Tutorial on Quasi-Polish Spaces[1]

Matthew de Brecht

Kyoto University

Dagstuhl-Seminar 21461 (November 14-19, 2021)

[1] This work was supported by JSPS KAKENHI Grant Number 18K11166.

Table of Contents

Table of Contents

Descriptive set theory

- Quasi-Polish spaces are a generalization of Polish spaces that includes non-metrizable spaces
  - "Quasi-Polish"
    = "countably based & completely quasi-metrizable"

(The original characterization used Smyth-complete quasi-metrics, but J. Goubault-Larrecq showed they can also be characterized using Yoneda-complete quasi-metrics).

Who cares about non-metrizable spaces?

- Descriptive set theory (DST) rarely needs metrizability.
  - Polish and quasi-Polish spaces have essentially the same DST.
  - Many (but not all) proofs that use complete metrics can be replaced with complete quasi-metrics or other proof methods.
- Non-metrizable spaces have many important applications.
  - The Gandy-Harrington space (effective descriptive set theory).
  - Non-trivial solutions to domain equations such as $D \cong D^D$.
  - Spectrums of commutative rings with the Zariski topology.
- You need non-metrizable spaces to topologize topology.
  - There is no Hausdorff topology on $\mathbf{O}(1)$ (the open subsets of the singleton space $1 = \{*\}$) that makes countable unions $\bigcup\colon \mathbf{O}(1)^{\mathbb{N}} \to \mathbf{O}(1)$ a continuous function.
  - Topologizing the powerset of the natural numbers as $2^{\mathbb{N}}$ (Cantor space) is a topological representation of $\mathbf{\Delta}^0_1(\mathbb{N})$, but not $\mathbf{\Sigma}^0_1(\mathbb{N})$, since the latter is closed under countable joins.

The following is a modification of the Borel Hierarchy, due to V. Selivanov, which is needed for non-metrizable spaces.

Definition (Borel - Selivanov)

Let X be a topological space. For each ordinal α $(1 \leq \alpha < \omega_1)$ define $\mathbf{\Sigma}^0_\alpha(X)$ inductively as follows:

- $\mathbf{\Sigma}^0_1(X)$ is the set of open subsets of X,
- For $\alpha > 1$, $A \in \mathbf{\Sigma}^0_\alpha(X)$ iff A can be expressed in the form

$$A = \bigcup_{i \in \mathbb{N}} U_i \setminus V_i,$$

  where $U_i, V_i \in \mathbf{\Sigma}^0_{\beta_i}(X)$ for some $\beta_i < \alpha$.

Furthermore, $A \in \mathbf{\Pi}^0_\alpha(X) \iff X \setminus A \in \mathbf{\Sigma}^0_\alpha(X)$ and $A \in \mathbf{\Delta}^0_\alpha(X) \iff A \in \mathbf{\Sigma}^0_\alpha(X) \cap \mathbf{\Pi}^0_\alpha(X)$

$\mathbf{B}(X) = \bigcup_{\alpha<\omega_1} \mathbf{\Sigma}^0_\alpha(X)$ is the set of Borel subsets of X.

Borel Hierarchy

- Traditionally, the $\mathbf{\Sigma}^0_2$-sets are defined as $\mathcal{F}_\sigma$ (countable unions of closed sets), and the $\mathbf{\Pi}^0_2$-sets are defined as $\mathcal{G}_\delta$ (countable intersections of open sets).
- However, this is problematic for non-metrizable spaces:
  - Define the Sierpinski space to be the set $\mathbb{S} = \{\bot, \top\}$ with topology $\mathbf{O}(\mathbb{S}) = \{\emptyset, \{\top\}, \mathbb{S}\}$.
  - Then $\{\top\} \in \mathbf{\Sigma}^0_1$ but $\{\top\} \notin \mathcal{F}_\sigma$, and $\{\bot\} \in \mathbf{\Pi}^0_1$ but $\{\bot\} \notin \mathcal{G}_\delta$. (so under the traditional definition, $\{\top\} \notin \mathbf{\Delta}^0_n$ for all $n < \omega$.)
- Using V. Selivanov's modified definition we obtain an actual hierarchy (which still agrees with the classical definition for metrizable spaces).

Quasi-Polish spaces

Definition

A topological space is quasi-Polish iff it satisfies any of the following equivalent properties:

- It is a countably based space with a topology generated by a (Smyth-) complete quasi-metric
- It is homeomorphic to a $\mathbf{\Pi}^0_2$-subspace of $\mathbb{S}^{\mathbb{N}}$
  - $\mathbb{S}^{\mathbb{N}} \cong \mathcal{P}(\mathbb{N})$, the powerset of the natural numbers with the Scott-topology
- It is a T_0-space and the image of a Polish space under an open continuous function
- It is countably based and has an admissible representation with Polish domain
- and more...

