A note on the closed prime spectrums of coPolish commutative rings

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A topological space X is *coPolish* if it is the direct limit of an increasing sequence of compact metrizable subspaces $(X_k)_{k\in\mathbb{N}}$. CoPolish spaces were introduced and studied by M. Schröder in [11] in the context of Type-2 complexity theory, and have also appeared in work on the Weihrauch complexity of overt choice [4] and the descriptive complexity of non-countably based spaces [1]. A countably based space is coPolish if and only if it is a locally compact Polish space, and there are many natural examples of non-countably based coPolish spaces, such as the free topological group generated by $\mathbb{R}$ and the space $\mathbb{R}[X]$ of polynomials[1].

A coPolish ring is a ring equipped with a coPolish topology making it a topological group with respect to addition and a topological monoid with respect to multiplication[2]. We will mainly be concerned with commutative rings in this note. A subset I of a commutative ring R is an *ideal* if it is an additive subgroup of R such that $rx \in I$ for every $r \in R$ and $x \in I$. An ideal I is *prime* if it does not equal the whole ring and $xy \in I$ implies $x \in I$ or $y \in I$.

The prime spectrum $\mathbf{Spec}(R)$ of a commutative ring R is defined to be the set of all prime ideals of R equipped with the Zariski topology. Although it plays a fundamental role in modern algebraic geometry [5], the prime spectrum of many important coPolish rings are not QCB_0-spaces, such as $\mathbf{Spec}(2^{\mathbb{Z}})$ (assuming the axiom of choice) and $\mathbf{Spec}(\mathbb{R}[X])$, hence it is not suitable from a computability theoretic perspective [10]. As a replacement, we define the *closed prime spectrum* $\mathbf{cSpec}(R)$ of a coPolish commutative ring R to be the set of topologically closed prime ideals of R with the topology generated by basic open sets of the form $B_K = \{I \in \mathbf{cSpec}(R) \mid I \cap K = \emptyset\}$, where K varies over compact subsets of R. The next proposition suggests it is reasonable to restrict attention to closed ideals when working with coPolish commutative rings.

Proposition 1. *Let R be a coPolish commutative ring and $I \subseteq R$ an ideal. The following hold:*

1. *The topological closure of I is an ideal.*
2. *I with the subspace topology is coPolish if and only if I is topologically closed.*
3. *The quotient ring R/I with the quotient topology is a coPolish ring if and only if I is topologically closed.*

Proof. 1. Let C be the topological closure of I. It is a standard result for topological groups that C is an additive subgroup of R. If $r, x \in R$ and $rx \notin C$, then $W = \{y \in R \mid ry \notin C\}$ is an open neighborhood of x that is disjoint from I, hence $x \notin C$. Therefore, C is an ideal.

2. By an unpublished result of M. Schröder (personal communication), a subspace of a coPolish space is coPolish if and only if it is locally closed, so we only need to show that every locally closed ideal $I \subseteq R$ is closed. Let C be the topological closure of I and let $U \subseteq R$ be open such that $I = C \cap U$. Assume for a contradiction there is some $x \in C \setminus I$. Let $W = \{y \in R \mid x + y \in U\}$. If $y \in W \cap I$, then using the fact that I and C are ideals we would have $-y \in I$ and $x + y \in I$ hence $x \in I$, a contradiction. Therefore, W is an open neighborhood of x disjoint from I, which contradicts x being in the closure of I. It follows that $I = C$ is a closed subset of R.

3. Using standard techniques for topological groups, R/I is a Hausdorff topological ring if and only if I is closed. The claim follows because every coPolish space is Hausdorff and coPolish spaces are closed under Hausdorff quotients. ◻

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1 [9, Theorem 5.2] is stated for k_ω-spaces, but the same argument shows that coPolish spaces are closed under the construction of free topological groups. Similarly, $\mathbb{R}[X]$ with the topology described in [1] is the free topological commutative $\mathbb{R}$-algebra generated by the singleton $\{X\}$.

2 Topological products, sequential products, and localic products all coincide for coPolish spaces, so there is no ambiguity about the continuity of binary operations on coPolish spaces.

If R has the discrete topology then $\mathbf{cSpec}(R)$ and $\mathbf{Spec}(R)$ are identical, but they differ in general. The next result shows $\mathbf{cSpec}(R)$ is more suitable from a computability theoretic perspective (see [2, 7, 4, 6, 3]).

