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Base-Complexity Classifications of Qcb$_0$-Spaces*

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Abstract. We define and study new classifications of qcb$_0$-spaces based on the idea to measure the complexity of their bases. The new classifications complement those given by recently introduced hierarchies of qcb$_0$-spaces and provide new tools to investigate non-countably based qcb$_0$-spaces. As a by-product, we show that there is no universal qcb$_0$-space and establish several new properties of the Kleene-Kreisel continuous functionals of countable types.

Key words. Qcb$_0$-spaces, Y-based spaces, hyperspaces, Scott topology, hyperprojective hierarchy, Kleene-Kreisel continuous functionals.

1. Introduction

A basic notion of Computable Analysis [3, 32] is the notion of an *admissible representation* of a topological space X. This is a partial continuous surjection δ from the Baire space $\mathcal{N}$ onto X satisfying a certain universality property (see Subsection 2.3 for some more details). Such a representation of X often induces a reasonable computability theory on X, and the class of admissibly represented spaces is wide enough to include most spaces of interest for Analysis or Numerical Mathematics. As shown by the second author [20], this class coincides with the class of the so-called qcb$_0$-spaces, i.e. T_0-spaces which are quotients of countably based spaces, and it forms a cartesian closed category with the continuous functions as morphisms. Thus, among qcb$_0$-spaces one meets many important function spaces including the continuous functionals of finite types [13, 14] interesting for several branches of logic and computability theory. In addition to being cartesian closed, the category QCB_0 of qcb$_0$-spaces is also closed under countable limits, countable colimits, and many other important constructions, making it a very convenient category of topological spaces. However, along with the benefits of this generality comes the challenge of developing comprehensive theories that provide a deeper understanding of arbitrary qcb$_0$-spaces.

Classical descriptive set theory [12] has proven to be extremely useful for classifying and studying separable metrizable spaces. Every separable metrizable space can be topologically embedded into a Polish space (a separable completely metrizable space), for example by taking the completion of a compatible metric. We can therefore classify a separable metrizable space according to the complexity of defining it as a subspace of some Polish space, where topological complexity can be quantified using natural hierarchies such as the Borel or Luzin (projective) hierarchies. This method of classification is topologically invariant for complexity levels of at least $\mathbf{\Pi}^0_2$ in the Borel hierarchy (it does not depend on which Polish space we embed into) because of the remarkable fact that a subspace of a Polish space is Polish if and only if it is a $\mathbf{\Pi}^0_2$ subspace. We can even generalize this approach to the entire class of countably based T_0-spaces (abbreviated cb$_0$-spaces) by using quasi-Polish spaces [5], which have the same $\mathbf{\Pi}^0_2$ absoluteness property as Polish spaces. In fact, for classifying cb$_0$-spaces we can restrict ourselves to the algebraic domain $P\omega$ of all subsets of natural numbers (denoted ω), which is quasi-Polish and universal for cb$_0$-spaces.

Unfortunately, this approach to classifying topological spaces does not immediately generalize to the entire category of qcb$_0$-spaces. First of all, as we will see in this paper, there is no universal qcb$_0$-space to serve as a basis

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for comparing topological complexity. A second critical problem is that the $\mathbf{\Pi}^0_2$ absoluteness property of Polish and quasi-Polish spaces does not apply to subspaces of non-countably based spaces. For example, in [30] it is shown that the space $\mathcal{O}(\mathcal{N})$, the lattice of open subsets of $\mathcal{N}$ with the Scott-topology, contains *singleton* subsets which are $\mathbf{\Pi}^1_1$-complete even though they are trivially Polish with respect to the subspace topology. It is possible to use similar methods to construct qcb$_0$-spaces that have singleton subsets of arbitrarily high complexity in the hyperprojective hierarchy.

Important progress towards classifying qcb$_0$-spaces was made in [25] and [26], where the Borel, projective, and hyperprojective hierarchies of qcb$_0$-spaces were introduced. The major insight was to classify qcb$_0$-spaces according to the complexity of the equivalence relation on the elements of $\mathcal{N}$ induced by an admissible representation of the space, which elegantly sidesteps the problem of finding a universal space. This approach works well because the universal property of admissible representations causes them to reflect many important topological properties of the underlying space. In fact, it was shown in [25, 26] that for cb$_0$-spaces, the newly introduced classification approach using admissible representations is equivalent to the approach described above that uses topological embeddings into $P\omega$.

However, the hierarchies defined in [25, 26] do not differentiate between countably based qcb$_0$-spaces and non-countably based spaces. In particular, the problem of placing an upper bound on the relative complexity of even very simple subsets (such as singletons) of non-countably based spaces can not be settled using this approach. Thus, although the Borel, projective, and hyperprojective hierarchies quantify one important aspect of the complexity of qcb$_0$-spaces, there appears to be an additional dimension of complexity that is mostly apparent in the large difference between countably based and non-countably based spaces.

In this paper we attempt to capture this additional dimension of complexity by introducing methods to classify a topological space according to the complexity of defining a basis for its topology. Our hope is that by combining the basis-complexity measures introduced in this paper with the hierarchies defined in [25, 26], we can obtain a more complete measure of the topological complexity of qcb$_0$-spaces.

The basic idea of our approach is a natural generalization of the definition of a countable basis. Given a topological space X, a countable basis for X can be viewed as a mapping ϕ from ω to the set $\mathcal{O}(X)$ of open subsets of X such that the range of ϕ is a basis for the topology of X. As a first approach to generalizing this definition to non-countably based spaces, we can replace the index set ω with an arbitrary topological space Y and consider whether or not a basis for X can be indexed by some mapping $\phi\colon Y \to \mathcal{O}(X)$ which is continuous with respect to the Scott-topology on $\mathcal{O}(X)$. The class of spaces that have such an indexing for a basis will be called *Y-based spaces*, and the complexity of Y according to the hierarchies in [25, 26] provides an indication of the complexity of the spaces in this class. This definition is very natural and we will show that it has several useful properties, but unfortunately it can be difficult to use in practice. We therefore also introduce a second related concept that we call *sequentially Y-based* spaces, which requires a more complicated definition but behaves much better when working with sequential spaces. In particular, we will show that universal spaces exist for the class of sequentially Y-based spaces for each qcb$_0$-space Y. We expect this observation will be useful for future development of a descriptive theory of qcb$_0$-spaces that avoids the problems mentioned earlier in this introduction.

We will provide a detailed analysis of the relationship between the proposed hierarchies and the previous ones, and provide some applications. The newly introduced basis-complexity classifications can be particularly useful when determining whether one space can be embedded into another space. We will demonstrate this claim by investigating the existence of certain classes of universal qcb$_0$-spaces, by showing that every qcb$_0$-space can be embedded into a space with a total admissible representation, and by establishing several apparently new properties of the Kleene-Kreisel continuous functionals of countable types.

After recalling some definitions and known facts in the next section, we discuss the notions of topological and sequential embeddings in Section 3. In Section 4 we establish some basic facts on the hyperspace of open subsets of a qcb$_0$-space, and in Section 5 we investigate the hyperspace of compact subsets. In Sections 6 and 7 we first introduce and study some versions of the notion of a Y-based space, in particular we characterize the qcb$_0$-spaces in these terms, and then define and investigate the two relevant classifications of qcb$_0$-spaces. In Section 8 we study which levels of the the new and old hierarchies have a universal (or sequentially universal) space, and we conclude in Section 9.

