbourhoods are all in $U$

+ *a compatibility condition* $$\text{cmp}\frac{a \ltimes V \quad a \lhd U}{\exists b \in A\,.\,b \lhd U \wedge b \,\varepsilon\, U}$$

# (Co)Inductive methods in Formal Topology

A powerful method to generate the aforementioned relations is to consider a so-called *Axiom Set*

$$I(x) \text{ set } [x \in A] \qquad S(x, y) \in \mathscr{P}(A) \; [x \in A, y \in I(x)]$$

and then (co)inductively generate:

- the *smallest basic cover* satisfying $a \lhd S(a, i)$ for each $a \in A, i \in I(a)$
- the *greatest positivity relation* satisfying $a \ltimes S(a, i)$ for each $a \in A, i \in I(a)$

*(compatibility follows for free)*

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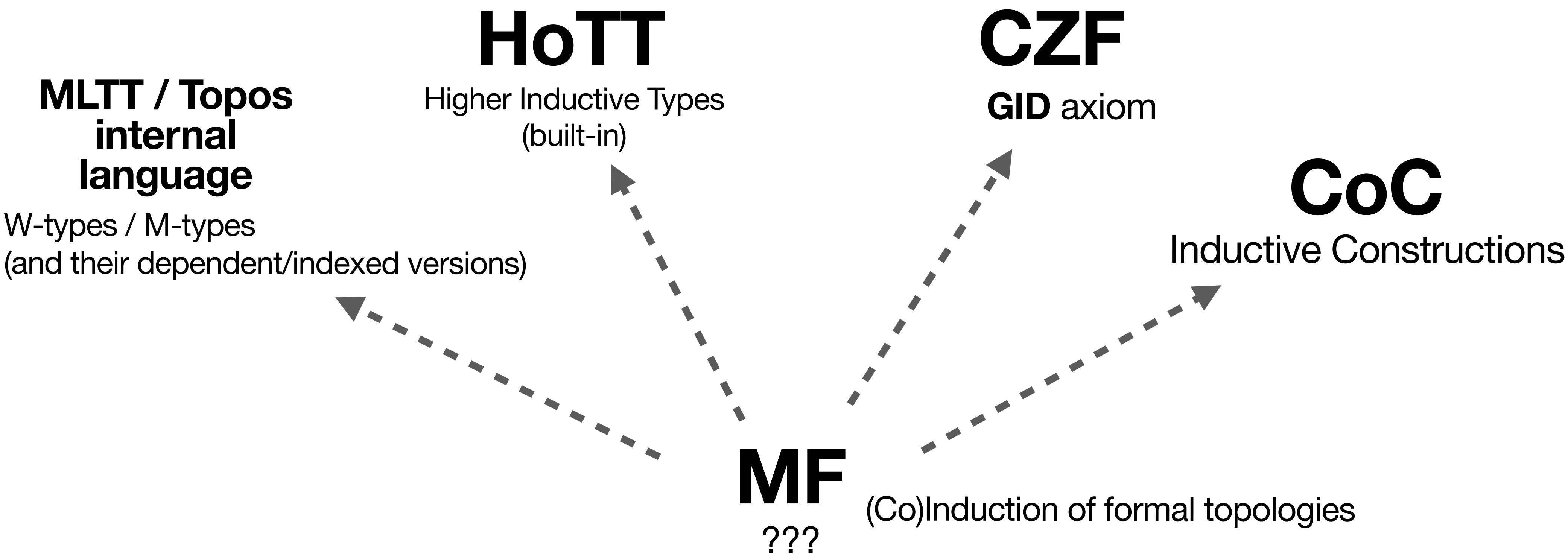
**Inductively generated formal topologies.**
Coquand, Sambin, Smith, and Valentini, 2003

# (Co)Induction in the various foundations




*Common pattern:* least and greatest fixed point of an endofunctor

# (Co)Induction and its logical interpretation

$A$ set

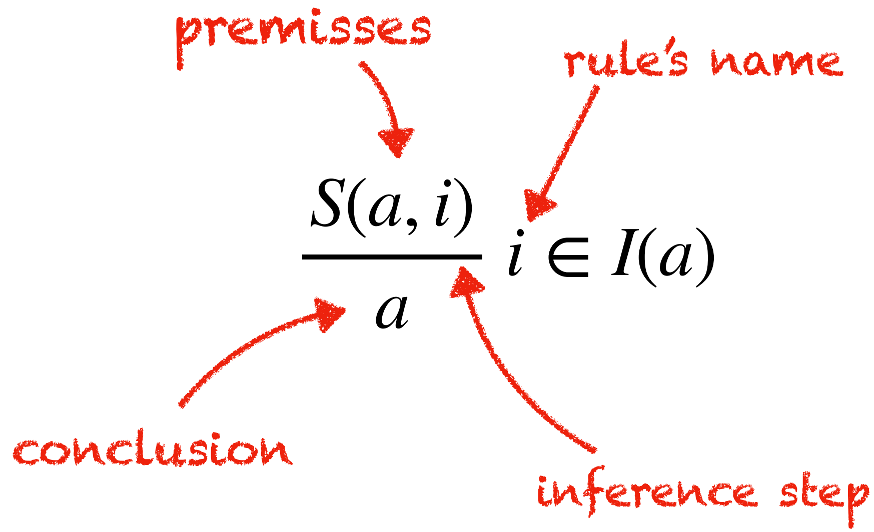
$I(x)$ set $[x \in A]$

$S(x, y) \in \mathscr{P}(A)$ $[x \in A, y \in I(x)]$

$\Gamma, \Delta : \mathsf{Pred}(A) \to \mathsf{Pred}(A)$

$\Gamma(P)(a) := \exists i \in I(a) \forall b \, \varepsilon \, S(a, i) \, . \, P(b)$

