The Compact Hyperspace Monad: a Constructive Approach

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Let $\mathcal{P}\colon \mathbf{Set} \to \mathbf{Set}$ be the powerset functor. Then it is well known that $(\mathcal{P}, \eta, U)$ is a monad, where for a set X, $x \in X$ and $Y \in \mathcal{P}^2(X)$,

$$\eta_X(x) = \{x\} \qquad \text{and} \qquad U_X(Y) = \bigcup Y.$$

The same is true in case of the compact hyperspace functor $\mathcal{K}$ mapping a compact Hausdorff space to its hyperspace of nonempty compact subsets.

In joint work with U. Berger a general framework for extracting algorithms computing with elements of compact metric spaces as well as compact subsets of such from proofs in a many-sorted intuitionistic first-order logic extended by strictly positive inductive and coinductive definitions has been introduced.

The approach is computationally equivalent to Weihrauch's Type-Two Theory of Effectivity.

However, contrary to this approach it is purely logical and representation-free. Representations of the computed objects are obtained via a realisability interpretation of the logic.

Note that though the logic is basically intuitionistic, a fair amount of classical logic is available: any true disjunction-free formula can be used as axiom.

1. Digit spaces

Definition

Let (X, μ) be a compact metric space and $D = \{d_1, \dots, d_n\}$ be a finite set of contracting maps $d\colon X^{\mathrm{ar}(d)} \to X$ of finite arity $\mathrm{ar}(d)$. Then (X, D) is called digit space if

$$X = \bigcup \{ \mathrm{range}(d) \mid d \in D \}. \tag{1}$$

The $d \in D$ are called digits. In case of (1) we also say they are covering.

Example (Signed digit representation)

Let

$$X = [-1, 1] \qquad \text{and} \qquad D = \{ \mathrm{av}_i \mid i = -1, 0, 1 \},$$

where

$$\mathrm{av}_i(x) = (x + i)/2.$$

Then $\mathrm{range}(\mathrm{av}_i) = [(-1 + i)/2, (1 + i)/2]$ and hence (1) holds.

This is also an example of a digit space such that all digits have arity 1. Such spaces are called proper.

Let $\mathbb{C}_X$ be the coinductively largest subset of X such that

$$x \in \mathbb{C}_X \to (\exists d) \bigvee_{\sigma=1}^{n} d = d_\sigma \wedge$$

$$(\exists y_1, \ldots, y_{\mathrm{ar}(d)}) \bigwedge_{\kappa=1}^{\mathrm{ar}(d)} y_\kappa \in \mathbb{C}_X \wedge x = d(y_1, \ldots, y_{\mathrm{ar}(d)}). \quad (2)$$

Then $X = \mathbb{C}_X$.
Since X may be a classically defined object, this equality is true only classically.
In the constructive approach one works with $\mathbb{C}_X$ instead of X, where in (2) the disjunction has to be interpreted constructively.

Thus, if e.g. all digits have arity 1, given $x_0 \in X$, one obtains some $d_0 \in D$ and some $x_1 \in X$ with $x_0 = d_0(x_1)$. By iterating this procedure, one therefore receives a sequence $(d_i)_{i \in \mathbb{N}}$ of self-maps in D so that for each $i \in \mathbb{N}$,

$$x_0 \in \mathrm{range}(d_0 \circ \cdots \circ d_i).$$

Let (X, D) be proper and $d_1, \dots, d_r \in D$ be pairwise distinct. Define $[d_1, \dots .d_r] \colon \mathcal{K}(X)^r \to \mathcal{K}(X)$ by

$$[d_1, \dots, d_r](K_1, \dots, K_r) := \bigcup_{\nu=1}^{r} \mathcal{K}(d_\nu)(K_\nu),$$

where $\mathcal{K}(d)(K) = d[K]$.

Let $\mathcal{K}(D)$ be the finite set of all such maps $[d_1, \dots, d_r]$.

As is well known $\mathcal{K}(X)$ is a compact metric space with respect to the Hausdorff metric.

Assume that all $d \in D$ have contraction factor q. If $\mathcal{K}(X)^r$ is endowed with the maximum metric the maps $[d_1, \dots, d_r]$ are also contracting with contraction factor q.