2. Notation and preliminaries

2.1. Notation

We freely use standard set-theoretic notations like $dom(f)$, $rng(f)$ and $graph(f)$ for the domain, range and graph of a function f, respectively, $X \times Y$ for the Cartesian product, $X \oplus Y$ for the disjoint union of sets X and Y, Y^X for the set of functions $f\colon X \to Y$ (but in the case when X, Y are qcb$_0$-spaces we use the same notation to denote the set of continuous functions from X to Y), and $P(X)$ for the set of all subsets of X. For $A \subseteq X$, $\overline{A}$ denotes the complement $X \setminus A$ of A in X. We identify the set of natural numbers with the first infinite ordinal ω. The first uncountable ordinal is denoted by ω_1. The notation $f : X \to Y$ means that f is a total function from a set X to a set Y.

2.2. Topological spaces

We assume the reader to be familiar with the basic notions of topology. The collection of all open subsets of a topological space X (i.e. the topology of X) is denoted by $\mathcal{O}(X)$; for the underlying set of X we will write X in abuse of notation. We will usually abbreviate "topological space" to "space". A space is *zero-dimensional* if it has a basis of clopen sets. Recall that a *basis* for the topology on X is a set $\mathcal{B}$ of open subsets of X such that for every $x \in X$ and open U containing x there is $B \in \mathcal{B}$ satisfying $x \in B \subseteq U$.

Let ω be the space of non-negative integers with the discrete topology. Of course, the spaces $\omega \times \omega = \omega^2$, and $\omega \oplus \omega$ are homeomorphic to ω, the first homeomorphism is realized by the Cantor pairing function $\langle \cdot, \cdot \rangle$. We denote the one-point compactification of ω by $\mathbb{N}_\infty$; ∞ stands for its point at infinity.

Let $\mathcal{N} = \omega^\omega$ be the set of all infinite sequences of natural numbers (i.e., of all functions $\xi\colon \omega \to \omega$). Let ω^* be the set of finite sequences of elements of ω, including the empty sequence. For $\sigma \in \omega^*$ and $\xi \in \mathcal{N}$, we write $\sigma \sqsubseteq \xi$ to denote that σ is an initial segment of the sequence ξ. By $\sigma\xi = \sigma \cdot \xi$ we denote the concatenation of σ and ξ, and by $\sigma \cdot \mathcal{N}$ the set of all extensions of σ in $\mathcal{N}$. For $x \in \mathcal{N}$, we can write $x = x(0)x(1)\ldots$ where $x(i) \in \omega$ for each $i < \omega$. For $x \in \mathcal{N}$ and $n < \omega$, let $x^{<n} = x(0)\ldots x(n-1)$ denote the initial segment of x of length n. Notations in the style of regular expressions like 0^ω, 0^*1 or 0^m1^n have the obvious standard meaning.

By endowing $\mathcal{N}$ with the product of the discrete topologies on ω, we obtain the so-called *Baire space*. The product topology coincides with the topology generated by the collection of sets of the form $\sigma \cdot \mathcal{N}$ for $\sigma \in \omega^*$. The Baire space is of primary importance for Descriptive Set Theory and Computable Analysis. The importance stems from the fact that many countable objects are coded straightforwardly by elements of $\mathcal{N}$, and it has very specific topological properties. In particular, it is a perfect zero-dimensional space and the spaces $\mathcal{N}^2$, $\mathcal{N}^\omega$, $\omega \times \mathcal{N} = \mathcal{N} \oplus \mathcal{N} \oplus \cdots$ (endowed with the product topology) are all homeomorphic to $\mathcal{N}$. Let $(x, y) \mapsto \langle x, y \rangle$ be a homeomorphism between $\mathcal{N}^2$ and $\mathcal{N}$. The Baire space $\mathcal{N}$ has the following universality property for zero-dimensional cb$_0$-spaces:

Proposition 2.1. *[12, Theorems 1.1 and 7.8] A topological space X embeds into $\mathcal{N}$ iff X is a zero-dimensional cb$_0$-space.*

The subspace $\mathcal{C} := 2^\omega$ of $\mathcal{N}$ formed by the infinite binary strings (endowed with the relative topology inherited from $\mathcal{N}$) is known as the *Cantor space*.

An important role in this paper is played by the *Sierpinski space* $\mathbb{S} = \{\bot, \top\}$, where the set $\{\top\}$ is open but not closed.

The space $P\omega$ is formed by the set of subsets of ω equipped with the Scott topology. A countable base of the Scott topology is formed by the sets $\{A \subseteq \omega \mid F \subseteq A\}$, where F ranges over the finite subsets of ω. Note that $P\omega = \mathcal{O}(\omega)$.

The importance of $P\omega$ is explained by its following well-known properties. First, $P\omega$ is universal for *cb*$_0$-spaces.

Proposition 2.2. *A topological space X embeds into $P\omega$ iff X is a cb$_0$-space.*

The second property shows that $P\omega$ is an injective object in the category of all topological spaces.

Proposition 2.3. *[9, Proposition 3.5] Let Y be a topological space and X be a topological subspace of Y. Then any continuous function $f : X \to P\omega$ can be extended to a continuous function $g : Y \to P\omega$.*

Remember that a space X is *Polish* if it is countably based and metrizable with a metric d such that (X, d) is a complete metric space. Important examples of Polish spaces are ω, $\mathcal{N}$, $\mathcal{C}$, the space of reals $\mathbb{R}$ and its Cartesian powers $\mathbb{R}^n$ ($n < \omega$), the closed unit interval $[0,1]$, the Hilbert cube $[0,1]^\omega$ and the Hilbert space $\mathbb{R}^\omega$. Simple examples of non-Polish spaces are $\mathbb{S}$, $P\omega$ and the space $\mathbb{Q}$ of rationals.

Sometimes we also mention quasi-Polish spaces, which were introduced and studied in [5]. Quasi-Polish spaces are defined as the countably based spaces which have a topology induced by a (Smyth-) complete quasi-metric. The descriptive set theory of quasi-Polish spaces is very similar to the classical theory for Polish spaces, but the class of quasi-Polish spaces contains many useful spaces in addition to Polish spaces, such as all ω-continuous domains and some non-Hausdorff spaces that are important to algebraic geometry. The spaces $\mathbb{S}$, $\mathcal{N}$, $P\omega$, and $Spec(\mathbb{Z})$, the spectrum of the integers with the Zariski topology, are all quasi-Polish while the space $\mathbb{Q}$ is not.

2.3. Admissible representations and qcb$_0$-spaces

A *representation* of a space X is a surjection of a subspace of the Baire space $\mathcal{N}$ onto X. A basic notion of Computable Analysis is the notion of admissible representation. A representation δ of X is *admissible*, if it is continuous and any continuous function $\nu : Z \to X$ from a subset $Z \subseteq \mathcal{N}$ to X is continuously reducible to δ, i.e. $\nu = \delta g$ for some continuous function $g : Z \to \mathcal{N}$. A topological space is *admissibly representable*, if it has an admissible representation.

The notion of admissibility was introduced in [15] for representations of cb$_0$-spaces (in a different but equivalent formulation) and was extensively studied by many authors. In [2] a close relation of admissible representations of countably based spaces to open continuous representations was established. In [19, 20] the notion was extended to non-countably based spaces and a nice characterization of the admissibly represented spaces was achieved. Namely, the admissibly represented sequential topological spaces coincide with the qcb$_0$-spaces, i.e., T_0-spaces which are topological quotients of countably based spaces.

The category QCB_0 of qcb$_0$-spaces as objects and continuous functions as morphisms is known to be cartesian closed (cf. [8, 20]). Products and function spaces are formed as in the supercategory Seq, which is the category of sequential topological spaces and of continuous functions. The topology of the sequential product to sequential spaces X and Y, which we denote by $X \times Y$, is the sequentialisation of the classical Tychonoff topology on the cartesian product of the respective underlying sets. By the *sequentialisation* of a topology τ we mean the collection of all sequentially open sets pertaining to this topology. This collection forms a topology which is finer than (or equal to) the original topology. Remember that *sequentially open* sets are defined to be the complements of the sets that are closed under forming limits of converging sequences.

