A note on the descriptive complexity of the upper and double powerspaces

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Given a topological space X, $\mathbf{O}(X)$ denotes the lattice of open subsets of X with the Scott-topology, $\mathbf{K}(X)$ denotes the space of saturated compact subsets of X with the upper-Vietoris topology, and $\mathbf{A}(X)$ denotes the space of closed subsets of X with the lower-Vietoris topology (see [5]). If X is countably based and sober, then $\mathbf{A}(X)$, $\mathbf{K}(X)$, and $\mathbf{O}(\mathbf{O}(X))$ are all countably based and sober. These and other powerspaces frequently appear in computable analysis and the theory of represented spaces [1, 9, 8, 7, 4]. Our main result is about the complexity of $\mathbf{K}(X)$ and $\mathbf{O}(\mathbf{O}(X))$ when X is a countably based co-analytic sober space.

A subspace $S \subseteq X$ of a quasi-Polish space X is *analytic* (or $\mathbf{\Sigma}^1_1$) if there exists a continuous function $f\colon \mathbb{N}^{\mathbb{N}} \to X$ such that the range of f is equal to S (see [2]). A subspace of a quasi-Polish space is *co-analytic* (or $\mathbf{\Pi}^1_1$) if its complement is analytic. A countably based space is analytic (co-analytic) if it is homeomorphic to an analytic (co-analytic) subspace of a quasi-Polish space.

A space is a *Baire space* if the intersection of any countable sequence of dense open subsets is dense. A space is *completely Baire* if each of its closed subspaces is a Baire space. If X is a countably based completely Baire space, then so is every $\mathbf{\Pi}^0_2$-subspace of X ([3], Theorem 4.1).

Our main result is the following:

Theorem 1. *The following are equivalent for every countably based co-analytic sober space X:*

1. $\mathbf{K}(X)$ *is analytic,*
2. $\mathbf{K}(X)$ *is completely Baire,*
3. $\mathbf{O}(\mathbf{O}(X))$ *is analytic,*
4. $\mathbf{O}(\mathbf{O}(X))$ *is completely Baire,*
5. X *is quasi-Polish.*

□

It is well-known that a similar result (restricted to metrizable spaces) holds for the Vietoris powerspace (see Exercise 33.5 in [6]). Note that the Vietoris powerspace has a strictly finer topology than the powerspace $\mathbf{K}(X)$ defined here.

The proof of Theorem 1 will easily follow from previous results and Lemma 1 below. In the following, we will write $X \mathrel{\widetilde{\in}} \mathbf{\Pi}^0_2(Y)$ to mean that X is homeomorphic to a $\mathbf{\Pi}^0_2$-subspace of Y. We also recall the definition of the countable space S_0 from [3]. The underlying set of S_0 is $\mathbb{N}^{<\mathbb{N}}$ (all finite strings of natural numbers), and a subbasis for the *closed* subsets of S_0 is given by sets of the form $\{\tau \in \mathbb{N}^{<\mathbb{N}} | \sigma \preceq \tau\}$, where $\sigma \in \mathbb{N}^{<\mathbb{N}}$ and $\preceq$ is the prefix relation. Note that the specialization order of S_0 is the inverse of the prefix relation.

Lemma 1. *If $S \subseteq \mathbb{N}^{\mathbb{N}}$ is co-analytic then $S \mathrel{\widetilde{\in}} \mathbf{\Pi}^0_2(\mathbf{K}(S_0))$.*

Proof. For $\sigma \in \mathbb{N}^{<\mathbb{N}}$, let $\langle\sigma\rangle_0$ be the substring of σ of elements with even indices, and let $\langle\sigma\rangle_1$ be the odd elements. We write $\sigma \diamond \tau$ for the concatenation of σ and τ, and $|\sigma|$ for the length of σ. Our notation will treat $\mathbb{N}$ and $\mathbb{N}^2$ as subspaces of $\mathbb{N}^{<\mathbb{N}}$. Fix an enumeration $\{\tau_n\}_{n\in\mathbb{N}}$ of $\mathbb{N}^{<\mathbb{N}}$.