Moreover, the $\vec{d} \in \mathcal{K}(D))$ are covering. Therefore, $(\mathcal{K}(X), \mathcal{K}(D))$ is digit space and hence we have

$$\mathcal{K}(X) = \mathbb{C}_{\mathcal{K}(X)}.$$

One would like to iterate the process described above to obtain coinductive characterisations of the higher compact hyperspaces $\mathcal{K}^n(X)$ with $n > 1$. Let us hereto switch to a more general setting.

Definition

For compact Hausdorff spaces X, Y and finite sets F of continuous maps $f\colon X^{\mathrm{ar}(f)} \to Y$ we write

$$X \xrightarrow{F} Y$$

to mean that for every $y \in Y$ there are $f \in F$ and $x_1, \dots, x_{\mathrm{ar}(f)} \in X$ with $y = f(x_1, \dots, x_{\mathrm{ar}(f)})$.

- Products

  Let $X \xrightarrow{F} Y$ and $X' \xrightarrow{F'} Y'$. Without restriction let all maps in $F \cup F'$ have the same arity, say s (introduce redundant arguments).
  For $f \in F$ and $f' \in F'$ set

$$\langle f, f' \rangle (x, x') = (f(x), f'(x'))$$

  and let $\langle F, F' \rangle$ be the set of all such pairs. Then

$$X \times X' \xrightarrow{\langle F, F' \rangle} Y \times Y'.$$

  Instead of $\langle F, \ldots, F \rangle$ (n times) we write $\Pi_n F$.

- Compact hyperspaces

  Let $X \xrightarrow{F} Y$ and $\mathcal{K}(F)$ be as defined earlier. We assume that all $\vec{f} \in \mathcal{K}(F)$ have the same arity $\|F\|$. Then

$$\mathcal{K}(X)^{\|F\|} \xrightarrow{\mathcal{K}(F)} \mathcal{K}(Y).$$

If (X, D) is a digit space, we assume again that all digits have the same arity s_D. By unfolding the coinductive definition of $\mathbb{C}_X$ we obtain a sequence

$$X_0 \xleftarrow{D_0} X_1 \xleftarrow{D_1} X_2 \xleftarrow{D_2} X_3 \leftarrow \cdots$$

where

$$X_0 = X,\ X_{i+1} = X_i^{s_D}, \quad D_0 = D,\ D_{i+1} = \Pi_{s_D} D_i.$$

Note here and in the next diagram that we have introduced the powers so that the maps involved are unary.

Then we obtain

$$\begin{aligned}\mathcal{K}(X_0) &\xleftarrow{\mathcal{K}(D_0)} \mathcal{K}(X_1)^{\|D_0\|} \\ &\xleftarrow{\Pi_{\|D_0\|}\mathcal{K}(D_1)} (\mathcal{K}(X_2)^{\|D_1\|})^{\|D_0\|} \\ &\xleftarrow{\Pi_{\|D_0\|}(\Pi_{\|D_1\|}\mathcal{K}(D_2))} (((\mathcal{K}(X_3)^{\|D_2\|})^{\|D_1\|})^{\|D_0\|} \\ &\leftarrow \cdots\end{aligned}$$

If we assume that (X, D) is proper, then $s_D = 1$, $X_i = X$, and $D_i = D$, for all $i \in \mathbb{N}$ and hence

$$\mathcal{K}(X_0) = \mathcal{K}(X), \mathcal{K}(X_1) = \mathcal{K}(X)^{\|D\|}, \mathcal{K}(X_2) = \mathcal{K}(X)^{\|D\|^2}, \ldots.$$

Therefore

$$\mathcal{K}^2(X_0) = \mathcal{K}^2(X), \mathcal{K}^2(X_1) = \mathcal{K}(\mathcal{K}(X_1)) = \mathcal{K}(\mathcal{K}(X)^{\|D\|}).$$

Hence the maps from $\mathcal{K}^2(X_1)$ to $\mathcal{K}^2(X_0)$ are no longer self-maps of $\mathcal{K}^2(X)$.

This shows that the digit space concept is too narrow to deal with the higher compact hyperspaces. The generalisation we just used, however, opens us a promising way to follow.