The exponential to X, Y in Seq, denoted by Y^X, has the set of all continuous functions from X to Y as the underlying set; its topology is equal to the sequentialisation of the compact-open topology on Y^X. The convergence relation of Y^X is *continuous convergence*: a sequence $(f_n)_n$ converges continuously to f_∞, if, whenever $(x_n)_n$ converges in X to x_∞, the sequence $(f_n(x_n))_n$ converges to $f_\infty(x_\infty)$ in Y. This is equivalent to the (sequential) continuity of the universal function $\tilde{f}\colon \mathbb{N}_\infty \times X \to Y$ defined by $\tilde{f}(n,x) = f_n(x)$. Further information can be found in e.g. [8, 20]. Note that a function between sequential spaces is topologically continuous if, and only if, it is sequentially continuous.

We will also use the following well-known facts (see e.g. [20, 32]).

Proposition 2.4. *Let X, Y be qcb$_0$-spaces, let δ be a continuous representation of X and let γ be an admissible representation of Y. Then any continuous function $f\colon X \to Y$ has a continuous* realiser *g, i.e., g is a partial continuous function on $\mathcal{N}$ satisfying $f\delta(p) = \gamma g(p)$ for all $p \in dom(\delta)$. If δ is additionally an admissible representation of X, then any function $f\colon X \to Y$ is continuous if, and only if, it has a continuous realiser.*

Apart from being cartesian-closed, the category QCB_0 is closed under countable products and countable coproducts. The product of a sequence $X_0, X_1, \ldots$ of qcb$_0$-spaces is denoted by $\prod_{n\in\omega} X_n$, the coproduct by $\bigoplus_{n\in\omega} X_n$. The category QCB_0 is also closed under countable limits and countable colimits.

As is well known, every Polish (and even every quasi-Polish) space X has a total admissible representation $\xi : \mathcal{N} \to X$ (cf. [5]).

2.4. Hierarchies of sets

A *pointclass* on a countably based space X is simply a collection $\mathbf{\Gamma}(X)$ of subsets of X. A *family of pointclasses* [30] is a family $\mathbf{\Gamma} = \{\mathbf{\Gamma}(X)\}$ indexed by countably based topological spaces X such that each $\mathbf{\Gamma}(X)$ is a pointclass on X and $\mathbf{\Gamma}$ is closed under continuous preimages, i.e. $f^{-1}(A) \in \mathbf{\Gamma}(X)$ for every $A \in \mathbf{\Gamma}(Y)$ and every continuous function $f\colon X \to Y$. A basic example of a family of pointclasses is given by the family $\mathcal{O} = \{\tau_X\}$ of the topologies of all the spaces X.

We will use some operations on families of pointclasses. First, the usual set-theoretic operations will be applied to the families of pointclasses pointwise: for example, the union $\bigcup_i \mathbf{\Gamma}_i$ of the families of pointclasses $\mathbf{\Gamma}_0, \mathbf{\Gamma}_1, \ldots$ is defined by $(\bigcup_i \mathbf{\Gamma}_i)(X) = \bigcup_i \mathbf{\Gamma}_i(X)$.

Second, a large class of such operations is induced by the set-theoretic operations of L.V. Kantorovich and E.M. Livenson (see e.g. [30] for the general definition). Among them are the operation $\mathbf{\Gamma} \mapsto \mathbf{\Gamma}_\sigma$, where $\mathbf{\Gamma}(X)_\sigma$ is the set of all countable unions of sets in $\mathbf{\Gamma}(X)$, the operation $\mathbf{\Gamma} \mapsto \mathbf{\Gamma}_\delta$, where $\mathbf{\Gamma}(X)_\delta$ is the set of all countable intersections of sets in $\mathbf{\Gamma}(X)$, the operation $\mathbf{\Gamma} \mapsto \mathbf{\Gamma}_c$, where $\mathbf{\Gamma}(X)_c$ is the set of all complements of sets in $\mathbf{\Gamma}(X)$, the operation $\mathbf{\Gamma} \mapsto \mathbf{\Gamma}_d$, where $\mathbf{\Gamma}(X)_d$ is the set of all differences of sets in $\mathbf{\Gamma}(X)$, the operation $\mathbf{\Gamma} \mapsto \mathbf{\Gamma}_\exists$ defined by $\mathbf{\Gamma}_\exists(X) := \{\exists^{\mathcal{N}}(A) \mid A \in \mathbf{\Gamma}(\mathcal{N} \times X)\}$, where $\exists^{\mathcal{N}}(A) := \{x \in X \mid \exists p \in \mathcal{N}.(p,x) \in A\}$ is the projection of $A \subseteq \mathcal{N} \times X$ along the axis $\mathcal{N}$, and finally the operation $\mathbf{\Gamma} \mapsto \mathbf{\Gamma}_\forall$ defined by $\mathbf{\Gamma}_\forall(X) := \{\forall^{\mathcal{N}}(A) \mid A \in \mathbf{\Gamma}(\mathcal{N} \times X)\}$, where $\forall^{\mathcal{N}}(A) := \{x \in X \mid \forall p \in \mathcal{N}.(p,x) \in A\}$.

The operations on families of pointclasses enable to provide short uniform descriptions of the classical hierarchies in arbitrary spaces. E.g., the Borel hierarchy is the family of pointclasses $\{\mathbf{\Sigma}^0_\alpha\}_{\alpha<\omega_1}$ defined by induction on α as follows [5, 29]: $\mathbf{\Sigma}^0_0(X) := \{\emptyset\}$, $\mathbf{\Sigma}^0_1 := \mathcal{O}$, $\mathbf{\Sigma}^0_2 := (\mathbf{\Sigma}^0_1)_{d\sigma}$, and $\mathbf{\Sigma}^0_\alpha(X) := (\bigcup_{\beta<\alpha} \mathbf{\Sigma}^0_\beta(X))_{c\sigma}$ for $\alpha > 2$. The sequence $\{\mathbf{\Sigma}^0_\alpha(X)\}_{\alpha<\omega_1}$ is called *the Borel hierarchy* in X. We also let $\mathbf{\Pi}^0_\beta(X) := (\mathbf{\Sigma}^0_\beta(X))_c$ and $\mathbf{\Delta}^0_\alpha(X) := \mathbf{\Sigma}^0_\alpha(X) \cap \mathbf{\Pi}^0_\alpha(X)$. The classes $\mathbf{\Sigma}^0_\alpha(X), \mathbf{\Pi}^0_\alpha(X), \mathbf{\Delta}^0_\alpha(X)$ are called the *levels* of the Borel hierarchy in X.

For this paper, the hyperprojective hierarchy is of main interest. The *hyperprojective hierarchy* is the family of pointclasses $\{\mathbf{\Sigma}^1_\alpha\}_{\alpha<\omega_1}$ defined by induction on α as follows: $\mathbf{\Sigma}^1_0 = \mathbf{\Sigma}^0_2$, $\mathbf{\Sigma}^1_{\alpha+1} = (\mathbf{\Sigma}^1_\alpha)_{c\exists}$, $\mathbf{\Sigma}^1_\lambda = (\mathbf{\Sigma}^1_{<\lambda})_{\delta\exists}$, where $\alpha, \lambda < \omega_1$, λ is a limit ordinal, and $\mathbf{\Sigma}^1_{<\lambda}(X) := \bigcup_{\alpha<\lambda} \mathbf{\Sigma}^1_\alpha(X)$.