Examples

The following are quasi-Polish:

- Polish spaces
  - $\mathbb{N}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{N}^{\mathbb{N}}$, etc.
- Countably based spaces that are locally homeomorphic to some Polish space
  - the line with two origins
  - countably based non-Hausdorff topological manifolds
  - etc.
- ω-continuous domains
  - $\mathbb{S}$, $\mathbb{N}_{\perp}$, $\mathcal{P}(\mathbb{N})$, etc.
- Countably based spectral spaces
  - Spec$(\mathbb{Z})$, Spec$(\mathbb{Q}[x_1, \ldots, x_n])$, etc.
- Countably based locally compact sober spaces
  - (Contains the last two categories)

(There are quasi-Polish spaces which do not fit into any of the above categories).

Counter-examples

The following are not quasi-Polish:

- Non-Polish metric spaces
  - S_2: $\mathbb{Q}$ with the subspace topology inherited from $\mathbb{R}$
  - etc.
- Non-sober spaces
  - S_1: $\mathbb{N}$ with the cofinite topology
  - S_D: $(\mathbb{N}, <)$ with the Scott-topology
  - etc.
- And some others
  - S_0: $(\mathbb{N}^{<\infty}, \preceq_{\text{prefix}})$ with the lower topology
  - the Gandy-Harrington space
  - etc.

Theorem (Generalized Hurewicz Theorem)

Any $\mathbf{\Pi}^1_1$ subspace of a quasi-Polish space which is not quasi-Polish will contain a $\mathbf{\Pi}^0_2$-subset homeomorphic to one of the four spaces (S_2, S_1, S_D, or S_0) highlighted above.

Some basic DST results

- A space is Polish if and only if it is a metrizable quasi-Polish space.
- Every countably based T_0-space embeds into a quasi-Polish space.
- If X is quasi-Polish, then $A \subseteq X$ is quasi-Polish iff $A \in \mathbf{\Pi}^0_2(X)$.
- QPol is the smallest (up to equivalence) full subcategory of Top that contains $\mathbb{S}$ and is closed under countable limits.
- Every quasi-Polish space is a Baire space, which means that the intersection of countably many dense open sets is dense.
  - They are completely Baire: every closed subspace is Baire.
- A partial continuous function into a quasi-Polish space can be extended to a continuous function with $\mathbf{\Pi}^0_2$-domain.
- Extending a quasi-Polish topology with countably many $\mathbf{\Delta}^0_2$-sets results in a quasi-Polish topology.
- If X is quasi-Polish and $A \subseteq X$ is Borel, then there is a quasi-Polish topology that refines the topology on X and such that A is open in the refinement.

Table of Contents

Domain Theory

- Quasi-Polish spaces are a generalization of ω-continuous domains
  - A space is quasi-Polish iff it is homeomorphic to the non-compact elements of an ω-continuous domain.
  - A space is quasi-Polish iff it is a convergent approximation space (V. Becher & S. Grigorieff, 2014).
  - They can also be characterized as spaces of ideals, similar to the "abstract basis" characterization of domains, but without the finite interpolation requirement.

Spaces of ideals

Definition

Let $\prec$ be a transitive relation on $\mathbb{N}$. A subset $I \subseteq \mathbb{N}$ is an ideal (with respect to $\prec$) if and only if:

1. $I \neq \emptyset$, *(I is non-empty)*
2. $(\forall a \in I)(\forall b \in \mathbb{N})\,(b \prec a \Rightarrow b \in I)$, *($I$ is a lower set)*
3. $(\forall a, b \in I)(\exists c \in I)\,(a \prec c \,\&\, b \prec c)$. *($I$ is directed)*

The collection $\mathbf{I}(\prec)$ of all ideals has the topology generated by basic open sets of the form $[n]_\prec = \{I \in \mathbf{I}(\prec) \mid n \in I\}$ for $n \in \mathbb{N}$.

- $\mathbb{N}$ encodes pieces of information about points in an abstract space.
  - The relation $a \prec b$ means b contains more information than a.
  - A point (i.e., an ideal $I \in \mathbf{I}(\prec)$) is any consistent collection of arbitrarily precise information.

Spaces of Ideals

The following was shown in joint work with A. Pauly & M. Schröder when investigating *computable* quasi-Polish spaces.

Theorem (d., A. Pauly, & M. Schröder, 2019)

A space is quasi-Polish if and only if it is homeomorphic to a space of the form $\mathbf{I}(\prec)$ for some transitive relation $\prec$ on $\mathbb{N}$.

- If $\prec$ is a partial order, then we get ω-algebraic domains.
- If $\prec$ satisfies the following *finite interpolation property*, then we get ω-continuous domains.
    - For every finite $F \subseteq \mathbb{N}$ and $z \in \mathbb{N}$,

$$F \prec z \text{ implies } (\exists y \in \mathbb{N})\, F \prec y \prec z$$

where $F \prec z$ is shorthand for $(\forall x \in F)\, x \prec z$.

Example

If $=$ is the equality relation on $\mathbb{N}$, then $\mathbf{I}(=)$ is homeomorphic to $\mathbb{N}$ with the discrete topology.