For each cochain $(Y_{i+1} \xrightarrow{F_i} Y_i)_{i \in \mathbb{N}}$ let

- $\mathfrak{Y} = \sum_{i \in \mathbb{N}} Y_i$ be the topological sum of the $(Y_i)_{i \in \mathbb{N}}$ and
- $\mathfrak{F} = \bigcup_{i \in \mathbb{N}} \{i\} \times F_i$ be the disjoint union of the F_i.

Then $(\mathfrak{Y}, \mathfrak{F})$ is a locally finite, infinite (extended) IFS. The maps in $\mathfrak{F}$ operate only locally on the components, i.e. for $(i, f) \in \mathfrak{F}$ and $(j, y_\kappa) \in \mathfrak{Y}$.

$$(i, f)((j_1, y_1), \ldots, (j_{\mathrm{ar}(f)}, y_{\mathrm{ar}(f)})) = \begin{cases} (i, f(y_1, \ldots, y_{\mathrm{ar}(f)})) & \\ \quad \text{if } j_\kappa = i + 1, \ (1 \leqslant \kappa \leqslant \mathrm{ar}(f)), & \\ \text{undefined} & \text{otherwise.} \end{cases}$$

For what follows assume that the Y_i are compact metric spaces and the $f \in F_i$ are contractions. Then $\mathfrak{Y}$ carries a canonical ∞-metric coinciding with the given metrics on the components.

Let $\mathbb{C}_{\mathfrak{Y}}$ be the coinductively largest subset of $\mathfrak{Y}$ such that

$$(i, y) \in \mathbb{C}_{\mathfrak{Y}} \to (\exists f)\, f \in F_i \wedge (\exists z_1, \ldots, z_{\mathrm{ar}(f)}) \bigwedge_{\kappa=1}^{\mathrm{ar}(f)} (i{+}1, z_\kappa) \in \mathbb{C}_{\mathfrak{Y}} \wedge (i, y) = (i, f)((i{+}1, z_1), \ldots, (i{+}1, z_{\mathrm{ar}(f)})).$$

Then (classically)

$$\mathfrak{Y} = \mathbb{C}_{\mathfrak{Y}}.$$

Set

$$\mathbb{C}_{\mathfrak{Y}}^{\langle 0 \rangle} = \{\, y \mid (0, y) \in \mathbb{C}_{\mathfrak{Y}} \,\}.$$

This will be the objects of our category.

Remark. Note that the family $(F_i)_{i \in \mathbb{N}}$ of predicates has to be definable in the underlying logic.

2. Morphisms

Let $(X_{i+1} \xrightarrow{D_i} X_i)_{i\in\mathbb{N}}, (Y_{i+1} \xrightarrow{E_i} Y_i)_{i\in\mathbb{N}}$ be cochains and $(\mathfrak{X}, \mathfrak{D})$, $(\mathfrak{Y}, \mathfrak{E})$ the associated infinite IFS.
Moreover, for $m > 0$, $j \in \mathbb{N}$, and $j_1 \leqslant \cdots \leqslant j_m \in \mathbb{N}$ let

$$\mathbb{F}(\mathfrak{X},\mathfrak{Y})^{(j)}_{j_1,\dots,j_m} = \{\, f\colon \mathfrak{X}^m \to \mathfrak{Y} \mid \\ \mathrm{dom}(f) = \bigtimes_{\nu=1}^{m} (\{j_\nu\} \times X_{j_\nu}) \wedge \mathrm{range}(f) \subseteq \{j\} \times Y_j \,\},$$

$$\mathbb{F}(\mathfrak{X},\mathfrak{Y})_{j_1,\dots,j_m} = \bigcup \{\, \mathbb{F}(\mathfrak{X},\mathfrak{Y})^{(j)}_{j_1,\dots,j_m} \mid j \in \mathbb{N} \,\},$$

$$\mathbb{F}(\mathfrak{X},\mathfrak{Y})^{(j)} = \bigcup \{\, \mathbb{F}(\mathfrak{X},\mathfrak{Y})^{(j)}_{j_1,\dots,j_m} \mid j_1 \leqslant \cdots \leqslant j_m \in \mathbb{N} \,\},$$

$$\mathbb{F}(\mathfrak{X},\mathfrak{Y}) = \bigcup_{m>0, j\in\mathbb{N}} \; \bigcup_{j_1 \leqslant \cdots \leqslant j_m} \mathbb{F}(\mathfrak{X},\mathfrak{Y})^{(j)}_{j_1,\dots,j_m}.$$

The following is a generalisation of U. Berger's coinductive-inductive characterisation of the uniformly continuous functions on the unit interval.

