A note on the spatiality of localic products of countably based sober spaces

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Let **Top** be the category of topological spaces and continuous maps, **Loc** the category of locales, and $\mathbf{\Omega}\colon \mathbf{Top} \to \mathbf{Loc}$ the usual functor mapping spaces to locales. $\mathbf{\Omega}$ preserves colimits (since it has a right adjoint $\mathbf{pt}\colon \mathbf{Loc} \to \mathbf{Top}$), but $\mathbf{\Omega}$ does not preserve finite products in general. The purpose of this note is to investigate subcategories of countably based sober spaces for which the restriction of $\mathbf{\Omega}$ does preserve finite products.

Let S_0 be the countable space defined in [2]. The underlying set of S_0 is $\mathbb{N}^{<\mathbb{N}}$, the set of finite sequences of natural numbers. A subbasis for the open subsets of S_0 is given by sets of the form $\{\tau \in \mathbb{N}^{<\mathbb{N}} \mid \sigma \not\preceq \tau\}$, where $\sigma \in \mathbb{N}^{<\mathbb{N}}$ and $\preceq$ is the prefix relation. Note that S_0 is a countably based sober space, its specialization order is the reverse of the prefix relation, and that S_0 has uncountably many distinct open sets ([2]; Proposition 6.1).

We first show that the localic product $\mathbf{\Omega}(S_0) \times_l \mathbf{\Omega}(S_0)$ is not spatial by describing a winning strategy for Player I in the game $\mathcal{G}(S_0, S_0)$ defined by T. Plewe (see Theorem 1.1 in [5] and the paragraph above it for a definition of the game). The proof strategy for the following lemma is essentially the same as P. Johnstone's proof that $\mathbf{\Omega}(\mathbb{Q}) \times_l \mathbf{\Omega}(\mathbb{Q})$ is not spatial (see Proposition II-2.14 of [4]), but the game theoretic approach allows us to hide the use of transfinite ordinals.

Lemma 1. *The localic product $\mathbf{\Omega}(S_0) \times_l \mathbf{\Omega}(S_0)$ is not spatial.*

Proof. We denote the length of $\sigma \in \mathbb{N}^{<\mathbb{N}}$ by $|\sigma|$. The empty string is denoted as ε, and the string consisting of m zeros is written $0^{(m)}$. The string obtained by appending $n \in \mathbb{N}$ to $\sigma \in \mathbb{N}^{<\mathbb{N}}$ is written $\sigma \diamond n$. We also write $\sigma \diamond \tau$ for the concatenation of strings. For $\sigma, \tau \in \mathbb{N}^{<\mathbb{N}}$, define

$$F_{\sigma,\tau} = \{s \diamond n \mid n \in \mathbb{N} \;\&\; s \in \mathbb{N}^{<\mathbb{N}} \;\&\; s \preceq \sigma \;\&\; s \diamond n \not\preceq \sigma \;\&\; n \leq |\sigma| + |\tau|\}, \text{ and}$$
$$U_{\sigma,\tau} = \{t \in \mathbb{N}^{<\mathbb{N}} \mid (\forall s \in F_{\sigma,\tau})\, s \not\preceq t\}.$$

Then $\sigma \in U_{\sigma,\tau}$, hence $\mathcal{U} = \{U_{\sigma,\tau} \times U_{\tau,\sigma} \mid \sigma, \tau \in \mathbb{N}^{<\mathbb{N}}\}$ is an open cover of $S_0 \times S_0$.

Observe that if $s \in U_{\sigma,\tau}$ and every element of s is less than or equal to $|\sigma| + |\tau|$ then $s \preceq \sigma$. Also note that if $U \subseteq S_0$ is open and $\sigma \in U$, then there exist infinitely many $n \in \mathbb{N}$ such that every string that has $\sigma \diamond n$ as a prefix is also in U. This is because there must be a finite $F \subseteq S_0$ such that the basic open set $W = \{\tau \in \mathbb{N}^{<\mathbb{N}} \mid (\forall s \in F)\, s \not\preceq \tau\}$ satisfies $\sigma \in W \subseteq U$. Fix any $n \in \mathbb{N}$ that is strictly larger than any element contained in any of the strings in F. Then for each $s \in F$, we have that $s \not\preceq \sigma \diamond n$ and $\sigma \diamond n \not\preceq s$, hence no extension of $\sigma \diamond n$ has s as a prefix. It follows that every extension of $\sigma \diamond n$ must be in W and therefore also in U.

Player I initializes the game by playing $S_0 \times S_0$ and the open covering $\mathcal{U}$ of $S_0 \times S_0$. The game begins with round 1. For convenience, define $V_0 = W_0 = S_0$, and $x_0 = y_0 = \varepsilon$, and $m_0 = n_0 = 0$. Player I's strategy for the i-th round ($i \geq 1$) proceeds as follows:

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1. Player I chooses $m_i \in \mathbb{N}$ such that every sequence extending $y_{i-1} \diamond m_i$ is in W_{i-1}. Player I then plays $x_i = x_{i-1} \diamond n_{i-1} \diamond 0^{(m_i)}$.
2. Player II must respond with an open subset $V_i \subseteq V_{i-1}$ containing x_i.
3. Next, Player I finds distinct n_i and n'_i in $\mathbb{N}$ such that any sequence that has either $x_i \diamond n_i$ or $x_i \diamond n'_i$ as a prefix is in V_i. Player I plays $y_i = y_{i-1} \diamond m_i \diamond 0^{(n_i + n'_i)}$.
4. Player II must respond with an open subset $W_i \subseteq W_{i-1}$ containing y_i.