We also consider relations on other countable sets (encoded by $\mathbb{N}$)

Example

If $\prec$ is the strict prefix relation on the set $\mathbb{N}^{<\infty}$ of finite sequences of natural numbers, then $\mathbf{I}(\prec)$ is homeomorphic to the Baire space $\mathbb{N}^{\mathbb{N}}$.

Example

If $\subseteq$ is the usual subset relation on the set $\mathcal{P}_{\text{fin}}(\mathbb{N})$ of finite subsets of $\mathbb{N}$, then $\mathbf{I}(\subseteq)$ is homeomorphic to $\mathcal{P}(\mathbb{N})$, the powerset of the natural numbers with the Scott-topology.

Examples: Completion of separable metric spaces

- Let (X, d) be a separable metric space. Fix a countable dense subset $D \subseteq X$, and define a transitive relation $\prec$ on $P = D \times \mathbb{N}$ as

$$\langle x, n\rangle \prec \langle y, m\rangle \iff d(x, y) < 2^{-n} - 2^{-m}.$$

  - This definition guarantees that the open ball with center x and radius 2^{-n} contains the closed ball with center y and radius 2^{-m}.
- $\mathbf{I}(\prec)$ is homeomorphic to the metric completion of (X, d).
  - This is related to the *formal ball* models in domain theory.
  - (Note that the metrizable spaces of ideals are precisely the Polish spaces)

Upper and lower powerspaces

The upper and lower powerspaces are used for

- (Topology) Constructing multi-valued functions
- (Computer science) Modeling non-deterministic programs
- (Logic) Providing semantics for modal logics

Definition

Given a topological space X with topology $\mathbf{O}(X)$, define the topological spaces $\mathbf{A}(X)$ and $\mathbf{K}(X)$ as follows:

- $\mathbf{A}(X)$ (Lower powerspace):
  - Set of closed subsets of X with lower Vietoris topology, which has subbasis $\Diamond U := \{A \in \mathbf{A}(X) \mid A \cap U \neq \emptyset\}$ for $U \in \mathbf{O}(X)$
- $\mathbf{K}(X)$ (Upper powerspace):
  - Set of saturated compact subsets of X with upper Vietoris topology, which has subbasis $\Box U := \{K \in \mathbf{K}(X) \mid K \subseteq U\}$ for $U \in \mathbf{O}(X)$

Note: $S \subseteq X$ is saturated iff $S = \bigcap\{W \in \mathbf{O}(X) \mid S \subseteq W\}$.
(Every subset of a T_1-space is saturated).

Upper and lower powerspaces

Somewhat surprisingly, several constructions on ω-algebraic domains due to M. Smyth still apply in this generality.

Theorem

Let $\prec$ be a binary transitive relation on $\mathbb{N}$. Define binary transitive relations $\prec_L, \prec_U$ on $\mathcal{P}_{\text{fin}}(\mathbb{N})$ as follows:

- $A \prec_L B$ iff $(\forall a \in A)(\exists b \in B)\, a \prec b$,
- $A \prec_U B$ iff $(\forall b \in B)(\exists a \in A)\, a \prec b$.

Then

- $\mathbf{I}(\prec_L) \cong \mathbf{A}(\mathbf{I}(\prec))$, the lower powerspace of $\mathbf{I}(\prec)$
  (i.e. the set of closed sets with the lower Vietoris topology)
- $\mathbf{I}(\prec_U) \cong \mathbf{K}(\mathbf{I}(\prec))$, the upper powerspace of $\mathbf{I}(\prec)$
  (i.e. the set of saturated compact sets with the upper Vietoris topology)

Double powerspace

Definition (Double powerspace)

- $\mathbb{S}^{\mathbb{S}^X}$ is the space of continuous functions from $\mathbb{S}^X$ to $\mathbb{S}$
(The notation can be justified by embedding QPol into the cartesian closed category QCB_0.)
  - The exponentials $\mathbb{S}^X$ and $\mathbb{S}^{\mathbb{S}^X}$ in QCB_0 both have the Scott-topology, which is equivalent to the compact-open topology when X is quasi-Polish. If X is quasi-Polish then $\mathbb{S}^X$ is quasi-Polish if and only if X is locally compact.

- This is realized by composing $\mathbf{A}$ and $\mathbf{K}$, because $\mathbb{S}^{\mathbb{S}^X} \cong \mathbf{A}(\mathbf{K}(X)) \cong \mathbf{K}(\mathbf{A}(X))$ when X is quasi-Polish (d. & T. Kawai, 2019).
  - This is closely related to work by S. Vickers on the double powerlocale and work by P. Taylor on Abstract Stone Duality.
  - See also recent work by E. Neumann investigating applications of the upper, lower, and double powerspace functors on effective represented spaces.

Valuations powerspace

Definition (Valuations)

- A valuation on X is a continuous function $\nu\colon \mathbf{O}(X) \to \overline{\mathbb{R}}_+$ satisfying:
  1. $\nu(\emptyset) = 0$, and (*strictness*)
  2. $\nu(U) + \nu(V) = \nu(U \cup V) + \nu(U \cap V)$. (*modularity*)
- $\mathbf{V}(X)$ (valuations powerspace) is the set of all valuations on X with the topology induced by subbasic opens of the form $\langle U, q\rangle := \{\nu \in \mathbf{V}(X) \mid \nu(U) > q\}$ with $U \in \mathbf{O}(X)$ and $q \in \overline{\mathbb{R}}_+ \setminus \{\infty\}$.