Define $\Phi : \mathcal{P}(\mathbb{F}(\mathfrak{X}, \mathfrak{Y})) \to (\mathcal{P}(\mathbb{F}(\mathfrak{X}, \mathfrak{Y})) \to \mathcal{P}(\mathbb{F}(\mathfrak{X}, \mathfrak{Y})))$ by

$$
\begin{aligned}
\Phi(F)(G) = \{\, f \in \mathbb{F}(\mathfrak{X}, \mathfrak{Y}) \mid \\
& [(\exists (i, e) \in \mathfrak{E})(\exists h_1, \ldots, h_{\mathrm{ar}(e)} \in F \cap \mathbb{F}(\mathfrak{X}, \mathfrak{Y})^{(i+1)}) \\
& \qquad f = (i, e) \circ (h_1 \times \cdots \times h_{\mathrm{ar}(e)})] \vee \\
& [(\exists j_1 \leqslant \cdots \leqslant j_{\mathrm{ar}(f)} \in \mathbb{N})\, f \in \mathbb{F}(\mathfrak{X}, \mathfrak{Y})_{j_1, \ldots, j_{\mathrm{ar}(f)}} \wedge \\
& \quad (\exists 1 \leqslant \nu \leqslant \mathrm{ar}(f))(\forall d \in D_{j_\nu})\, f \circ (j_\nu, d^{(\nu, \mathrm{ar}(f))}) \in G] \,\}
\end{aligned}
$$

where

$$
\begin{aligned}
& d^{(\nu,m)}((j_1, x_1), \ldots, (j_m, x_m)) = \\
& \quad ((j_1, x_1), \ldots, (j_{\nu-1}, x_{\nu-1}), (j_\nu, d(x_\nu)), (j_{\nu+1}, x_{\nu+1}), \ldots, (j_m, x_m)),
\end{aligned}
$$

for $x_\kappa \in X_{j_\kappa}$ $(\kappa \in \{j_1, \ldots, j_m\} \backslash \{j_\nu\})$ and $x_\nu \in X_{j_\nu + 1}$.

Set

$$\mathcal{J}(F) = \mu\Phi(F).$$

Then $\mathcal{J}(F)$ is the least subset G of $\mathbb{F}(\mathfrak{X}, \mathfrak{Y})$ so that

W If $(i, e) \in \mathfrak{E}$ and $\vec{h} \in (F \cap \mathbb{F}(\mathfrak{X}, \mathfrak{Y})^{(i+1)})^{\mathrm{ar}(e)}$ then $(i, e) \circ \vec{h} \in G$.

R If $f \in \mathbb{F}(\mathfrak{X}, \mathfrak{Y})$ and $\nu, j_1, \ldots, j_{\mathrm{ar}(f)} \in \mathbb{N}$ so that

- $j_1 \leqslant \cdots \leqslant j_{\mathrm{ar}(f)}$ and $f \in \mathbb{F}(\mathfrak{X}, \mathfrak{Y})_{j_1, \ldots, j_{\mathrm{ar}(f)}}$
- $1 \leqslant \nu \leqslant \mathrm{ar}(f)$ and for all $d \in D_{j_\nu}$, $f \circ d^{(\nu, \mathrm{ar}(f))} \in G$,

then $f \in G$.

Set

$$\mathbb{C}_{\mathbb{F}(\mathfrak{X},\mathfrak{Y})} = \nu\mathcal{J} \quad \text{and} \quad \mathbb{C}_{\mathbb{F}(\mathfrak{X}^{\langle 0 \rangle},\mathfrak{Y}^{\langle 0 \rangle})} = \mathbb{C}_{\mathbb{F}(\mathfrak{X},\mathfrak{Y})} \cap \bigcup_{m>0} \mathbb{F}(\mathfrak{X}, \mathfrak{Y})^{(0)}_{0^{(m)}}$$

where $x^{(m)} = (x, \ldots, x)$ (m times).

