A note on making analytic sets open

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Let X be an effective quasi-Polish space, and let $\widehat{X}$ be the refinement of X that adds each (lightface) Σ^1_1-subset as a c.e.-open subset. Although $\widehat{X}$ is not quasi-Polish in general, in this note we will show that there is a "nice" computable embedding of $\widehat{X}$ into an effective quasi-Polish space Y.

Theorem 1. *There is a computable procedure which converts a code for an effective quasi-Polish space X into a code for an effective quasi-Polish space Y and a code for a computable retraction $f\colon Y \to X$ such that (i) each fiber of f has a unique maximal element (w.r.t. the specialization order of Y), (ii) the subspace $M \subseteq Y$ of maximal elements of the fibers of f is computably homeomorphic to $\widehat{X}$, and (iii) the restriction of f to M corresponds to the canonical map from $\widehat{X}$ to X.* □

In particular, if X is a T_1-space then $\widehat{X}$ embeds as the maximal elements of Y. Since the subspace of maximal elements of a quasi-Polish space is a strong Choquet space, this result generalizes Theorem 25.18 in [7]. In the special case that X is the Baire space, then $\widehat{X}$ is the Gandy-Harrington space, which has important applications in descriptive set theory (e.g. [6]) and more recently in the study of enumeration degrees [8]. It was already known that Theorem 6.1 of [9] implies the Gandy-Harrington space embeds as the maximal elements of a quasi-Polish space, but this note provides a direct computable construction.

In this note we define quasi-Polish spaces [1] using a characterization from [5] (see also [2], [3], and [4][1]).

Definition 1 ([5]). *Let $\prec$ be a transitive relation on $\mathbb{N}$. A subset $I \subseteq \mathbb{N}$ is an* ideal *(with respect to $\prec$) if and only if it is* (1) non-empty ($I \neq \emptyset$)*,* (2) a lower set ($a \prec b \in I \implies a \in I$)*, and* (3) directed ($a, b \in I \implies (\exists c \in I)[a \prec c \,\&\, b \prec c]$)*. The collection $\mathbf{I}(\prec)$ of all ideals has the topology generated by basic open sets of the form $[n]_\prec = \{I \in \mathbf{I}(\prec) \mid n \in I\}$. A space is* (effectively) quasi-Polish *if and only if it is (computably) homeomorphic to $\mathbf{I}(\prec)$ for some (c.e.) transitive relation $\prec$ on $\mathbb{N}$.* □

These definitions also apply to relations on sets that are computably isomorphic to a c.e. subset of $\mathbb{N}$. We call $\prec$ a *code* for a space X when X is computably homeomorphic to $\mathbf{I}(\prec)$. For example, the strict prefix relation $\prec_{\mathbb{N}^\mathbb{N}}$ on the set $\mathbb{N}^{<\mathbb{N}}$ of all finite sequences of natural numbers is a code for the Baire space $\mathbb{N}^\mathbb{N}$.

Let $\prec_1$ and $\prec_2$ be c.e. transitive relations on $\mathbb{N}$. Each c.e. subset $R \subseteq \mathbb{N} \times \mathbb{N}$ is a *code* for a computable partial function $\ulcorner R \urcorner :\subseteq \mathbf{I}(\prec_1) \to \mathbf{I}(\prec_2)$ defined as $\ulcorner R \urcorner(I) = \{n \in \mathbb{N} \mid (\exists m \in I)\, \langle m, n \rangle \in R\}$. Note that the domain of $\ulcorner R \urcorner$ is (lightface) Π^0_2. It was shown in [2][2] that a total function $f\colon \mathbf{I}(\prec_1) \to \mathbf{I}(\prec_2)$ is computable if and only if there is a c.e. code R with $f = \ulcorner R \urcorner$. We say a code R is $(\prec_1, \prec_2)$-*closed* if for each $\langle m, n \rangle \in R$, if $m \prec_1 m'$ then $\langle m', n \rangle \in R$ and if $n' \prec_2 n$ then $\langle m, n' \rangle \in R$. The $(\prec_1, \prec_2)$-closure of R can be enumerated given R, $\prec_1$, and $\prec_2$, and taking the closure does not change the interpretation of codes for total functions.

