The non-normal abyss in Kleene's Computability Theory

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Why? The '$\exists^2$-side' deals (exactly) with function classes that have a built-in approximation-device for function values

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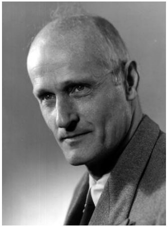
Turing machines may or may not produce an output after finitely many steps: partiality and the Halting problem.

Turing and Kleene



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 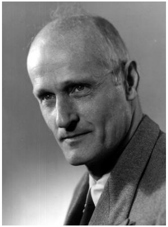

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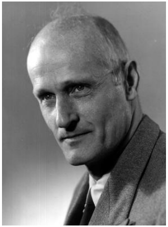

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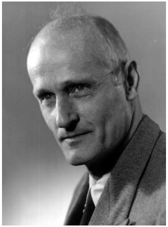

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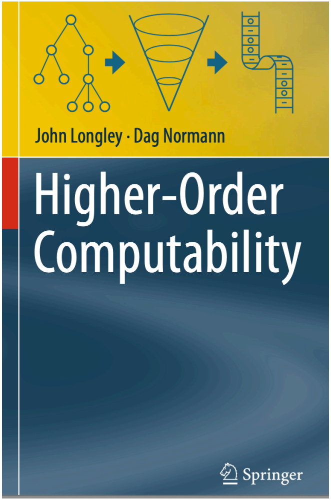
S1-S8 merely provide a kind of primitive recursion while S9 hard-codes the recursion theorem in an ad hoc way.

For details, consult:



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Normann-Sanders, JLC22, https://arxiv.org/abs/2203.05250.

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Both have (at most) countably many points of discontinuity and a rich history (PDE, probability, Bourbaki, Scheeffer, ...).

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NB: right-continuity as in $f(x) = f(x-)$ allows us to approximate $f(x)$ given only $f(q)$ for all $q \in \mathbb{Q} \cap [0,1]$.

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Baire (1905) notes that Baire 2 functions can be represented as iterated limits.

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$\exists^3$ computes $\sup_{x\in[p,q]} f(x)$ given Baire 2 $f : [0,1] \to [0,1]$, $p, q \in [0,1]$, and the associated sequence of Baire 1 functions.

Sharp: the functional S^2_k, which decides Π^1_k-formulas, cannot in general compute suprema for Baire 2 functions (holds for any k).

Borderline: the Suslin functional S^2_1 computes $\sup_{x\in[p,q]} f(x)$ given effectively Baire 2 $f : [0,1] \to [0,1]$, $p, q \in [0,1]$, and the associated double sequence of continuous functions.

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Note that quasi-continuity allows us to approximate $f(x)$ given only $f(q)$ for all $q \in \mathbb{Q} \cap [0,1]$.

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Many such decompositions exist, with numerous similar examples.

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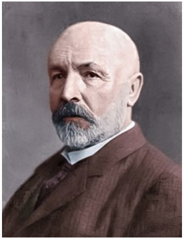
Finally, how do we prove our negative results?

The uncountability of $\mathbb{R}$

Cantor's first set theory paper (1874): uncountability of $\mathbb{R}$.



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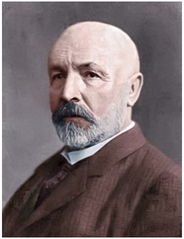

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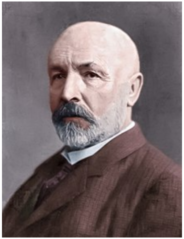

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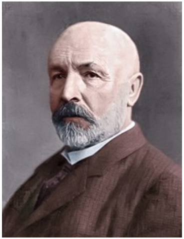

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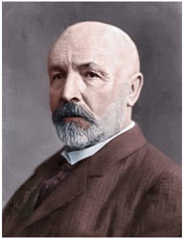

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Our negative results are obtained by computing a CR from the functionals at hand using:

$$f(x) := \begin{cases} \frac{1}{2^{Y(x)+1}} & \text{if } x \in X \\ 0 & \text{otherwise} \end{cases}$$

which is BV, semi-continuous, cliquish, ... and is found in the literature.

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Mathematically close (or equivalent) notions can land on either side of the abyss!

Thanks!

Questions?

Funded by the Klaus Tschira Foundation, German DFG, and RUB Bochum.