In this way, we obtain for any topological space X the sequence $\{\mathbf{\Sigma}^1_\alpha(X)\}_{\alpha<\omega_1}$, which we call here *the hyperprojective hierarchy in X*. The pointclasses $\mathbf{\Sigma}^1_\alpha(X)$, $\mathbf{\Pi}^1_\alpha(X) := (\mathbf{\Sigma}^1_\alpha(X))_c$ and $\mathbf{\Delta}^1_\alpha(X) := \mathbf{\Sigma}^1_\alpha(X) \cap \mathbf{\Pi}^1_\alpha(X)$ are called *levels of the hyperprojective hierarchy in X*. The finite non-zero levels of the hyperprojective hierarchy coincide with the corresponding levels of Luzin's projective hierarchy [5, 25]. The class of *hyperprojective sets* in X is defined as the union of all levels of the hyperprojective hierarchy in X. For more information on the hyperprojective hierarchy see [11, 12, 26].

Remark 2.1. (1) If X is Polish then one can equivalently take $\mathbf{\Sigma}^1_0 = \mathbf{\Sigma}^0_1$ in the definition of the hyperprojective hierarchy and obtain the same non-zero levels as above. For non-Polish spaces our definition guarantees the "right" inclusions of the levels, as the first item of the next proposition states.

(2) In the case of Polish spaces our "hyperprojective hierarchy" is in fact an initial segment of the hyperprojective hierarchy from [11], so "ω_1-hyperprojective" would be a more precise name for our hierarchy; nevertheless, we prefer to use the easier term "hyperprojective".

(3) In the literature one can find two slightly different definitions of hyperprojective hierarchy. Our definition corresponds to that in [11]. The other one (see e.g. Exercise 39.18 in [12]) differs from ours only for limit levels, namely it takes $(\mathbf{\Sigma}^1_{<\lambda})_\sigma$ instead of our $\mathbf{\Sigma}^1_\lambda$.

The next assertion collects some properties of the hyperprojective hierarchy. They are proved just in the same way as for the classical projective hierarchy in Polish spaces [12].

Proposition 2.5. (1) *For any $\alpha < \beta < \omega_1$, $\mathbf{\Sigma}^1_\alpha \cup \mathbf{\Pi}^1_\alpha \subseteq \mathbf{\Delta}^1_\beta$.*

(2) *For any limit countable ordinal λ, $\mathbf{\Sigma}^1_{<\lambda} = \mathbf{\Pi}^1_{<\lambda}$ and $(\mathbf{\Sigma}^1_{<\lambda})_\delta = (\mathbf{\Pi}^1_{<\lambda})_{\sigma c}$.*

(3) *For any non-zero $\alpha < \omega_1$, $\mathbf{\Sigma}^1_\alpha = (\mathbf{\Sigma}^1_\alpha)_\sigma = (\mathbf{\Sigma}^1_\alpha)_\delta = (\mathbf{\Sigma}^1_\alpha)_\exists$. In particular, the class $\mathbf{\Sigma}^1_\alpha(\mathcal{N})$ is closed under countable unions, countable intersections, continuous images, and continuous preimages of functions with a $\mathbf{\Pi}^0_2$-domain.*

(4) *For any non-zero $\alpha < \omega_1$, $\mathbf{\Pi}^1_\alpha = (\mathbf{\Pi}^1_\alpha)_\sigma = (\mathbf{\Pi}^1_\alpha)_\delta = (\mathbf{\Pi}^1_\alpha)_\forall$. In particular, the class $\mathbf{\Pi}^1_\alpha(\mathcal{N})$ is closed under countable unions and countable intersections, and continuous preimages of functions with a $\mathbf{\Pi}^0_2$-domain.*

(5) *For any uncountable Polish space (and also for any uncountable quasi-Polish space [5]) X, the hyperprojective hierarchy in X does not collapse, i.e. $\mathbf{\Sigma}^1_\alpha(X) \not\subseteq \mathbf{\Pi}^1_\alpha(X)$ for each $\alpha < \omega_1$.*

Remark 2.2. It is known that the Borel and hyperprojective hierarchies behave quite well for some classes of cb$_0$-spaces, namely for the Polish and quasi-Polish spaces [5, 12]. For non-countably based spaces, the situation is more complicated. Although the levels of these hierarchies are natural pointclasses in arbitrary topological spaces, their behaviour (even for qcb$_0$-spaces) may be quite different from what one expects from the classical Descriptive Set Theory.

2.5. Hyperprojective hierarchy of qcb$_0$-spaces

For any representation δ of a space X, let $EQ(\delta) := \{\langle p, q\rangle \in \mathcal{N} \mid p, q \in dom(\delta) \wedge \delta(p) = \delta(q)\}$. Let $\mathbf{\Gamma}$ be a family of pointclasses. A topological space X is called $\mathbf{\Gamma}$-*representable*, if X has an admissible representation δ with $EQ(\delta) \in \mathbf{\Gamma}(\mathcal{N})$. The class of all $\mathbf{\Gamma}$-representable spaces is denoted $\mathsf{QCB}_0(\mathbf{\Gamma})$. This notion from [25] enables to transfer hierarchies of sets to the corresponding hierarchies of qcb$_0$-spaces. In particular, we arrive at the following definition.

Definition 2.6. The sequence $\{\mathsf{QCB}_0(\mathbf{\Sigma}^1_\alpha)\}_{\alpha<\omega_1}$ is called the *hyperprojective hierarchy* of qcb$_0$-spaces. By *levels* of this hierarchy we mean the classes $\mathsf{QCB}_0(\mathbf{\Sigma}^1_\alpha)$ as well as the classes $\mathsf{QCB}_0(\mathbf{\Pi}^1_\alpha)$ and $\mathsf{QCB}_0(\mathbf{\Delta}^1_\alpha)$.

This hierarchy has many nice properties [26], in particular the full subcategory of QCB_0 formed by the hyperprojective qcb$_0$-spaces is the smallest full subcategory of QCB_0 which contains the Sierpinski space as an object and is closed under forming function spaces, countable limits and countable colimits in QCB_0.

We will cite the following facts from [26]:

Proposition 2.7. (1) *Let* $\mathbf{\Gamma} \in \{\mathbf{\Sigma}^1_\alpha, \mathbf{\Pi}^1_\alpha \mid 0 \leq \alpha < \omega_1\}$ *and let* X *be a Hausdorff space. Then* X *is* $\mathbf{\Gamma}$-*representable, if* X *has an admissible representation* δ *with* $dom(\delta) \in \mathbf{\Gamma}(\mathcal{N})$.

(2) *For any* $\mathbf{\Gamma} \in \{\mathbf{\Sigma}^1_\alpha, \mathbf{\Pi}^1_\alpha \mid 1 \leq \alpha < \omega_1\}$, *we have* $\mathsf{QCB}_0(\mathbf{\Gamma}) \cap \mathsf{CB}_0 = \mathsf{CB}_0(\mathbf{\Gamma})$, *where* $\mathsf{CB}_0(\mathbf{\Gamma})$ *is the class of spaces homeomorphic to a* $\mathbf{\Gamma}$-*subspace of* $P\omega$.

Proposition 2.8. *Let* $1 \leq \alpha < \omega_1$, $X \in \mathsf{QCB}_0(\mathbf{\Sigma}^1_\alpha)$ *and* $Y \in \mathsf{QCB}_0(\mathbf{\Pi}^1_\alpha)$. *Then* $Y^X \in \mathsf{QCB}_0(\mathbf{\Pi}^1_\alpha)$.