- $\mathbf{O}(X)$ and $\overline{\mathbb{R}}_+ = [0, \infty]$ are assumed to have the Scott-topology.
- Valuations are commonly used instead of Borel measures in computable topology and constructive logic.
  - Every (locally finite) valuation on a quasi-Polish space extends (uniquely) to a Borel measure. Conversely, restricting any Borel measure to the open subsets results in a valuation.

Valuations powerspace

Theorem

Let $\prec$ be a binary transitive relation on $\mathbb{N}$. Define the binary transitive relation $\prec_V$ on the (countable) set $\{r :\subseteq \mathbb{N} \to \mathbb{Q}_{>0} \mid dom(r)$ is finite $\}$ as $r \prec_V s$ iff

$\sum_{b \in F} r(b) < \sum\{s(c) \mid c \in dom(s) \,\&\, (\exists b \in F)\, b \prec c\}$ for every non-empty $F \subseteq dom(r)$.

Then $\mathbf{I}(\prec_V) \cong \mathbf{V}(\mathbf{I}(\prec))$.

- This is related to work by C. Jones on the probabilistic powerdomain in domain theory, which is used to model probabilistic computations.

Table of Contents

Computable topology

- Quasi-Polish spaces are "complete" computable topological spaces
    - The *effective* part of the definition of a computable topological space uniquely determines a *complete* computable topological space, which is computably equivalent to a space of ideals over a c.e. relation.
        - This leads to effective notions of quasi-Polish spaces that are equivalent to earlier proposals.
    - The advantage of the "spaces of ideals" approach to computable topology is that it requires fewer assumptions on the mathematical foundations.
        - For example, it can be formalized within second-order arithmetic.

Effective version of quasi-Polish spaces

Theorem (d., A. Pauly, & M. Schröder, 2019)

The following are equivalent for a represented space $\mathbf{X}$:

- $\mathbf{X}$ is computably isomorphic to $\mathbf{I}(\prec)$ for some c.e. transitive relation $\prec$ on $\mathbb{N}$.
- $\mathbf{X}$ is computably isomorphic to a Π^0_2-subspace of $\mathbf{O}(\mathbb{N})$.
- $\mathbf{X}$ has an effectively fiber-overt computably admissible representation with domain in $\Pi^0_2(\mathbb{N}^\mathbb{N})$.

- This is equivalent to what M. Korovina & O. Kudinov (2017) called *computable quasi-Polish spaces*.
- It is also related to the *effective convergent approximation spaces* considered by V. Selivanov (2015).
- d., A. Pauly, & M. Schröder (2019) called them *precomputable quasi-Polish spaces*.

Effective version of overt quasi-Polish spaces

The following is from d., A. Pauly, & M. Schröder (2019) and M. Hoyrup, C. Rojas, V. Selivanov, & D. Stull (2019).

Theorem

The following are equivalent for an admissibly represented space $\mathbf{X}$:

- $\mathbf{X} \cong \mathbf{I}(\prec)$, where $\prec$ and $E_\prec = \{n \in \mathbb{N} \mid [n]_\prec \neq \emptyset\}$ are both c.e.
- $\mathbf{X}$ is computably isomorphic to an overt Π^0_2-subspace of $\mathbf{O}(\mathbb{N})$.
- $\mathbf{X}$ is empty or has an effectively fiber-overt computably admissible total representation.
- $\mathbf{X}$ is empty or it is the image of $\mathbb{N}^\mathbb{N}$ under a computable effectively open map.

- d., A. Pauly, & M. Schröder (2019) called them *computable quasi-Polish spaces*.
- M. Hoyrup, C. Rojas, V. Selivanov, & D. Stull (2019) called them *effective quasi-Polish spaces*.
- They correspond to *effectively enumerable computable quasi-Polish spaces* in the terminology of M. Korovina & O. Kudinov (2017).

Computable functions

Definition

Let $\prec$ and $\sqsubset$ be transitive relations on $\mathbb{N}$.

- A code for a partial function is any subset $R \subseteq \mathbb{N} \times \mathbb{N}$.
- Each code R represents the partial function $\ulcorner R \urcorner :\subseteq \mathbf{I}(\prec) \to \mathbf{I}(\sqsubset)$ defined as

$$\begin{aligned} \ulcorner R \urcorner (I) &= \{n \in \mathbb{N} \mid (\exists m \in I)\, \langle m, n \rangle \in R\}, \\ dom(\ulcorner R \urcorner) &= \{I \in \mathbf{I}(\prec) \mid \ulcorner R \urcorner (I) \in \mathbf{I}(\sqsubset)\}. \end{aligned}$$

Note that $dom(\ulcorner R \urcorner)$ is a $\mathbf{\Pi}^0_2$-subset of $\mathbf{I}(\prec)$.

