Extending continuous valuations on quasi-Polish spaces to Borel measures

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Continuous valuations are a useful alternative to Borel measures for developing a constructive approach to probability and measure theory [5, 3, 4, 8]. Some justification for using valuations in place of measures is given by the numerous valuation extension results (such as [7], [1], and [6]), which show that in many important cases a valuation can be uniquely extended to a Borel measure. Some of the most general results are due to M. Alvarez-Manilla [1], who showed that every locally finite continuous valuation on a regular or locally compact sober space has a unique extension to a Borel measure.

Here we show that every locally finite continuous valuation on a quasi-Polish space [2] uniquely extends to a Borel measure. Our result is not subsumed by M. Alvarez-Manilla's results because of the existence of non-regular non-locally-compact quasi-Polish spaces such as $\mathcal{K}(\omega^\omega)$ (the space of compact saturated subsets of the Baire space with the upper Vietoris topology). On the other hand, our result only applies to countably based spaces, and it does not include non-Polish regular spaces, such as the rationals $\mathbb{Q}$, which are covered by M. Alvarez-Manilla's results.

The main merit of our result is that it is a nice combination of simplicity and generality. Quasi-Polish spaces include many spaces of interest in measure theory, such as Polish spaces and ω-continuous domains, but our proof only requires the valuation extension property for $\mathcal{P}(\omega)$, the natural numbers with the Scott-topology, and a simple preservation lemma for $\mathbf{\Pi}^0_2$-subspaces.

We let $\overline{\mathbb{R}}_+$ denote the space of extended non-negative reals $[0, +\infty]$ with the Scott-topology. Given a topological space X, we let $\mathcal{O}(X)$ denote the lattice of open subsets of X with the Scott-topology.

A *valuation* on a topological space X is a function $\nu\colon \mathcal{O}(X) \to \overline{\mathbb{R}}_+$ satisfying:

1. $\nu(\emptyset) = 0$ (strictness),
2. $\nu(U \cup V) + \nu(U \cap V) = \nu(U) + \nu(V)$ for all opens $U, V \in \mathcal{O}(X)$ (modularity),
3. $U \subseteq V$ implies $\nu(U) \leq \nu(V)$ for all opens $U, V \in \mathcal{O}(X)$ (monotonicity).

A *continuous valuation* on X is a valuation which is continuous with respect to the Scott-topologies on $\mathcal{O}(X)$ and $\overline{\mathbb{R}}_+$. A valuation ν on X is *bounded* or *finite* if $\nu(X) < \infty$, and ν is *locally finite* if for every $x \in X$ there is an open neighborhood U of x such that $\nu(U) < \infty$.

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If X is a countably based space, then every σ-additive measure defined on the Borel subsets of X restricts to a continuous valuation on X. The converse does not hold in general without additional constraints on X. We will say that a topological space X has the *valuation extension property* if every locally finite continuous valuation on X has a unique extension to a σ-additive measure defined on the Borel subsets of X. We restrict attention to locally finite valuations in order to guarantee uniqueness of the extension.

The next lemma generalizes an observation mentioned at the top of Section 4 in [6] concerning the preservation of valuation extension results to G_δ-subspaces. Note that $\mathbf{\Pi}^0_2$-sets are strictly more general than G_δ-sets when working with non-metrizable spaces (see [2], for example).

Lemma 1. *If X is a countably based T_0-space with the valuation extension property, then every $\mathbf{\Pi}^0_2$-subspace of X has the valuation extension property.*

Proof. Assume $Y \in \mathbf{\Pi}^0_2(X)$ and let $\nu\colon \mathcal{O}(Y) \to \overline{\mathbb{R}}_+$ be a locally finite continuous valuation.

As shown by M. Alvarez-Manilla (Lemma 4.1 in [1]), every locally finite continuous valuation on a space X has a unique extension to a Borel measure if and only if every finite continuous valuation on X has a unique extension. Therefore, it suffices to prove our $\mathbf{\Pi}^0_2$-preservation result under the assumption that $\nu(Y) < \infty$. This assumption is necessary because our extension of ν to ν_X in the next paragraph preserves finiteness but may not preserve local finiteness of valuations.