2.6. Continuous functionals of countable type

The hyperprojective hierarchy of qcb$_0$-spaces is closely related to the continuous functionals of countable types over ω defined by induction on countable ordinals α as follows [26]:

$$\mathbb{N}\langle 0\rangle := \omega,\ \mathbb{N}\langle \alpha+1\rangle := \omega^{\mathbb{N}\langle\alpha\rangle} \text{ and } \mathbb{N}\langle\lambda\rangle := \prod_{\alpha<\lambda} \mathbb{N}\langle\alpha\rangle\,,$$

where ω denotes the space of natural numbers endowed with the discrete topology, $\alpha, \lambda < \omega_1$ and λ is a limit ordinal. We call $\mathbb{N}\langle\alpha\rangle$ *the space of continuous functionals of type* α over ω. Obviously, for $k < \omega$ the space $\mathbb{N}\langle k\rangle$ coincides with the space of Kleene-Kreisel continuous functionals of type k extensively studied in the literature, and $\mathbb{N}\langle 1\rangle$ coincides with the Baire space $\mathcal{N}$. We will also deal with the coproduct spaces

$$\mathbb{N}\langle{<}\lambda\rangle := \bigoplus_{\alpha<\lambda} \mathbb{N}\langle\alpha\rangle,$$

where λ is a countable ordinal limit.

The following propositions list facts which were established in [26].

Proposition 2.9. *For every countable successor ordinal* $\alpha \geq 2$, $\mathbb{N}\langle\alpha\rangle \in \mathsf{QCB}_0(\mathbf{\Pi}^1_{\alpha-1}) \setminus \mathsf{QCB}_0(\mathbf{\Sigma}^1_{\alpha-1})$. *For any countable limit ordinal* α, $\mathbb{N}\langle\alpha\rangle \in \mathsf{QCB}_0((\mathbf{\Pi}^1_{<\alpha})_\delta) \setminus \mathsf{QCB}_0((\mathbf{\Sigma}^1_{<\alpha})_\sigma)$. *Moreover,* $\mathbb{N}\langle 1\rangle \in \mathsf{QCB}_0(\{\mathcal{N}\})$ *and* $\mathbb{N}\langle 0\rangle \in \mathsf{QCB}_0(\{\{n0^\omega \mid n \in \omega\}\})$.

For every countable ordinal α we fix an admissible representation δ_α of $\mathbb{N}\langle\alpha\rangle$ witnessing Proposition 2.9 and denote the domain of δ_α by D_α (see [26] for an explicit construction).

Proposition 2.10. *For any countable successor ordinal* $\alpha \geq 2$, $D_\alpha \in \mathbf{\Pi}^1_{\alpha-1}(\mathcal{N}) \setminus \mathbf{\Sigma}^1_{\alpha-1}(\mathcal{N})$. *For any countable limit ordinal* λ, $D_\lambda \in (\mathbf{\Pi}^1_{<\lambda})_\delta(\mathcal{N}) \setminus (\mathbf{\Sigma}^1_{<\lambda})_\sigma(\mathcal{N})$.

Proposition 2.11 relates the spaces $\mathbb{N}\langle\alpha\rangle$ to the countable hyperprojective hierarchy over $\mathcal{N}$.

Proposition 2.11. *Let* α *be a non-zero countable ordinal and* B *a non-empty subset of* $\mathcal{N}$. *Then* $B \in \mathbf{\Sigma}^1_\alpha(\mathcal{N})$ *iff there is a continuous function* $f\colon \mathbb{N}\langle\alpha\rangle \to \mathcal{N}$ *with* $rng(f) = B$.

In the following $X \cong Y$ denotes that the spaces X and Y are homeomorphic.

Proposition 2.12. *For all countable ordinals $\alpha_0, \alpha_1, \ldots$, we have $\prod_{i\in\omega} \mathbb{N}\langle\alpha_i\rangle \cong \mathbb{N}\langle\sup\{1, \alpha_i \mid i \in \omega\}\rangle$. For all $\alpha \le \beta < \omega_1$, $\mathbb{N}\langle\alpha\rangle \times \mathbb{N}\langle\beta\rangle \cong \mathbb{N}\langle\beta\rangle$ and $\mathbb{N}\langle\alpha\rangle$ is a retract of $\mathbb{N}\langle\beta\rangle$.*

We add the following lemma about coproducts.

Lemma 2.13. (1) *For any countable ordinal α, $\mathbb{N}\langle\alpha\rangle \cong \mathbb{N}\langle\alpha\rangle \oplus \mathbb{N}\langle\alpha\rangle \cong \bigoplus_{i\in\omega} \mathbb{N}\langle\alpha\rangle$.*

(2) *For any $\alpha < \omega_1$, $\mathbb{N}\langle\alpha\rangle^{\mathbb{N}_\infty} \cong \mathbb{N}\langle\alpha\rangle$.*

(3) *For any non-empty qcb$_0$-space X, $\omega^X \oplus \omega^X \cong \omega^X$ and $\omega \times \omega^X \cong \omega^X$.*

(4) *Let $\lambda < \omega_1$ be a limit ordinal and $\{\alpha_k\}$ be a sequence of pairwise distinct ordinals with $\bigcup_k \alpha_k = \lambda$. Then $(\bigoplus_k \mathbb{N}\langle\alpha_k\rangle)^{\mathbb{N}_\infty} \cong \bigoplus_k \mathbb{N}\langle\alpha_k\rangle$, in particular $\mathbb{N}\langle<\lambda\rangle^{\mathbb{N}_\infty} \cong \mathbb{N}\langle<\lambda\rangle$.*

(5) *In notation of the previous item, the spaces $\bigoplus_k \mathbb{N}\langle\alpha_k\rangle$ and $\mathbb{N}\langle<\lambda\rangle$ are retracts (in particular, continuous images) of each other.*

Proof. (1) Let 2 be a two point discrete space. Proposition 2.10 yields us

$$\mathbb{N}\langle\alpha\rangle \cong \omega \times \mathbb{N}\langle\alpha\rangle \cong (2 \times \omega) \times \mathbb{N}\langle\alpha\rangle \cong 2 \times (\omega \times \mathbb{N}\langle\alpha\rangle) \cong 2 \times \mathbb{N}\langle\alpha\rangle\,.$$

In the category of sequential spaces, we have $X \oplus X \cong 2 \times X$ and $\bigoplus_{i\in\omega} X \cong \omega \times X$. We obtain

$$\mathbb{N}\langle\alpha\rangle \oplus \mathbb{N}\langle\alpha\rangle \cong 2 \times \mathbb{N}\langle\alpha\rangle \cong \mathbb{N}\langle\alpha\rangle \quad \text{and} \quad \bigoplus_{i\in\omega} \mathbb{N}\langle\alpha\rangle \cong \omega \times \mathbb{N}\langle\alpha\rangle \cong \mathbb{N}\langle\alpha\rangle\,.$$

(2) This assertion is checked by induction. Since $\mathbb{N}_\infty$ is a compact zero-dimensional metric space and ω is countable and discrete, $\omega^{\mathbb{N}_\infty}$ is countable and discrete as well. Hence $\omega^{\mathbb{N}_\infty}$ is homeomorphic to the discrete space of natural numbers ω, which proves the assertion for $\alpha = 0$.