Sets of the form $\{K \in \mathbf{K}(S_0) \mid \tau_n \notin K\}$ (for $n \in \mathbb{N}$) form a subbase for the *open* subsets of $\mathbf{K}(S_0)$. It is easy to see that the elements of $\mathbf{K}(S_0)$ are precisely the well founded trees on $\mathbb{N}$, which is the underlying set of a standard example of a $\mathbf{\Pi}^1_1$-complete set [6] (note that the topology on $\mathbf{K}(S_0)$ is strictly weaker than the more standard zero-dimensional topology used in [6]).

We use the notation $\uparrow\!\sigma = \{x \in \mathbb{N}^{\mathbb{N}} | \sigma \preceq x\}$ for basic clopen subsets of $\mathbb{N}^{\mathbb{N}}$. Let $U \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ be open such that $S = \{x \in \mathbb{N}^{\mathbb{N}} \mid (\forall y \in \mathbb{N}^{\mathbb{N}})\langle x, y\rangle \in U\}$. For $x \in \mathbb{N}^{\mathbb{N}}$ define

$$\alpha(x) = \{0 \diamond \sigma \in \mathbb{N}^{<\mathbb{N}} \mid \langle\sigma\rangle_0 \preceq x \;\&\; \uparrow\!\langle\sigma\rangle_0 \times \uparrow\!\langle\sigma\rangle_1 \not\subseteq U]\}$$
$$\beta(x) = \{1 \diamond n \in \mathbb{N}^2 \mid \tau_n \preceq x\} \cup \{2 \diamond n \in \mathbb{N}^2 \mid \tau_n \not\preceq x\}$$

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Define $\gamma\colon S \to \mathbf{K}(S_0)$ so that $\gamma(x)$ is the saturation of $\alpha(x) \cup \beta(x)$. It is easy to see that γ is continuous and well-defined (the choice of U guarantees that $\alpha(x)$ does not contain an infinite branch if and only if $x \in S$). Furthermore, $\tau_n \preceq x$ if and only if $2 \diamond n \notin \gamma(x)$, hence γ is an embedding.

Let G be the set of all $K \in \mathbf{K}(S_0)$ such that the following hold ($n, m \in \mathbb{N}$ and $\sigma \in \mathbb{N}^{<\mathbb{N}}$):

1. $(\forall n)[1 \diamond n \in K \iff 2 \diamond n \notin K]$
2. $(\forall m, n$ satisfying $\tau_n \preceq \tau_m)[1 \diamond m \in K \Rightarrow 1 \diamond n \in K]$
3. $(\forall m, n$ satisfying $\tau_n \not\preceq \tau_m)[1 \diamond m \in K \Rightarrow 1 \diamond n \notin K]$
4. $(\forall m)(\exists n)[m \leq |\tau_n| \;\&\; 2 \diamond n \notin K]$
5. $(\forall \sigma$ satisfying $|\sigma| > 1)[1 \diamond \sigma \notin K \;\&\; 2 \diamond \sigma \notin K]$
6. $(\forall n > 2)\, n \notin K$
7. $(\forall \sigma, n$ satisfying $\tau_n = \langle\sigma\rangle_0)[0 \diamond \sigma \in K \iff (1 \diamond n \in K \;\&\; \uparrow\langle\sigma\rangle_0 \times \uparrow\langle\sigma\rangle_1 \not\subseteq U)]$

Each of the conditions above correspond to a countable intersection of finite boolean combinations of open subsets of $\mathbf{K}(S_0)$, hence G is a $\mathbf{\Pi}^0_2$-subspace of $\mathbf{K}(S_0)$. It is easy to verify that $\gamma(x) \in G$ for each $x \in S$.