$\Delta(P)(a) := \forall i \in I(a) \exists b \, \varepsilon \, S(a, i) \, . \, P(b)$




We postulate the existence of two $A$-subsets:

- **Ind**: the least fixed point of $\Gamma$ — the provables
- **CoInd**: the greatest fixed point of $\Delta$ — the confutables

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**Generalized Inductive Definitions in Constructive Set Theory**
M. Rathjen, 2010

# Relation between (Co)Inductions

$A$ set $\qquad I(x)$ set $[x \in A] \qquad S(x, y) \in \mathscr{P}(A)\ [x \in A, y \in I(x)]$

| **According to…** | **…it is an** |
| --- | --- |
| Type Theory (MLTT/HoTT/Topos) | Indexed Container |
| Set Theory (CZF) | (Co)Inductive Definition (local and conclusion bounded) |
| Formal Topology | Axiom Set |
| Minimalist Foundation | *All the above!* |

# Relation to (Co)Inductive Topologies

**Theorems.**

**MF** + Ind *is equivalent to* **MF** + inductive $\lhd$

**MF** + CoInd *is equivalent to* **MF** + coinductive $\ltimes$

*Equivalent* here means that the two constructors' rules are mutually derivable.

In this way, the +Ind+CoInd extension inherits:

- extension of the setoid interpretation linking the two levels;
- an interpretation in **CZF** + $\mathrm{REA}_{\bigcup}$ + RDC;
- a realizability interpretation of the extended intensional level.

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**Inductive and Coinductive Topological Generation with Church's thesis and the Axiom of Choice**
M. E. Maietti, S. Maschio, M. Rathjen, 2021

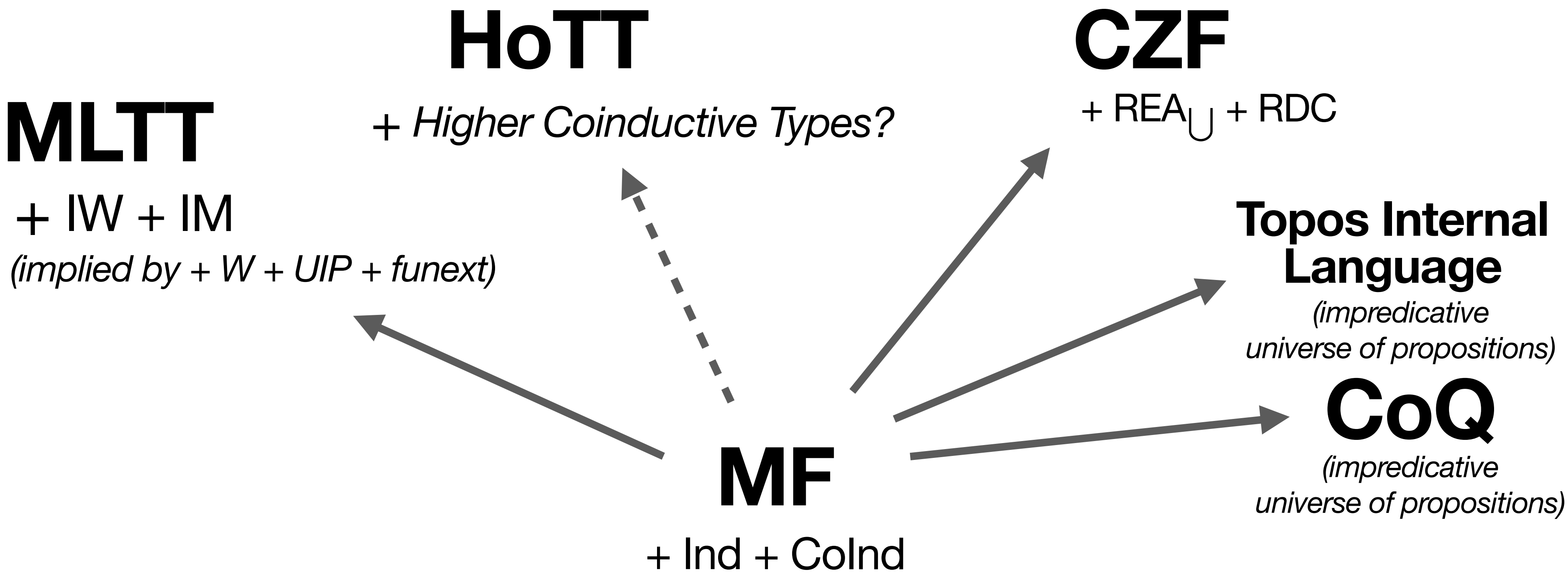
# Relation to W-types/M-types

**Theorems.**

| | | |
|---|---|---|
| **iMF** + Ind | *is compatible with* | **MLTT** + IW |
| **iMF** + CoInd | *is compatible with* | **MLTT** + IM |
| **iMF** + CoInd | *is interpretable in* | **HoTT** |

*Compatible* means that there exists an interpretation preserving the meaning of the logical and mathematical entities.

# Compatibility of (Co)Induction

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**Indexed Containers**
T. Altenkirch, N. Ghani, P. Hancock, C. McBride, P. Morris, 2014