By the cartesian closedness of QCB_0, we obtain for $\alpha < \omega_1$:

$$\mathbb{N}\langle\alpha+1\rangle^{\mathbb{N}_\infty} \cong (\omega^{\mathbb{N}\langle\alpha\rangle})^{\mathbb{N}_\infty} \cong (\omega^{\mathbb{N}_\infty})^{\mathbb{N}\langle\alpha\rangle} \cong \omega^{\mathbb{N}\langle\alpha\rangle} \cong \mathbb{N}\langle\alpha+1\rangle.$$

Using Proposition 2.12, for a limit countable ordinal λ we obtain:

$$\begin{aligned}\mathbb{N}\langle\lambda\rangle^{\mathbb{N}_\infty} &\cong \Big(\prod_{\alpha<\lambda} \mathbb{N}\langle\alpha+1\rangle\Big)^{\mathbb{N}_\infty} \cong \Big(\prod_{\alpha<\lambda} \omega^{\mathbb{N}\langle\alpha\rangle}\Big)^{\mathbb{N}_\infty} \\ &\cong \prod_{\alpha<\lambda} (\omega^{\mathbb{N}\langle\alpha\rangle})^{\mathbb{N}_\infty} \cong \prod_{\alpha<\lambda} (\omega^{\mathbb{N}_\infty})^{\mathbb{N}\langle\alpha\rangle} \cong \prod_{\alpha<\lambda} \omega^{\mathbb{N}\langle\alpha\rangle} \cong \prod_{1\le\alpha<\lambda} \mathbb{N}\langle\alpha\rangle \cong \prod_{\alpha<\lambda} \mathbb{N}\langle\alpha\rangle \cong \mathbb{N}\langle\lambda\rangle.\end{aligned}$$

(3) Fix $x_0 \in X$ and let $Y_n := \{f \in \omega^X \mid f(x_0) = n\}$ for each $n < \omega$. Then $\{Y_n\}$ is a clopen partition of ω^X, hence $\omega^X \cong \bigoplus_n Y_n$. By symmetry, $Y_m \cong Y_n$ for all $m, n < \omega$. Therefore, the usual bijections between $\omega \oplus \omega$ and ω, and between $\omega \times \omega$ and ω induce the desired homeomorphisms.

(4) Let $X := \bigoplus_k \mathbb{N}\langle\alpha_k\rangle$ and $Y := X^{\mathbb{N}_\infty}$, we have to show $X \cong Y$. For any $k < \omega$, let $M_k := \{f \in Y \mid f(\infty) \in \{k\} \times \mathbb{N}\langle\alpha_k\rangle\}$. Then $\{M_k\}$ is a clopen partition of Y, hence $Y \cong \bigoplus_k M_k$.

Let $\mathcal{F}$ be the set of finite subsets of ω and $S^k_F := \{f \in M_k \mid f^{-1}(\{k\} \times \mathbb{N}\langle\alpha_k\rangle) = \mathbb{N}_\infty \setminus F\}$ for any $F \in \mathcal{F}$. Since $\mathbb{N}_\infty$ is compact, $\{S^k_F \mid F \in \mathcal{F}\}$ is a clopen partition of M_k, hence $M_k \cong \bigoplus_{F\in\mathcal{F}} S^k_F$.

The equality $f = f|_{\mathbb{N}_\infty\setminus F} \cup f|_F$ induces a homeomorphism $f \mapsto (f|_{\mathbb{N}_\infty\setminus F}, f|_F)$ between S^k_F and $\mathbb{N}\langle\alpha_k\rangle^{\mathbb{N}_\infty\setminus F} \times (\bigoplus_{n\neq k} \mathbb{N}\langle\alpha_n\rangle)^F$. Since $\mathbb{N}_\infty \setminus F \cong \mathbb{N}_\infty$ and F is a discrete subspace of $\mathbb{N}_\infty$,

$$S^k_F \cong \mathbb{N}\langle\alpha_k\rangle^{\mathbb{N}_\infty} \times \Big(\bigoplus_{n\neq k} \mathbb{N}\langle\alpha_n\rangle\Big)^m$$

where m is the number of elements in F. By (2), $\mathbb{N}\langle\alpha_k\rangle^{\mathbb{N}_\infty} \cong \mathbb{N}\langle\alpha_k\rangle$, hence $S^k_\emptyset \cong \mathbb{N}\langle\alpha_k\rangle$. Since $\{\mathbb{N}\langle\alpha_n\rangle \mid n \neq k\}$ is closed under the binary product by Proposition 2.12, we obtain by (1)

$$\Big(\bigoplus_{n\neq k} \mathbb{N}\langle\alpha_n\rangle\Big)^m \cong \bigoplus_{n\neq k} \mathbb{N}\langle\alpha_n\rangle$$

for any $m \neq 0$. Therefore,

$$S_F^k \cong \mathbb{N}\langle\alpha_k\rangle \times (\bigoplus_{n\neq k} \mathbb{N}\langle\alpha_n\rangle) \cong \bigoplus_{n\neq k}(\mathbb{N}\langle\alpha_k\rangle \times \mathbb{N}\langle\alpha_n\rangle).$$

From Item (3) we obtain $S_F^k \cong \bigoplus_{n\geq k} \mathbb{N}\langle\alpha_n\rangle$ for $F \neq \emptyset$. Altogether we obtain (using again Item (3)):

$$Y \cong \bigoplus_k (\bigoplus_{F\in\mathcal{F}} S_F^k) \cong (\bigoplus_k S_\emptyset^k) \oplus (\bigoplus_{k,F\neq\emptyset} S_F^k) \cong \bigoplus_k (\omega \times \mathbb{N}\langle\alpha_k\rangle) \cong X.$$

(5) The assertion follows easily from Proposition 2.12.

□

3. Topological embeddings versus sequential embeddings

In this section we briefly discuss two notions of embedding for sequential spaces relevant to this paper. The first one is the usual topological embedding which is used in Section 6. The second one is a lesser known sequential embedding which is more natural for sequential spaces and results in a more satisfactory theory in Section 7 than the theory based on topological embeddings.

Recall that any subset X of a topological space Y may be considered as a topological space (which is called a *topological subspace* of Y) with the induced topology $\{V \cap X \mid V \in \mathcal{O}(Y)\}$. We say that a space X *embeds topologically* into Y, if X is homeomorphic to a topological subspace M of Y; the corresponding homeomorphism seen as a function e from X to Y is called a *topological embedding* of X into Y.

When dealing with sequential spaces (in particular, qcb$_0$-spaces), it is natural to consider the following modification of topological embeddings:

Definition 3.1. Let X, Y be sequential spaces.

(1) The space X is *a sequential subspace* of Y, if $X \subseteq Y$ and, whenever $(x_n)_n$ is a sequence in X and $x_\infty \in X$, convergence of $(x_n)_n$ to x_∞ in X is equivalent to convergence of $(x_n)_n$ to x_∞ in Y.

(2) We say that X embeds sequentially into Y, if there is an injection $e\colon X \to Y$ such that convergence of $(x_n)_n$ to x_∞ in X is equivalent to convergence of $(e(x_n))_n$ to $e(x_\infty)$ in Y. In this case we call e a *sequential embedding of X into Y*.

The distinction between topological subspace and sequential subspace is subtle, but very important. For example, Proposition 2.3 does not hold in general if "topological subspace" is replaced by "sequential subspace".

It can be shown that if X and Y are sequential spaces, then X embeds sequentially into Y if and only if there is a topological subspace $S \subseteq Y$ such that X is homeomorphic to the sequentialisation of S.

It is easy to check that, for all sequential spaces X, Y, if $e\colon X \to Y$ is a topological embedding then it is also a sequential embedding, but the converse does not hold in general. If $e\colon X \to Y$ is a surjective sequential embedding, then e is a homeomorphism.