Proposition

Let $(X_{i+1} \xrightarrow{D_i} X_i)_{i\in\mathbb{N}}$, $(Y_{i+1} \xrightarrow{E_i} Y_i)_{i\in\mathbb{N}}$ *and* $(Z_{i+1} \xrightarrow{C_i} Z_i)_{i\in\mathbb{N}}$ *be cochains and* $(\mathfrak{X}, \mathfrak{D})$, $(\mathfrak{Y}, \mathfrak{E})$, $(\mathfrak{Z}, \mathfrak{C})$ *be the associated infinite IFS. If* $f \in \mathbb{C}_{\mathbb{F}(\mathfrak{Y}^{\langle 0\rangle},\mathfrak{Z}^{\langle 0\rangle})}$ *and* $g_1, \ldots, g_{\mathrm{ar}(f)} \in \mathbb{C}_{\mathbb{F}(\mathfrak{X}^{\langle 0\rangle},\mathfrak{Y}^{\langle 0\rangle})}$*, then* $f \circ (g_1, \ldots, g_{\mathrm{ar}(f)}) \in \mathbb{C}_{\mathbb{F}(\mathfrak{X}^{\langle 0\rangle},\mathfrak{Z}^{\langle 0\rangle})}$.

Proof Let

$$F = \{\, f \circ (g_1, \ldots, g_{\mathrm{ar}(f)}) \mid f \in \mathbb{C}_{\mathbb{F}(\mathfrak{Y},\mathfrak{Z})} \wedge g_1, \ldots, g_{\mathrm{ar}(f)} \in \mathbb{C}_{\mathbb{F}(\mathfrak{X},\mathfrak{Y})} \,\}.$$

Then, by coinduction according to the definition of $\mathbb{C}_{\mathbb{F}(\mathfrak{X},\mathfrak{Z})}$ one has to show that

$$F \subseteq \mathcal{J}^{\mathfrak{X},\mathfrak{Z}}(F).$$

That is one needs to show that $\mathbb{C}_{\mathbb{F}(\mathfrak{Y},\mathfrak{Z})} \subseteq G$, where

$$G = \{\, f \in \mathbb{F}(\mathfrak{Y}, \mathfrak{Z}) \mid (\forall g_1, \ldots, g_{\mathrm{ar}(f)} \in \mathbb{C}_{\mathbb{F}(\mathfrak{X},\mathfrak{Y})}) \; f \circ (g_1 \times \cdots \times g_{\mathrm{ar}(f)}) \in \mathcal{J}^{\mathfrak{X},\mathfrak{Z}}(F) \,\}.$$

Since $\mathbb{C}_{\mathbb{F}(\mathfrak{Y},\mathfrak{Z})} = \mathcal{J}^{\mathfrak{Y},\mathfrak{Z}}(\mathbb{C}_{\mathbb{F}(\mathfrak{Y},\mathfrak{Z})})$, it suffices to show

$$\mathcal{J}^{\mathfrak{Y},\mathfrak{Z}}(\mathbb{C}_{\mathbb{F}(\mathfrak{Y},\mathfrak{Z})}) \subseteq G.$$

By the inductive definition of $\mathcal{J}^{\mathfrak{Y},\mathfrak{Z}}(\mathbb{C}_{\mathbb{F}(\mathfrak{Y},\mathfrak{Z})})$ it is therefore sufficient to demonstrate that

$$\Phi^{\mathfrak{Y},\mathfrak{Z}}(\mathbb{C}_{\mathbb{F}(\mathfrak{Y},\mathfrak{Z})})(G) \subseteq G,$$

which means that one has to show that the corresponding Rules (W) and (R) hold.
This kind of proof is typical for all the next results.

Lemma

$\mathrm{id}_{\mathfrak{X}^{\langle 0\rangle}} \in \mathbb{C}_{\mathbb{F}(\mathfrak{X}^{\langle 0\rangle},\mathfrak{X}^{\langle 0\rangle})}$.