Computable functions

- Computability of functions between spaces of ideals can be easily defined in a way that is compatible with TTE.
  - An admissible representation $\delta :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbf{I}(\prec)$ is given by defining $\delta(p) = I \iff range(p) = I$. In other words, a name of an ideal in $\mathbf{I}(\prec)$ is just an enumeration of its elements.

Theorem

Let $\prec$ and $\sqsubset$ be c.e. transitive relations on $\mathbb{N}$. A total function $f \colon \mathbf{I}(\prec) \to \mathbf{I}(\sqsubset)$ is computable if and only if there is a c.e. code $R \subseteq \mathbb{N} \times \mathbb{N}$ such that $f = \ulcorner R \urcorner$.

Intuitively, a function $f \colon \mathbf{I}(\prec) \to \mathbf{I}(\sqsubset)$ is computable if and only if there is an algorithm that produces an enumeration of $f(I) \in \mathbf{I}(\sqsubset)$ when given an enumeration of $I \in \mathbf{I}(\prec)$.

Compatibility with the powerspace functors

For any total $\mathbf{I}(\prec) \xrightarrow{\ulcorner R \urcorner} \mathbf{I}(\sqsubset)$ and $\mathbf{P} \in \{\mathbf{A}, \mathbf{K}, \mathbf{V}\}$, the composition

$\mathbf{P}(\mathbf{I}(\prec)) \cong \mathbf{I}(\prec_P) \xrightarrow{\ulcorner R_P \urcorner} \mathbf{I}(\sqsubset_P) \cong \mathbf{P}(\mathbf{I}(\sqsubset))$ is equal to $\mathbf{P}(\ulcorner R \urcorner)$.

- Lower powerspace:
  - $\mathbf{A}(f)(A) = Cl_Y(\{f(x) \mid x \in A\})$.
  - $A \prec_L B \iff (\forall a \in A)(\exists b \in B)\, a \prec b$ for $A, B \in \mathcal{P}_{\text{fin}}(\mathbb{N})$.
  - $R_L = \{\langle A, B \rangle \mid (\forall b \in B)(\exists a \in A)\, \langle a, b \rangle \in R\}$.
- Upper powerspace:
  - $\mathbf{K}(f)(K) = Sat_Y(\{f(x) \mid x \in K\})$.
  - $A \prec_U B \iff (\forall b \in B)(\exists a \in A)\, a \prec b$ for $A, B \in \mathcal{P}_{\text{fin}}(\mathbb{N})$.
  - $R_U = \{\langle A, B \rangle \mid (\forall a \in A)(\exists b \in B)\, \langle a, b \rangle \in R\}$.
- Valuations powerspace:
  - $\mathbf{V}(f)(\nu) = \lambda U.\nu(f^{-1}(U))$
  - $R_V = \left\{\langle r, s \rangle \,\middle|\, (\forall G \subseteq dom(s))\ \left[G \neq \emptyset \Rightarrow \sum_{a \in A_G} r(a) > \sum_{b \in G} s(b)\right]\right\}$
    where
    $A_G = \{a \in dom(r) \mid (\exists a_0 \in \mathbb{N})(\exists b \in G)\, [a_0 \prec a \,\&\, \langle a_0, b \rangle \in R]\}$.

Computable topological spaces

Definition

A (countably based) computable topological space is a tuple (X, φ, S) where:

1. X is a T_0-space,
2. $\varphi \colon \mathbb{N} \to \mathbf{O}(X)$ is an enumeration of a basis for X,
3. $S \subseteq \mathbb{N}^3$ is a c.e. set satisfying $\varphi(n) \cap \varphi(m) = \bigcup\{\varphi(k) \mid \langle n, m, k\rangle \in S\}$ for each $n, m \in \mathbb{N}$.

- The only effective aspect of this definition is the c.e. set S.
- In particular, if (X, φ, S) is a computable topological space, and $e \colon Y \to X$ is *any* embedding, then (Y, ψ, S) is also a computable topological space, where $\psi = \mathbf{O}(e) \circ \varphi$.
  - $\mathbf{O}(e) \colon \mathbf{O}(X) \to \mathbf{O}(Y)$ maps $U \mapsto e^{-1}(U)$ (ψ just restricts the basic open subsets of X enumerated by φ to the subspace Y)

Completion of computable topological spaces

Definition

Let $S \subseteq \mathbb{N}^3$ be a c.e. set. A computable topological space (X, φ, S) is complete if and only if for any computable topological space (Y, ψ, S) there is a *unique* computable embedding $e\colon Y \to X$ satisfying $\psi = \mathbf{O}(e) \circ \varphi$.

Intuitively, (X, φ, S) is a complete computable topological space if and only if all other computable topological spaces associated to S are essentially just subspace embeddings $e\colon Y \to X$ on the previous slide.