Lemma 3.2. *Let X, Y, Z be sequential T_0-spaces and $q\colon X \to Y$ a quotient map. Then the induced map $Z^q\colon Z^Y \to Z^X$ defined by $Z^q(f) := f \circ q$ is a sequential embedding of Z^Y into Z^X.*

Proof. By the cartesian closedness of Seq, Z^q is continuous. From the surjectivity of q it follows that Z^q is injective. Now let $(f_n)_{n\leq\infty}$ be a sequence in Z^Y such that $(Z^q(f_n))_n$ converges to $Z^q(f_\infty)$ in Z^X. This is equivalent to the continuity of the function $g\colon \mathbb{N}_\infty \times X \to Z$ defined by $g(n,x) := f_n(q(x))$. Since in the category of sequential spaces Seq quotient maps are preserved by products (see [27]), the function $\tilde{q}\colon \mathbb{N}_\infty \times X \to \mathbb{N}_\infty \times Y$ mapping (n,x) to $(n,q(x))$, is a quotient map as well. As the function $\tilde{f}\colon \mathbb{N}_\infty \times Y \to Z$ sending (n,y) to $f_n(y)$ satifies $g = \tilde{f} \circ \tilde{q}$ and $\tilde{q}$ is quotient, $\tilde{f}$ is continuous (cf. [7, Proposition 2.4.2]). This is equivalent to saying that $(f_n)_n$ converges to f_∞ in Z^Y. We conclude that Z^q is a sequential embedding. □

Given topological spaces X and Y and subsets $A \subseteq X$ and $B \subseteq Y$, we say that A *Wadge-reduces* to B, denoted $A \leq_W B$, if and only if there is a continuous function $f\colon X \to Y$ such that $A = f^{-1}(B)$.

We will apply the following easy fact to topological or sequential embeddings.

Lemma 3.3. *Let $\xi\colon \mathcal{N} \to X$ be a total continuous representation of $X \in \mathsf{QCB}_0$, $\delta\colon D \to Y$ an admissible representation of $Y \in \mathsf{QCB}_0$ and $e\colon X \to Y$ a continuous injection. Then $EQ(\xi) \leq_W EQ(\delta)$.*

Proof. Since ξ is continuous and δ is admissible, by Proposition 2.4 there is a continuous realizer $\tilde{e}\colon \mathcal{N} \to D$ of e. Then we have $\xi(a) = \xi(b)$ iff $e(\xi(a)) = e(\xi(b))$ iff $\delta(\tilde{e}(a)) = \delta(\tilde{e}(b))$. Thus the continuous function $\langle a,b\rangle \mapsto \langle \tilde{e}(a), \tilde{e}(b)\rangle$ Wadge-reduces $EQ(\xi)$ to $EQ(\delta)$. □

4. Hyperspaces of open sets

Topological spaces formed by pointclasses in a space X are sometimes referred to as "hyperspaces". In this section we discuss hyperspaces of open subsets and in Section 5 hyperspaces of compact subsets.

4.1. The ω-Scott topology on the open subsets

Let X be a sequential space. We define $\mathcal{O}(X)$ to be the space of open subsets of X topologized with the *ω-Scott topology* $\tau_{\omega\mathrm{Scott}}$ defined on the complete lattice $(\mathcal{O}(X);\subseteq)$. So a set $H \subseteq \mathcal{O}(X)$ belongs to $\tau_{\omega\mathrm{Scott}}$, if H is upwards closed in $(\mathcal{O}(X);\subseteq)$ and $\mathcal{D} \cap H \neq \emptyset$ for each countable directed subset $\mathcal{D}$ of $(\mathcal{O}(X);\subseteq)$ with $\bigcup \mathcal{D} \in H$. Elements of $\tau_{\omega\mathrm{Scott}}$ are called *ω-Scott open*. The more familiar *Scott topology* τ_{Scott} is defined similarly by considering all directed families of opens, not only the countable directed ones. The ω-Scott topology $\tau_{\omega\mathrm{Scott}}$ refines the Scott topology, i.e. $\tau_{\mathrm{Scott}} \subseteq \tau_{\omega\mathrm{Scott}}$.

The ω-Scott topology $\tau_{\omega\mathrm{Scott}}$ is known to be sequential. This is the reason why we equip the collection $\mathcal{O}(X)$ with the ω-Scott topology rather than with the Scott topology. However, both topologies induce the same convergence relation on the collection $\mathcal{O}(X)$. Moreover, if X is hereditarily Lindelöf (in particular if X is a qcb_0-space), then the ω-Scott topology coincides with the Scott topology. It is useful to identify a subset $W \subseteq X$ with its characteristic function $cf(W)\colon X \to \mathbb{S}$ defined by $cf(W)(x) = \top :\Longleftrightarrow x \in W$, where $\mathbb{S}$ is the Sierpinski space. Clearly, W is open if, and only if, the function $cf(W)$ is continuous.

For more details, we refer e.g. to Section 2 in [23].

Proposition 4.1. *Let X be a sequential space.*

1. *The function $cf\colon \mathcal{O}(X) \to \mathbb{S}^X$ is a homeomorphism between $\mathcal{O}(X)$ and the function space $\mathbb{S}^X$.*
2. *If X is a qcb_0-space, then $\mathcal{O}(X)$ is a qcb_0-space and the ω-Scott topology coincides with the Scott topology.*
3. *A sequence $(U_n)_n$ converges to U in $\mathcal{O}(X)$ iff $V_k := \bigcap_{n\geq k} U_n \cap U$ is open for all $k \in \omega$ and $U = \bigcup_{k\in\omega} V_k$.*

Proof. See e.g. Proposition 2.2 in [23]. □

Note that $\tau_{\omega\mathrm{Scott}}$ forms the underlying set of the sequential space $\mathcal{O}(\mathcal{O}(X))$.

4.2. Consonant spaces

We discuss some conditions on a space X which simplify understanding of the space $\mathcal{O}(X)$.

To this aim we consider yet another topology on the collection $\mathcal{O}(X)$ of open subsets, namely the *compact-open topology* τ_{CO}. It is generated by the subbasic opens $K^{\subseteq} := \{U \in \mathcal{O}(X) \mid K \subseteq U\}$, where K runs through the compact subsets of X. The name "compact-open topology" is motivated by the fact that it coincides with the usual compact-open topology on the function space $\mathbb{S}^X$ under the natural identification of an open subset $U \subseteq X$ with its continuous characteristic function $cf(U)$. Obviously, $\tau_{\mathrm{CO}} \subseteq \tau_{\mathrm{Scott}} \subseteq \tau_{\omega\mathrm{Scott}}$. If X is sequential, then τ_{CO} induces the same convergence relation on $\mathcal{O}(X)$ as both the ω-Scott topology and the Scott topology (see Proposition 2.2 in [23]).

A topological space X is called *consonant*, if the Scott topology on the collection $\mathcal{O}(X)$ coincides with the compact-open topology. Non-consonant space are usually called *dissonant*.

Proposition 4.2. (1) *Every quasi-Polish space is consonant.*

(2) *The space $\mathbb{Q}$ of rationals is dissonant.*

(3) *A metrizable space $X \in \mathsf{CB}_0(\mathbf{\Pi}^1_1)$ is consonant if, and only if, X is Polish.*

Proof. Item (1) follows from the known facts that Polish spaces are consonant (see Theorem 4.1 in [6]), that every (non-empty) quasi-Polish space is the image of $\mathcal{N}$ under an open continuous mapping (Lemma 38 in [5]), and that the image of a consonant space under a continuous open mapping is consonant (Theorem 8.2 in [17]).

Item (2) is known (see e.g. [4]).

To prove (3), let X be a metrizable space in $\mathsf{CB}_0(\mathbf{\Pi}^1_1)$ which is not Polish. By a theorem by Hurewicz (see e.g. Theorem 21.18 in [12]), $\mathbb{Q}$ is homeomorphic to a closed subspace of X. Since $\mathbb{Q}$ is dissonant, X is also dissonant by Proposition 4.2 in [6]. □

A. Bouziad [1] has shown that it is independent of ZFC whether or not every metrizable consonant space in $\mathsf{CB}_0(\mathbf{\Sigma}^1_1)$ is Polish.