Proposition

For $m \geqslant 0$ let $\mathrm{ev}\colon \mathbb{F}^{(m)}(\mathfrak{X},\mathfrak{Y})^{(0)}_{0^{(m)}} \times X_0^m \to Y_0$ *with* $\mathrm{ev}(f,\vec{x}) = f(\vec{x})$ *be the evaluation map. Then*

$$\mathrm{ev}[\mathbb{C}^{(m)}_{\mathbb{F}(\mathfrak{X}^{\langle 0\rangle},\mathfrak{Y}^{\langle 0\rangle})} \times (\mathbb{C}^{\langle 0\rangle}_{\mathfrak{X}})^m] \subseteq \mathbb{C}^{\langle 0\rangle}_{\mathfrak{Y}}.$$

That is the spaces $\mathbb{C}_{\mathbb{F}(\mathfrak{X},\mathfrak{Y})}$ behave properly with respect to evaluation.

For $f \in \mathbb{F}^{(1)}(\mathfrak{X}, \mathfrak{Y})_0^{(0)}$ and $K \in \mathcal{K}(X_0)$, $\mathcal{K}(f)(K) = f[K]$. If f is continuous, we know that $f[K] \in \mathcal{K}(Y_0)$ and $\mathcal{K}(f)$ is continuous as well. The next result is an analogue of this statement in the constructive framework presented here.

Theorem

Let $(X_{i+1} \xrightarrow{D_i} X_i)_{i \in \mathbb{N}}$ and $(Y_{i+1} \xrightarrow{E_i} Y_i)_{i \in \mathbb{N}}$ be cochains and $(\mathfrak{X}, \mathfrak{D})$, $(\mathfrak{Y}, \mathfrak{E})$ the associated infinite IFS. Then for all $f \in \mathbb{C}^{(1)}_{\mathbb{F}(\mathfrak{X}^{\langle 0 \rangle}, \mathfrak{Y}^{\langle 0 \rangle})}$,

$$\mathcal{K}(f) \in \mathbb{C}_{\mathbb{F}(\mathfrak{K}(\mathfrak{X})^{\langle 0 \rangle}, \mathfrak{K}(\mathfrak{Y})^{\langle 0 \rangle})}.$$

Here, $(\mathfrak{K}(\mathfrak{X}), \mathfrak{K}(\mathfrak{D}))$ is obtained from $(X_{i+1} \xrightarrow{D_i} X_i)_{i \in \mathbb{N}}$ in the following way. First, we have to introduce powers of the involved spaces so that the maps in the sets D_i all are unary:

$$X_i' = (\cdots((X_i^{s_{i-1}})^{s_{i-2}})\cdots)^{s_0}, \quad D_i' = \Pi_{s_{i-1}} \cdots \Pi_{s_0} D_i$$

where s_i is the maximal arity of the maps in D_i. Then by applying the functor $\mathcal{K}$ to each $X_{i+1}' \xrightarrow{D_i'} X_i'$, we obtain the cochain $(\mathcal{K}(X_{i+1}') \xrightarrow{\mathcal{K}(D_i')} \mathcal{K}(X_i'))_{i \in \mathbb{N}}$, of which, finally, $(\mathfrak{K}(\mathfrak{X}), \mathfrak{K}(\mathfrak{D}))$ is the associated infinite IFS.

It follows that the structure **CDS** with

- Objects: $\mathbb{C}_{\mathfrak{X}}^{\langle 0 \rangle}$, for cochains $(X_{i+1} \xrightarrow{D_i} X_i)_{i \in \mathbb{N}}$ with compact metric spaces X_i and finite sets D_i of contractions $d\colon D_{i+1} \to D_i$
- Morphisms: $\mathbb{C}^{(1)}_{\mathbb{F}(\mathfrak{X}^{\langle 0 \rangle}, \mathfrak{Y}^{\langle 0 \rangle})}$

is a category and $\mathfrak{K}\colon \mathbf{CDS} \to \mathbf{CDS}$ is a functor.

For a cochain $(X_{i+1} \xrightarrow{D_i} X_i)_{i\in\mathbb{N}}$ and its associated infinite IFS $(\mathfrak{X}, \mathfrak{D})$ set

$$\eta_{\mathfrak{X}}(i,x) := (i, \{(\cdots((x^{(s_{i-1}))})^{(s_{i-2})})\cdots)^{s_0}\}),$$

where for $z \in Z$, $z^{(n)} = (z, \ldots, z)$ (n times).