For the remainder of this note, fix a c.e. transitive relation $\prec$ on $\mathbb{N}$ and an enumeration $(R_i)_{i \in \mathbb{N}}$ of all $(\prec_{\mathbb{N}^\mathbb{N}}, \prec)$-closed c.e. subsets of $\mathbb{N} \times \mathbb{N}$. We obtain an enumeration $(A_i)_{i \in \mathbb{N}}$ of all Σ^1_1-subsets of $\mathbf{I}(\prec)$ by defining $A_i = \{I \in \mathbf{I}(\prec) \mid (\exists P \in \mathbf{I}(\prec_{\mathbb{N}^\mathbb{N}})) \ulcorner R_i \urcorner(P) = I\}$. For $i, n \in \mathbb{N}$, let $R_i^{(n)}$ be the finite subset of R_i enumerated within n computation steps. Given a set S, let $\mathcal{P}_{\mathrm{fin}}(S)$ be the set of all finite subsets of S. Define a relation $\sqsubset$ on $\mathbb{N} \times \mathbb{N} \times \mathcal{P}_{\mathrm{fin}}(\mathbb{N}^{<\mathbb{N}} \times \mathbb{N})$ as $\langle m, x, F \rangle \sqsubset \langle n, y, G \rangle$ if and only if the following all hold:

1. $m < n$, $x \prec y$, and $F \subseteq G$ (monotonicity),
2. $(\forall \langle \sigma, i \rangle \in F)(\forall \langle \rho, w \rangle \in R_i^{(m)})\, [\rho \prec_{\mathbb{N}^\mathbb{N}} \sigma \implies w \prec y]$, and
3. $(\forall \langle \sigma, i \rangle \in F)(\exists \langle \tau, i \rangle \in G)\, [\sigma \prec_{\mathbb{N}^\mathbb{N}} \tau \,\&\, \langle \tau, x \rangle \in R_i]$.

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[1] Theorem 1 of [4] contains a related refinement result that applies to Π^0_1-sets.

[2] We incorrectly omitted the requirement that $\prec_1$ be a c.e. relation in the original paper. We are grateful to Ivan Georgiev for pointing out this mistake and providing a counter example.

It is easy to see that $\sqsubset$ is c.e. and transitive. Define $f\colon \mathbf{I}(\sqsubset) \to \mathbf{I}(\prec)$ as $f(J) = \{x \in \mathbb{N} \mid (\exists m, F)\, \langle m, x, F\rangle \in J\}$ and define $g\colon \mathbf{I}(\prec) \to \mathbf{I}(\sqsubset)$ as $g(I) = \{\langle m, x, \emptyset\rangle \mid m \in \mathbb{N} \,\&\, x \in I\}$. The next lemma implies $f(J)$ is a lower set, and the other requirements for $f(J)$ and $g(I)$ to be ideals can be verified directly. Thus f and g are total computable functions satisfying $f(g(I)) = I$ for each $I \in \mathbf{I}(\prec)$, hence $\mathbf{I}(\prec)$ is a computable retract of $\mathbf{I}(\sqsubset)$.

Lemma 1. *If $J \in \mathbf{I}(\sqsubset)$ and $\langle n, y, G\rangle \in J$, then (i) $\langle m, y, G\rangle \in J$ for each $m \in \mathbb{N}$, (ii) $x \prec y$ implies $\langle n, x, G\rangle \in J$, and (iii) $F \subseteq G$ implies $\langle n, y, F\rangle \in J$. In particular, $g(I)$ is the unique minimal element of $f^{-1}(\{I\})$ for each $I \in \mathbf{I}(\prec)$.*

Proof. For (i), fix $m \in \mathbb{N}$. Using the directedness of J, we can find $\langle n', y', G'\rangle \in J$ with $\langle n, y, G\rangle \sqsubset \langle n', y', G'\rangle$ and $m < n'$. Then $\langle m, y, G\rangle \sqsubset \langle n', y', G'\rangle$ hence $\langle m, y, G\rangle \in J$. The proof for (ii) and (iii) are similar, but (ii) uses the assumption that R_i is $(\prec_{\mathbb{N}^\mathbb{N}}, \prec)$-closed to show the third item in the definition of $\sqsubset$ holds. □

Lemma 2. *For each $I \in \mathbf{I}(\prec)$, there is a unique maximal element $\hat{I}$ in $f^{-1}(\{I\})$.*

Proof. $f^{-1}(\{I\})$ is non-empty because it contains $g(I)$, and it is quasi-Polish because it is $\mathbf{\Pi}^0_2$. We show it is directed with respect to $\subseteq$, which is the specialization order on $f^{-1}(\{I\})$. Assume $f(J_1) = f(J_2) = I$. Set $J = \{\langle m, x, F_1 \cup F_2\rangle \mid \langle m, x, F_1\rangle \in J_1 \,\&\, \langle m, x, F_2\rangle \in J_2\}$. It is clear that J is non-empty, and the previous lemma can be used to show it is a lower set. To see J is directed, assume $\langle m, x, F_1 \cup F_2\rangle$ and $\langle m', x', F'_1 \cup F'_2\rangle$ are in J. Fix a $\sqsubset$-upper bound $\langle n_i, y_i, G_i\rangle$ of $\langle m, x, F_i\rangle$ and $\langle m', x', F'_i\rangle$ in J_i ($i \in \{1, 2\}$). Let y be a $\prec$-upper bound of y_1 and y_2 in I, and set $n = \max(n_1, n_2)$. It is straightforward to show that $\langle n, y, G_1 \cup G_2\rangle$ is in J, and it is a $\sqsubset$-upper bound of $\langle m, x, F_1 \cup F_2\rangle$ and $\langle m', x', F'_1 \cup F'_2\rangle$, hence J is directed. Thus $J_1, J_2 \subseteq J \in f^{-1}(\{I\})$, hence $f^{-1}(\{I\})$ is directed. It follows from the sobriety of $f^{-1}(\{I\})$ that it has a unique maximal element. □