Fix $K \in G$ and let $P = \{\tau_n \in \mathbb{N}^{<\mathbb{N}} \mid 1 \diamond n \in K\}$. The second and third conditions guarantee that P is closed under prefixes and linearly ordered by $\preceq$, and the first and fourth conditions guarantee that the lengths of the strings in P are unbounded. Hence there is a unique $x \in \mathbb{N}^{\mathbb{N}}$ such that $\tau_n \preceq x \iff \tau_n \in P \iff 1 \diamond n \in K$. Condition one now implies $2 \diamond n \in K \iff \tau_n \not\preceq x$, and using the fifth condition we have that the strings in K starting with 1 or 2 are exactly the strings in $\beta(x)$. The remaining non-empty strings in K must begin with 0 because of the sixth condition, and the seventh condition guarantees that these remaining strings are precisely the elements of $\alpha(x)$. The compactness of K implies $\alpha(x)$ has no infinite branch, hence $x \in S$ and $K = \gamma(x)$. Therefore, γ is a homeomorphism between S and the $\mathbf{\Pi}^0_2$-subspace G of $\mathbf{K}(S_0)$. ☐

Using Lemma 1, we can now prove Theorem 1 as follows. First note that if X is quasi-Polish, then $\mathbf{K}(X)$ and $\mathbf{O}(\mathbf{O}(X))$ are quasi-Polish (see [4, 5]), so items (1) through (4) all follow from (5). So assume that X is a countably based co-analytic sober space which is *not* quasi-Polish. Then X contains a $\mathbf{\Pi}^0_2$-subspace S which is homeomorphic to either $\mathbb{Q}$ or S_0 (see [3]). It was shown in ([4], Corollary 5.5) that $\mathbf{K}(\mathbb{Q})$ is not analytic, and Lemma 1 implies that $\mathbf{K}(S_0)$ is not analytic, hence $\mathbf{K}(S)$ is not analytic. Furthermore, $\mathbf{K}(S)$ is not completely Baire because $\mathbb{Q} \mathrel{\widetilde{\in}} \mathbf{\Pi}^0_2(\mathbf{K}(\mathbb{Q}))$ ([5], Proposition 8) and $\mathbb{Q} \mathrel{\widetilde{\in}} \mathbf{\Pi}^0_2(\mathbf{K}(S_0))$ (Lemma 1 above).

We have $\mathbf{K}(S) \mathrel{\widetilde{\in}} \mathbf{\Pi}^0_2(\mathbf{K}(X))$ by ([4], Theorem 5.3). Sobriety of X implies $X \mathrel{\widetilde{\in}} \mathbf{\Pi}^0_2(\mathbf{A}(X))$ ([5], Proposition 3), hence $S \mathrel{\widetilde{\in}} \mathbf{\Pi}^0_2(\mathbf{A}(X))$, which implies $\mathbf{K}(S) \mathrel{\widetilde{\in}} \mathbf{\Pi}^0_2(\mathbf{K}(\mathbf{A}(X)))$. Since $\mathbf{K}(\mathbf{A}(X))$ and $\mathbf{O}(\mathbf{O}(X))$ are homeomorphic ([5], Theorem 22) we have $\mathbf{K}(S) \mathrel{\widetilde{\in}} \mathbf{\Pi}^0_2(\mathbf{O}(\mathbf{O}(X)))$. Thus $\mathbf{K}(X)$ and $\mathbf{O}(\mathbf{O}(X))$ both have a $\mathbf{\Pi}^0_2$-subspace which is neither analytic nor completely Baire. Both of these properties are hereditary under $\mathbf{\Pi}^0_2$-subspaces, hence items (1) through (4) do not hold for X. This completes the proof of Theorem 1.

It would be interesting to see if an effective (light-faced) version of Lemma 1 holds, and also to have a full characterization of the $\mathbf{\Pi}^0_2$-subspaces of $\mathbf{K}(\mathbb{Q})$ and $\mathbf{K}(S_0)$. We also have the following question.

Question 1. If X is a QCB_0-space and $\mathbf{O}(X)$ is a countably based analytic space, then is $\mathbf{O}(X)$ quasi-Polish?

The author has learned from M. Schröder that his previous results ([10], Theorem 7.3) imply that the above question has a positive answer if X is Hausdorff, even when the "analytic" assumption is removed.

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