The game then continues on to round $i+1$.

We show that at the end of each round $i \geq 1$, the open rectangle $V_i \times W_i$ chosen by Player II is not a subset of any open rectangle in $\mathcal{U}$. Fix any $\sigma, \tau \in \mathbb{N}^{<\mathbb{N}}$ with $x_i \in U_{\sigma,\tau}$ and $y_i \in U_{\tau,\sigma}$. Since $y_{i-1} \preceq y_i$ we have $y_{i-1} \in U_{\tau,\sigma}$, and an inductive argument (keep reading) yields $y_{i-1} \preceq \tau$. Using the fact that $|y_{i-1}| \geq n_{i-1}$ it can be shown that every element occurring in x_i is less than or equal to $|y_{i-1}| \leq |\sigma| + |\tau|$, hence the assumption $x_i \in U_{\sigma,\tau}$ and the observation at the top of the second paragraph of this proof implies $x_i \preceq \sigma$. Similarly, every element of y_i is less than or equal to $|x_i| \leq |\tau| + |\sigma|$, hence $y_i \in U_{\tau,\sigma}$ implies $y_i \preceq \tau$ (thereby completing the inductive argument). Either $x_i \diamond n_i \not\preceq \sigma$ or $x_i \diamond n'_i \not\preceq \sigma$, and $n_i, n'_i \leq |y_i| \leq |\sigma| + |\tau|$, thus $x_i \diamond n_i \notin U_{\sigma,\tau}$ or $x_i \diamond n'_i \notin U_{\sigma,\tau}$, but both $x_i \diamond n_i$ and $x_i \diamond n'_i$ are in V_i, so we conclude that $V_i \times W_i \not\subseteq U_{\sigma,\tau} \times U_{\tau,\sigma}$. Therefore, the above strategy is winning for Player I, hence $\mathbf{\Omega}(S_0) \times_l \mathbf{\Omega}(S_0)$ is not spatial. □

Let $\omega\mathbf{Sob}$ be the category of countably based sober spaces, and let $\mathbf{QPol}$ be the category of quasi-Polish spaces [1]. $\omega\mathbf{Sob}$ and $\mathbf{QPol}$ are closed under countable limits (as defined in $\mathbf{Top}$), and the restriction of $\mathbf{\Omega}$ to $\mathbf{QPol}$ preserves all countable limits (Theorems 4.4 and 4.5 of [3]).

Theorem 1. *Assume $\mathcal{C}$ is a full subcategory of $\omega\mathbf{Sob}$ satisfying:*

(1) $\mathcal{C}$ is closed under finite limits (as defined in $\mathbf{Top}$),
(2) the restriction of $\mathbf{\Omega}$ to $\mathcal{C}$ preserves finite products,
(3) $\mathcal{C}$ contains $\mathcal{P}(\mathbb{N})$ (the powerset of $\mathbb{N}$ with the Scott-topology), and
(4) every space in $\mathcal{C}$ is co-analytic (i.e., homeomorphic to a $\mathbf{\Pi}^1_1$-subspace of $\mathcal{P}(\mathbb{N})$).

Then $\mathcal{C}$ is a full subcategory of $\mathbf{QPol}$.

Proof. Assume for a contradiction that there is some space X in $\mathcal{C}$ which is not quasi-Polish. X is co-analytic by (4), hence Theorem 7.2 of [2] implies there is a $\mathbf{\Pi}^0_2$-subspace Y of X which is homeomorphic to either S_0 or $\mathbb{Q}$ (the candidates S_D and S_1 mentioned in [2] can be omitted because X is sober). Since Y is a $\mathbf{\Pi}^0_2$-subspace of X, it is the equalizer of a pair of continuous functions $f, g\colon X \to \mathcal{P}(\mathbb{N})$ (see the concluding section of [1]), hence Y is in $\mathcal{C}$ by (1), (3), and the assumption that $\mathcal{C}$ is a full subcategory of $\omega\mathbf{Sob}$. Lemma 1 and the fact that $\mathbf{\Omega}(\mathbb{Q}) \times_l \mathbf{\Omega}(\mathbb{Q})$ is not spatial imply $\mathbf{\Omega}(Y \times Y) \neq \mathbf{\Omega}(Y) \times_l \mathbf{\Omega}(Y)$. Therefore, the restriction of $\mathbf{\Omega}$ to $\mathcal{C}$ does not preserve products, which contradicts (2). □

We conjecture that the above theorem still holds if (3) is omitted. It is consistent with ZFC to replace "co-analytic" in (4) with any level of the projective hierarchy, and we conjecture that it is consistent with ZF+(Dependent Choice) if (4) is removed completely.

References

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