Completion of computable topological spaces

Theorem

- For every c.e. subset $S \subseteq \mathbb{N}^3$, there is a Π^0_2-subspace $X \subseteq \mathcal{P}(\mathbb{N})$ such that $(X, \lambda n.\{x \in X \mid n \in x\}, S)$ is a complete computable topological space.
  - Hence X is computably homeomorphic to a space of the form $\mathbf{I}(\prec)$ for some transitive c.e. relation $\prec$ on $\mathbb{N}$.
- Conversely, for every transitive c.e. relation $\prec$ on $\mathbb{N}$, there is a c.e. $S \subseteq \mathbb{N}^3$ such that $(\mathbf{I}(\prec), \lambda n.[n]_\prec, S)$ is a complete computable topological space.

- As a corollary, we get a (computably) equivalent notion of *computable topological space* if we defined them to be a pair $(\prec, X)$, where $\prec$ is a transitive c.e. relation and $X \subseteq \mathbf{I}(\prec)$.
- In other words, the effective aspect of the definition of a computable topological space determines a quasi-Polish space. The non-effective aspect of the definition corresponds to some subspace of the quasi-Polish space.

Table of Contents

Geometric Logic

- Quasi-Polish spaces are countably axiomatized propositional geometric theories
  - Dual perspective due to R. Heckmann.
  - Points of the space are models of the theory.
  - Topological completeness corresponds to logical completeness.

Propositional geometric logic

- Propositional geometric formulas are built from propositional variables, constants $\bot$ and $\top$, finite conjunctions, and infinitary disjunctions, but not negation nor implication.
  - S. Abramsky, S. Vickers and others have described it as the logic of "observable properties"
- A propositional geometric theory is a set of sequents of the form $\phi \vdash \psi$, where ϕ and ψ are propositional geometric formulas.
- Rules of inference (see P. Johnstone or S. Vickers):
  - $\phi \vdash \phi$ (identity)
  - $\dfrac{\phi \vdash \psi \qquad \psi \vdash \chi}{\phi \vdash \chi}$ (cut)
  - $\phi \vdash \top$, $\phi \wedge \psi \vdash \phi$, $\phi \wedge \psi \vdash \psi$, $\dfrac{\phi \vdash \psi \qquad \phi \vdash \chi}{\phi \vdash \psi \wedge \chi}$
  - $\bot \vdash \psi$, $\phi \vdash \bigvee S$ $(\phi \in S)$, $\dfrac{\phi \vdash \psi \text{ (all } \phi \in S)}{\bigvee S \vdash \psi}$
  - $\phi \wedge \bigvee S \vdash \bigvee \{\phi \wedge \psi \mid \psi \in S\}$

Frames

The Lindenbaum algebra of a propositional geometric theory is a frame.

Definition

A frame is a complete lattice satisfying $a \wedge \bigvee_{i \in I} b_i = \bigvee_{i \in I}(a \wedge b_i)$ for arbitrary index sets I.

- (Equivalently, a frame is a complete Heyting algebra)

A frame homomorphism is a function between frames that preserves finite meets and arbitrary joins.

- (But it need not preserve implication)

Example

- If X is a topological space then $\mathbf{O}(X)$ is a frame.
- If $f\colon X \to Y$ is a continuous function between topological spaces then $f^{-1}\colon \mathbf{O}(Y) \to \mathbf{O}(X)$ is a frame homomorphism.

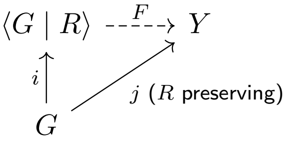
Frame presentations

A frame presentation consists of

- a set G of generators (= propositional variables), and
- a set R of relations (= axioms) of the form "$\phi \vdash \psi$", where ϕ and ψ are frame expressions involving the constants $\top, \bot$, elements of G, and the operators $\wedge, \bigvee$.

A presentation uniquely determines a frame $\langle G \mid R \rangle$ satisfying:

- There is $i\colon G \to \langle G \mid R \rangle$ that preserves the relations in R
  - If $p_0 \wedge \ldots \wedge p_m \vdash \bigvee_{i \in I} q_0^i \wedge \ldots \wedge q_{n_i}^i$ is in R, then $i(p_0) \wedge \ldots \wedge i(p_m) \leq \bigvee_{i \in I} i(q_0^i) \wedge \ldots \wedge i(q_{n_i}^i)$ holds in $\langle G \mid R \rangle$
- If Y is any frame and $j\colon G \to Y$ preserves R, then there is a unique frame homomorphism $F\colon \langle G \mid R \rangle \to Y$ such that $F \circ i = j$



(The frame $\langle G \mid R \rangle$ is the Lindenbaum algebra of the theory with variables G and axioms R)

Theorem (R. Heckmann)

A frame has a countable presentation (i.e., G and R are both countable) if and only if it is isomorphic to the open set lattice of a quasi-Polish space.

Quasi-Polish space = countable propositional geometric theory

- Open set lattice $\approx$ Lindenbaum algebra
- Open sets $\approx$ Propositions
- Points $\approx$ Models of the theory

Note: Recent work by R. Chen extends R. Heckmann's results to predicate logic and further develops connections between descriptive set theory and locale theory.