Let $M = \{\hat{I} \in \mathbf{I}(\sqsubset) \mid I \in \mathbf{I}(\prec)\}$, which is a Π^1_1-subset of $\mathbf{I}(\sqsubset)$. For $i \in \mathbb{N}$ and $\sigma \in \mathbb{N}^{<\mathbb{N}}$, let U^σ_i be the c.e.-open subset of all $J \in \mathbf{I}(\sqsubset)$ with $\langle m, x, \{\langle \sigma, i\rangle\}\rangle \in J$ for some $m, x \in \mathbb{N}$. For each $\sigma \in \mathbb{N}^{<\mathbb{N}}$ we also define $A^\sigma_i = \{I \in \mathbf{I}(\prec) \mid (\exists P \in \mathbf{I}(\prec_{\mathbb{N}^\mathbb{N}}))\,[\sigma \in P \,\&\, \ulcorner R_i \urcorner(P) = I]\}$. We can assume our enumeration of $(R_i)_{i\in\mathbb{N}}$ is reasonable enough that there is a computable function h satisfying $R_{h(\sigma,i)} = R_i \cap (\{\tau \in \mathbb{N}^{<\mathbb{N}} \mid \sigma \preceq_{\mathbb{N}^\mathbb{N}} \tau\} \times \mathbb{N})$ for each $\sigma \in \mathbb{N}^{<\mathbb{N}}$ and $i \in \mathbb{N}$, hence $A_{h(\sigma,i)} = A^\sigma_i$. The next lemma completes the proof of Theorem 1.

Lemma 3. *$U^\sigma_i \subseteq f^{-1}(A^\sigma_i)$ and $U^\sigma_i \cap M = f^{-1}(A^\sigma_i) \cap M$ for each $i \in \mathbb{N}$ and $\sigma \in \mathbb{N}^{<\mathbb{N}}$.*

Proof. Assume $J \in U^\sigma_i \cap M$ and fix a $\sqsubset$-ascending sequence $(\langle m_k, x_k, F_k\rangle)_{k\in\mathbb{N}}$ that is cofinal in J. We can assume without loss of generality that $\langle \sigma, i\rangle \in F_0$, so the definition of $\sqsubset$ implies there is a $\prec_{\mathbb{N}^\mathbb{N}}$-ascending sequence $(\sigma_k)_{k\in\mathbb{N}}$ with $\sigma_0 = \sigma$ and $\langle \sigma_k, i\rangle \in F_k$ and $\langle \sigma_{k+1}, x_k\rangle \in R_i$ for each $k \in \mathbb{N}$. Let $P \in \mathbf{I}(\prec_{\mathbb{N}^\mathbb{N}})$ be the ideal generated by the $\prec_{\mathbb{N}^\mathbb{N}}$-ascending sequence $(\sigma_k)_{k\in\mathbb{N}}$, and let $I \in \mathbf{I}(\prec)$ be the ideal generated by the $\prec$-ascending sequence $(x_k)_{k\in\mathbb{N}}$. Then $\sigma \in P$ and it can be shown that $\ulcorner R_i \urcorner(P) = I$ by using the assumption that R_i is $(\prec_{\mathbb{N}^\mathbb{N}}, \prec)$-closed, hence $I \in A^\sigma_i$. Clearly $I \subseteq f(J)$, and if $x \in f(J)$ then since $(\langle m_k, x_k, F_k\rangle)_{k\in\mathbb{N}}$ is cofinal in J there is $k \in \mathbb{N}$ with $x \prec x_k$, hence $x \in I$. Therefore, $f(J) = I \in A^\sigma_i$.

For the second claim, assume $\hat{I} \in f^{-1}(A^\sigma_i) \cap M$ and fix $P \in \mathbf{I}(\prec_{\mathbb{N}^\mathbb{N}})$ with $\sigma \in P$ and $\ulcorner R_i \urcorner(P) = I$. Define $J = \{\langle m, x, F\rangle \mid m \in \mathbb{N} \,\&\, x \in I \,\&\, F \subseteq P \times \{i\}$ is finite$\}$. Then $J \in U^\sigma_i$ and $f(J) = I$ hence $\hat{I} \in U^\sigma_i$. □

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