Then

$$\eta_{\mathfrak{X}}(i,x) \in \{i\} \times \mathcal{K}(X_i') = \{i\} \times \mathcal{K}((\cdots((X_i^{s_{i-1}})^{s_{i-2}})\cdots)^{s_0}).$$

Moreover,

$$\eta_{\mathfrak{X}} \in \mathbb{C}^{(1)}_{\mathbb{F}(\mathfrak{X}^{\langle 0\rangle}, \mathfrak{K}(\mathfrak{X})^{\langle 0\rangle})}.$$

In addition, define

$$U_{\mathfrak{X}}(i, \mathbb{K}) = (i, \langle \cdots \langle \langle \bigcup^{(\|D_{i-1}\|)} \rangle^{(\|D_1\|)} \rangle \cdots \rangle^{(\|D_0\|)} (\mathbb{K})),$$

where for a map f, $f^{\langle n \rangle} = f \times \cdots \times f$ (n-times).

Note that

$$\mathfrak{K}^2((X))_i = \{i\} \times \mathcal{K}((\cdots(\mathcal{K}((\cdots(X_i^{s_{i-1}})\cdots)^{s_0})^{\|D_{i-1}\|})\cdots)^{\|D_0\|}).$$

Then

$$U_{\mathfrak{X}} \in \mathbb{C}^{(1)}_{\mathbb{F}(\mathfrak{K}^2(\mathfrak{X})^{\langle 0 \rangle}, \mathfrak{K}(\mathfrak{X})^{\langle 0 \rangle})}.$$

Furthermore,

Theorem

$(\mathfrak{K}, \eta, U)$ *is a monad.*

3. Computable maps

Definition

Let (X, μ, Q) be a metric space with countable dense subspace Q. Then (X, μ, Q) is computable if the sets

$$\{ (u, v, r) \in Q \times Q \times \mathbb{Q} \mid \mu(u, v) < r \}$$
$$\{ (u, v, r) \in Q \times Q \times \mathbb{Q} \mid \mu(u, v) > r \}$$

are effectively enumerable.

Definition

Let (X, μ, Q) and (X', μ', Q') be metric spaces with countable dense subspaces Q and Q', respectively. A map $h\colon X^i \to X'$ is

1. uniformly continuous if there is a map $\xi\colon \mathbb{Q}_+ \to \mathbb{Q}_+$, called modulus of continuity, such that for all $\varepsilon \in \mathbb{Q}_+$, $\vec{x}, \vec{y} \in X^i$.

$$\mu(\vec{x}, \vec{y}) < \xi(\varepsilon) \to \mu'(h(\vec{x}), h(\vec{y})) < \varepsilon.$$

2. computable if it has a computable modulus of continuity and there is a procedure G_h, which given $\vec{u} \in Q^i$ and $n > 0$ computes some $v \in Q'$ with

$$\mu'(h(\vec{u}), v) < 2^{-n}.$$

Definition

A cochain $(X_{i+1} \xleftarrow{D_i} X_i)_{i \in \mathbb{N}}$ is computable if the underlying metric spaces X_i are computable and all $d \in D_i$ are computable, both uniformly in i.

Theorem

Let $((X_{i+1}, Q_{X_{i+1}}) \xleftarrow{D_i} (X_i, Q_{X_i}))_{i \in \mathbb{N}}$ *and* $((Y_{i+1}, Q_{Y_{i+1}}) \xleftarrow{E_i} (Y_i, Q_{Y_i}))_{i \in \mathbb{N}}$ *be computable cochains so that stronger for every* $i \in \mathbb{N}$*,*

1. $X_i = \bigcup \{ \mathrm{int}(\mathrm{range}(d)) \mid d \in D_i \}$,
2. *every* $d \in D_i$ *has a right inverse* d'*, uniformly computable in* i*, where* d' *is a right inverse of* d *if*

$$d \circ d' = \mathrm{id}_{\mathrm{range}(d)},$$

and similarly for (Y_i, E_i, Q_{Y_i})*. Then*

$$\mathbb{C}_{\mathbb{F}(\mathfrak{X}^{\langle 0 \rangle}, \mathfrak{Y}^{\langle 0 \rangle})} = \{ f \in \mathbb{F}(\mathfrak{X}, \mathfrak{Y})^{(0)}_{0^{(\mathrm{ar}(f))}} \mid f \text{ computable} \}.$$