Theorem 1. *If R is a coPolish commutative ring, then $\mathbf{cSpec}(R)$ is a quasi-Polish space.*

Proof. Let $\mathbb{S}$ be the Sierpinski space, with the two elements $\bot$ (*bottom* or *false*) and $\top$ (*top* or *true*). Since R is coPolish the QCB_0 exponentials $\mathbb{S}^R$ and $\mathbb{S}^{R\times R}$ are quasi-Polish [11, 4]. $\mathbf{O}(R)$, the open set lattice of R with the Scott-topology, is homeomorphic to $\mathbb{S}^R$ [12, Proposition 2.2], hence also quasi-Polish. By [8, Theorem 3.3], the Scott-topology on $\mathbf{O}(R)$ has a basis of open sets of the form $\triangledown K = \{U \in \mathbf{O}(R) \mid K \subseteq U\}$ with K varying over compact subsets of R. Therefore, the map sending $I \in \mathbf{cSpec}(R)$ to its complement in $\mathbf{O}(R)$ is a topological embedding. For $U \in \mathbf{O}(R)$, the complement of U is a prime ideal of R if and only if

1. $1 \in U$,
2. $0 \not\in U$,
3. $x + y \in U$ implies $x \in U$ or $y \in U$,
4. $rx \in U$ implies $x \in U$, and
5. $x \in U$ and $y \in U$ implies $xy \in U$.

Define continuous maps $f_1, g_1, f_2, g_2 \colon \mathbf{O}(R) \to \mathbb{S}$ and $f_3, g_3, f_4, g_4, f_5, g_5 \colon \mathbf{O}(R) \to \mathbb{S}^{R\times R}$ as

1. $f_1(U) = (1 \in U)$ and $g_1(U) = \top$,
2. $f_2(U) = (0 \in U)$ and $g_2(U) = \bot$,
3. $f_3(U) = \lambda\langle x, y\rangle.(x + y \in U)$ and $g_3(U) = \lambda\langle x, y\rangle.(x + y \in U) \wedge ((x \in U) \vee (y \in U))$,
4. $f_4(U) = \lambda\langle r, x\rangle.(rx \in U)$ and $g_4(U) = \lambda\langle r, x\rangle.(rx \in U) \wedge (x \in U)$,
5. $f_5(U) = \lambda\langle x, y\rangle.(x \in U) \wedge (y \in U)$ and $g_5(U) = \lambda\langle x, y\rangle.(x \in U) \wedge (y \in U) \wedge (xy \in U)$,

and define continuous maps $f, g \colon \mathbf{O}(R) \to \mathbb{S} \times \mathbb{S} \times \mathbb{S}^{R\times R} \times \mathbb{S}^{R\times R} \times \mathbb{S}^{R\times R}$ as

$$f(U) = \langle f_1(U), f_2(U), f_3(U), f_4(U), f_5(U)\rangle \quad \text{and} \quad g(U) = \langle g_1(U), g_2(U), g_3(U), g_4(U), g_5(U)\rangle.$$

Then $\mathbf{cSpec}(R)$ is the equalizer of f and g, which implies $\mathbf{cSpec}(R)$ is quasi-Polish. □

If the ring operations and constants $0, 1 \in R$ are computable and $\mathbf{O}(R)$ is precomputable in the sense of [4], then f and g are computable hence $\mathbf{cSpec}(R)$ is a precomputable quasi-Polish space.

A continuous ring homomorphism $f \colon R \to S$ between coPolish commutative rings determines a continuous function $\mathbf{cSpec}(f) \colon \mathbf{cSpec}(S) \to \mathbf{cSpec}(R)$ defined as $\mathbf{cSpec}(f)(I) = f^{-1}(I)$ for each $I \in \mathbf{cSpec}(S)$. The function $\mathbf{cSpec}(f)$ is well-defined because the preimage of a prime ideal under a ring homomorphism is a prime ideal, and the continuity of $\mathbf{cSpec}(f)$ follows from the continuity of f. It is clear that $\mathbf{cSpec}(f)$ is computable whenever f is, so we obtain the following.

Theorem 2. **cSpec** *is a functor from the category of coPolish commutative rings and continuous (computable) ring homomorphisms to the category of quasi-Polish spaces and continuous (computable) functions.*
□

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