Let $e\colon Y \to X$ be the topological embedding of Y into X. Then the preimage function $e^{-1}\colon \mathcal{O}(X) \to \mathcal{O}(Y)$ is continuous, hence the function $\nu_X\colon \mathcal{O}(X) \to \overline{\mathbb{R}}_+$ defined as $\nu_X = \nu \circ e^{-1}$ is continuous. Clearly, $\nu_X(X) < \infty$, and by using the fact that $\nu_X(U) = \nu(e^{-1}(U)) = \nu(U \cap Y)$ for each $U \in \mathcal{O}(X)$, it is easy to see that ν_X is a finite continuous valuation on X.

Using the valuation extension property of X, the valuation ν_X extends uniquely to a σ-additive measure μ_X on the Borel subsets of X. We let μ denote the restriction of μ_X to the Borel subsets of Y.

Since Y is a Borel subset of X, it is clear that μ is a well defined σ-additive measure on the Borel subsets of Y. Therefore, it only remains to show that μ extends ν. Towards this end, we first prove that $\mu_X(X \setminus Y) = 0$. As $X \setminus Y$ is a $\mathbf{\Sigma}^0_2$-subset of X we can represent it as a countable union

$$X \setminus Y = \bigcup_{i \in \omega} U_i \setminus V_i,$$

with U_i, V_i open subsets of X. Assume for a contradiction that $\mu_X(X \setminus Y) > 0$. The σ-additivity of μ_X implies there is $i \in \omega$ such that $\mu_X(U_i \setminus V_i) > 0$. It follows that

$$\mu_X(U_i \setminus V_i) = \nu_X(U_i) - \nu_X(U_i \cap V_i) = \nu(U_i \cap Y) - \nu(U_i \cap V_i \cap Y),$$

hence $\nu(U_i \cap Y) > \nu(U_i \cap V_i \cap Y)$. On the other hand, $U_i \cap Y \subseteq U_i \cap V_i \cap Y$ follows from the assumption that $U_i \setminus V_i$ is disjoint from Y. This contradicts the monotonicity property of valuations. Therefore, $\mu_X(X \setminus Y) = 0$, and it follows that any Borel subset of X disjoint from Y must also have a μ_X-measure of zero.

To finish the proof, let U be any given open subset of Y and let V be an open subset of X such that $V \cap Y = U$. Then

$$\nu(U) = \nu_X(V) = \mu_X(V) = \mu_X(U) + \mu_X(V \setminus Y) = \mu_X(U) + 0 = \mu(U),$$

hence μ is a σ-additive Borel measure extending ν. □

The valuation extension property for quasi-Polish spaces now follows from the well known fact that $\mathcal{P}(\omega)$ has the valuation extension property [7] and that every quasi-Polish space embeds into $\mathcal{P}(\omega)$ as a $\mathbf{\Pi}^0_2$-subspace [2].

Theorem 1. *Every quasi-Polish space has the valuation extension property.* □

We next show that the preservation result in Lemma 1 is in some sense optimal. Let S_1 denote the natural numbers with the cofinite topology, and let S_D denote the countable chain $(\omega, \leq)$ with the Scott-topology. The continuous valuation defined as $\nu(U) = 1$ if and only if U is non-empty demonstrates that both S_1 and S_D fail to have the valuation extension property. Neither S_1 nor S_D is sober, but they are $\mathbf{\Sigma}^0_2$-subspaces of their sobrifications, which both happen to be quasi-Polish. We have proven the following.

Proposition 1. *The valuation extension property can fail to be preserved by* $\mathbf{\Sigma}^0_2$*-subspaces.* □

It can be shown that every non-sober countably based T_0-space contains a $\mathbf{\Pi}^0_2$-subspace homeomorphic to either S_1 or S_D. We therefore obtain the following necessary condition for the valuation extension property, which was originally proved by M. Alvarez-Manilla [1] in a slightly more general form.

Proposition 2. *Every countably based* T_0*-space with the valuation extension property is sober.* □

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