Table of Contents

Topological Duality

- Quasi-Polish spaces are coPolish-presented locales
  - Putting the topology back into locale theory
  - Dual relationship with the coPolish spaces that were proposed by M. Schröder.

$\mathbb{S}^X$ as a topological frame

Proposition (d. & T. Kawai, 2019)

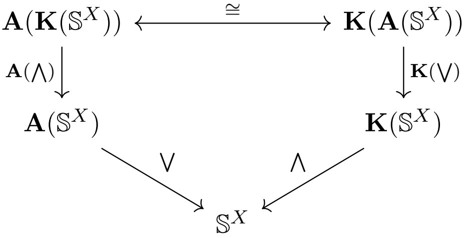
If X is quasi-Polish then the following are continuous:

- $\bigvee \colon \mathbf{A}(\mathbb{S}^X) \to \mathbb{S}^X, \quad \mathcal{A} \mapsto \bigcup_{U \in \mathcal{A}} U.$
- $\bigwedge \colon \mathbf{K}(\mathbb{S}^X) \to \mathbb{S}^X, \quad \mathcal{K} \mapsto \bigcap_{U \in \mathcal{K}} U.$

- $\mathbb{S}^X$ is an Eilenberg-Moore algebra of the monads $\mathbf{K}$ and $\mathbf{A}$ with the above structure maps.
  - Algebras of these monads were studied in depth by A. Schalk.
  - $\mathbb{S}^X$ is also an algebra of the double powerspace monad with structure map $\mathbb{S}^{\eta_X} : \mathbb{S}^{\mathbb{S}^{\mathbb{S}^X}} \to \mathbb{S}^X$ (see P. Taylor's work on ASD).
- The observation that a compact intersection of opens is open is due to M. Escardó (2004). The above proposition holds for all represented spaces (A. Pauly, 2015) if the QCB_0-version of the powerspaces are used (M. Schröder, 2002).

$\mathbb{S}^X$ as a topological frame

- By R. Heckmann's result, the spaces $\mathbb{S}^X$ for quasi-Polish X are precisely the countably presented frames with the Scott-topology.
- Since $\mathbb{S}^X$ is an algebra of both $\mathbf{A}$ and $\mathbf{K}$, we can view it as a topological frame with overt joins and compact meets.
    - This allows a precise connection between open sets and c.e.-sets, since it avoids the problematic "arbitrary" unions.
- Beck distributivity holds for $\mathbf{A}$ and $\mathbf{K}$ in this case, which implies overt joins and compact meets distribute over each other.
    - The maps below are equivalent to $\mathbb{S}^{\eta_X}$.



Countably presented frames as spaces

- Let G and R be countable sets (discrete spaces).
- Each pair of functions $\varphi, \psi \colon R \to \mathbb{S}^{\mathbb{S}^G} \cong \mathbf{A}(\mathbf{K}(G))$ determines the frame presentation with generators G and relations

$$\bigvee_{F \in \varphi(r)} \bigwedge F = \bigvee_{F' \in \psi(r)} \bigwedge F'$$

  for each $r \in R$ ($F, F' \in \mathbf{K}(G)$ are finite because G is discrete).
- Conversely, every countable frame presentation can be viewed as a pair of functions $\varphi, \psi \colon R \to \mathbb{S}^{\mathbb{S}^G} \cong \mathbf{A}(\mathbf{K}(G))$ for suitable G and R.

Countably presented frames as spaces

- We obtain the frame presented by $\varphi, \psi \colon R \to \mathbb{S}^{\mathbb{S}^G}$ as follows:
  - Let $\hat{\varphi}, \hat{\psi} \colon \mathbb{S}^G \to \mathbb{S}^R$ be the double transpose of φ, ψ.
    (i.e., $\hat{\varphi}(U)(r) = \varphi(r)(U)$ and $\hat{\psi}(U)(r) = \psi(r)(U)$)
  - The equalizer $e \colon X \to \mathbb{S}^G$ of $\hat{\varphi}$ and $\hat{\psi}$ is quasi-Polish.
  - Then $\mathbb{S}^X$ is the frame presented by φ, ψ.
- If we interpret "frame" to mean "double powerspace algebra", then we get the same result if G (or R) is any *space* such that $\mathbb{S}^G$ is a quasi-Polish double powerspace algebra.
  - It follows that quasi-Polish spaces are exactly the coPolish-presented locales.

Example: If X is quasi-Polish then

$$\mathbf{O}(\mathbf{A}(X)) \cong \langle \Diamond U (U \in \mathbf{O}(X)) \mid \Diamond\bot = \bot, \Diamond(U \vee V) = \Diamond U \vee \Diamond V \rangle$$
$$\mathbf{O}(\mathbf{K}(X)) \cong \langle \Box U (U \in \mathbf{O}(X)) \mid \Box\top = \top, \Box(U \wedge V) = \Box U \wedge \Box V \rangle$$

(Relations for preserving directed joins are unnecessary if all maps in the ambient category are continuous)

Regular frames

Since countably presented frames are topological frames, it makes sense to look at quasi-Polish frames.

Definition

- A frame X is regular iff each $x \in X$ satisfies
$$x = \bigvee \{y \in X \mid (\exists z \in X)[\, y \wedge z = \bot \text{ and } x \vee z = \top \,]\}.$$
- A quasi-Polish (regular) frame is a quasi-Polish space whose specialization order is a (regular) frame and such that binary joins and binary meets are continuous functions.

Note: If X is a sober space, then $\mathbf{O}(X)$ is a regular frame iff X is a regular Hausdorff space.

CoPolish spaces

Definition

A space is coPolish iff it is the direct limit of an increasing sequence of its compact metrizable subspaces.

M. Schröder introduced coPolish spaces when investigating Type-2 complexity theory, and has found many characterizations and natural examples of them.

Proposition (M. Schröder)

Every coPolish space is a regular Hausdorff QCB_0-space.

Example: The polynomial ring $\mathbb{R}[X]$ is coPolish but not countably based.

- $(p_i)_{i \in \mathbb{N}} \to p$ in $\mathbb{R}[X]$ iff $\{deg(p_i) \mid i \in \mathbb{N}\}$ is bounded and the coefficients of the p_i converge to the coefficients of p.

Quasi-Polish regular frames and coPolish spaces

It was known early on from M. Schröder's original work on coPolish spaces that the frame of a coPolish space (with the Scott-topology) is qusai-Polish. The following spatiality result is a converse to this observation.

Theorem

The following are equivalent for a quasi-Polish regular frame Y:

1. Y is an algebra of the double powerspace monad,
2. Y is homeomorphic to $\mathbf{O}(X)$ (with the Scott-topology) for some coPolish space X.

CoPolish spaces

Just as Polish spaces have many applications in analysis, coPolish spaces have many applications in topological algebra.

Proposition

If X is coPolish, then $\mathbf{G}(X)$, the free topological group generated by X, is coPolish.

(Other free algebra constructions are possible, such as $\mathbb{R}[X]$, which is the free commutative $\mathbb{R}$-algebra generated by a single indeterminate X.)

Proof: Every coPolish space is a k_ω-space, and it is known that $\mathbf{G}(-)$ preserves k_ω-spaces. Most constructions of $\mathbf{G}(X)$ in the literature are highly non-constructive, so it can be difficult to see that the size constraint is satisfied. A more constructive construction can be found in the J. Isbell et al. paper "Remarks on localic groups", which only requires frame coproducts and countable frame limits, so it can be carried out with quasi-Polish frames. The same J. Isbell et al. paper shows that localic products of k_ω-spaces are spatial, and that if the localic free group is spatial then it corresponds with the topological free group, so the localic and topological constructions agree.

Table of Contents

Quasi-Polish = Enough compact sets?

- It is well-known that the completely Baire property characterizes Polish spaces among "definable" countably based metrizable spaces.
- This does not extend to quasi-Polish spaces, because S_0 is a countably based completely Baire non-quasi-Polish space.
- However, $\mathbf{K}(X)$ is completely Baire iff X is quasi-Polish, whenever X is a "definable" countably based sober space.
- Consonance also says a space has "enough" compact sets, so does this also characterize quasi-Polish spaces?
  - A space X is consonant iff the Scott-topology and compact-open topology agree on $\mathbb{S}^X$.

Open Problem

Is it consistent with $\mathbf{ZF} + \mathbf{DC}$ that every countably based sober consonant space is quasi-Polish?

(It is known that a co-analytic countably based sober space is quasi-Polish iff every $\mathbf{\Pi}^0_2$-subspace is consonant).

Cartesian closed subcategories?

- For more comparisons with domains and other applications to computation, it is important to know more about the cartesian closed subcategories of QPol.
  - I am mainly interested in full sub-CCCs of QCB_0 that are contained in QPol, so exponentials will have the compact-open topology.
- $X \in$ QPol is exponentiable (i.e., $Y^X \in$ QPol for all $Y \in$ QPol) if and only if X is locally compact. However, the locally compact spaces do not form a cartesian closed subcategory.
- ωFS-domains (the largest cartesian closed full subcategory of ω-continuous domains) is a full sub-CCC of QCB_0 contained in QPol, but it is unknown if it is maximal in QPol.

Open Problem

Are ωFS-domains a maximal cartesian closed subcategory of QPol?

Spaces vs. Locales

- Locales are a generalization of quasi-Polish spaces, where uncountable sets of generators and relations are allowed.
    - R. Chen's work shows how locale theory is in many ways a generalization of descriptive set theory, but without the countability restrictions.
- Spatial locales and sober spaces have some similarities, until you start doing mathematics with them.
    - The theories of countably presented locales and quasi-Polish spaces are equivalent if you assume classical logic, but:
    - Quiz: Is addition on the subobject $\mathbb{Q}$ of $\mathbb{R}$ computable?

Open Problem

Is it consistent with $\mathbf{ZF} + \mathbf{DC}$ that $\mathbf{QPol}$ is the largest full subcategory of countably based sober spaces where localic products are spatial?

Some initial progress on this problem was presented at "Computability, Continuity, Constructivity (CCC 2019)".