Next we give some remarks on when the hyperspaces of open sets are countably based.

Proposition 4.3. *Let X be a countably based Hausdorff space.*

(1) *The space $\mathcal{O}(X)$ has a countable base if, and only if, X is locally compact.*
(2) *If $\mathcal{O}(X)$ has a countable base, then X is consonant.*

We remark that Statement (2) can be shown for any Hausdorff qcb-space.

Proof. The first statement follows from Theorem 7.3 and Corollary 7.4 in [21].

If X is locally compact, then there is a countable base $\mathcal{B}$ for X such that the closure $Cls(B)$ is compact for any base element $B \in \mathcal{B}$. Let H be Scott-open and $U \in H$. Then there is a sequence $(B_i)_i$ of base elements in $\mathcal{B}$ such that $U = \bigcup_{i\in\omega} B_i = \bigcup_{i\in\omega} Cls(B_i)$. Moreover there is some $k \in \omega$ such that $\bigcup_{i<k} B_i \in H$. Then $K := \bigcup_{i<k} Cls(B_i)$ is compact and satisfies $U \in K^{\subseteq} \subseteq H$. Therefore X is consonant. □

4.3. Representing open subsets of countably based spaces

Here we obtain some new information on admissible representations of the hyperspace of open sets. In [30] it was shown that for any countably based space X the space $\mathcal{O}(X)$ has a total admissible representation $\pi_X : \mathcal{N} \to \mathcal{O}(X)$. It is constructed as follows. We choose a numbered base $\{B_0, B_1, \ldots\}$ in X containing the empty set (say, $B_0 = \emptyset$), and define π_X by $\pi_X(a) := \bigcup_n B_{a(n)}$. Up to continuous equivalence of representations, π_X does not depend on the choice of such a numbered base.

We start with improving this result to the fact that there is an admissible representation $\gamma_X : \mathcal{C} \to \mathcal{O}(X)$ of $\mathcal{O}(X)$ by elements of the Cantor space $\mathcal{C}$ (this is indeed an improvement because if we have such γ_X then $\gamma_X \circ r$, where $r : \mathcal{N} \to \mathcal{C}$ is a continuous retraction, is an admissible representation of $\mathcal{O}(X)$ with domain $\mathcal{N}$).

To define γ_X, we choose a numbered base $\{B_0, B_1, \ldots\}$ of X such that any base set appears in the numbering infinitely often (this time it is not necessary to require that the base contains the empty set) and set $\gamma_X(p) := \bigcup\{B_n \mid p(n) = 1\}$. Up to the continuous equivalence of representations, γ_X does not depend on the choice of such a numbered base.

Proposition 4.4. *The representation γ_X is an admissible representation of $\mathcal{O}(X)$.*

Proof. Similarly to the proof of Theorem 6.5 in [30], γ_X is continuous. Since π_X is admissible (w.r.t. any of the topologies in Section 4) by Theorem 8.6 in [30], it suffices to show that π_X is topologically reducible to γ_X, i.e. to find a continuous function $f : \mathcal{N} \to \mathcal{C}$ such that $\pi_X = \gamma_X \circ f$. Let π_X be defined from a base $\{B_0, B_1, \ldots\}$ as specified above.

Define the numbered basis $\{B'_n \mid n < \omega\}$ of $\mathcal{O}(X)$ by $B'_{\langle m,n\rangle} = B_m$, so any element B_m of the former base appears infinitely many times in the new numbering. Choose now a continuous function f on $\mathcal{N}$ such that: $f(x) = 0^\omega$ if $x = 0^\omega$ has no non-zero elements, $f(x) = 0^{\langle k_0,n_0\rangle-1}1\cdots 0^{\langle k_l,n_l\rangle-1}10^\omega$ for suitable $n_0, \ldots, n_1$ if $x = 0^{m_0}(k_0 + 1)\cdots 0^{m_l}(k_l + 1)0^\omega$ has $l + 1$ non-zero elements, and $f(x) = 0^{\langle k_0,n_0\rangle-1}10^{\langle k_1,n_1\rangle-1}1\cdots$ for suitable $n_0, n_1, \ldots$ if $x = 0^{m_0}(k_0 + 1)0^{m_1}(k_1 + 1)\cdots$ has infinitely many non-zero elements. Such an f has the desired property. □

Below we will especially be interested in the particular case $X = \mathcal{N}$ and, more generally, in the case when X is a subspace of $\mathcal{N}$. Let $\sigma_0, \sigma_1, \ldots$ be an enumeration without repetition of the set ω^* such that σ_0 is the empty string. Then the sets $B_n = \sigma_n \cdot \mathcal{N}$ form a numbered base of $\mathcal{N}$ that has no empty set, while the sets $\tilde{B}_0 = \emptyset, \tilde{B}_{n+1} = B_n$ form a numbered base of $\mathcal{N}$ that has the empty set. According to the general construction above, we obtain admissible representations $\pi_{\mathcal{N}}\colon \mathcal{N} \to \mathcal{O}(\mathcal{N})$ (constructed from $\{\tilde{B}_n\}$) and $\gamma_{\mathcal{N}}\colon \mathcal{C} \to \mathcal{O}(\mathcal{N})$ (constructed from $\{B'_n\}$) of $\mathcal{O}(\mathcal{N})$.

For any $X \subseteq \mathcal{N}$, we can canonically define admissible representations $\pi_X\colon \mathcal{N} \to \mathcal{O}(X)$ and $\gamma_X\colon \mathcal{C} \to \mathcal{O}(X)$ of $\mathcal{O}(X)$ by $\pi_X(p) = X \cap \pi_{\mathcal{N}}(p)$ and $\gamma_X(p) = X \cap \gamma_{\mathcal{N}}(p)$ (the representations are admissible because both $\{X \cap \tilde{B}_n \mid n < \omega\}$ and $\{X \cap B'_n \mid n < \omega\}$ are suitable numbered bases in X).

Below we make use of the following estimation of Wadge degrees of $EQ(\pi_X)$ and $EQ(\gamma_X)$:

Lemma 4.5. *For any $\emptyset \neq X \subseteq \mathcal{N}$, $\mathcal{N} \setminus X$ is Wadge reducible to both $EQ(\pi_X)$ and $EQ(\gamma_X)$.*

Proof. Since π_X and γ_X are topologically equivalent, $EQ(\pi_X) \equiv_W EQ(\gamma_X)$, so it suffices to show $\mathcal{N} \setminus X \leq_W EQ(\pi_X)$. Since $\mathcal{N}$ is Hausdorff, the function $p \mapsto \mathcal{N} \setminus \{p\} \in \mathcal{O}(\mathcal{N})$ is continuous. By the admissibility of $\pi_{\mathcal{N}}$, there is a continuous function f on $\mathcal{N}$ such that $\pi_{\mathcal{N}}(f(p)) = \mathcal{N} \setminus \{p\}$. For $p \in \mathcal{N} \setminus X$ we have $\pi_X(f(p)) = X \cap (\mathcal{N} \setminus \{p\}) = X$, whereas for $p \in X$ we have

$$\pi_X(f(p)) = X \cap (\mathcal{N} \setminus \{p\}) = X \setminus \{p\} \neq X.$$

Thus, $\mathcal{N} \setminus X \leq_W EQ(\pi_X)$ via the continuous function $p \mapsto \langle f(p), g(p)\rangle$, where g is a constant function satisfying $\pi_X g(p) = X$ for all $p \in \mathcal{N}$. □

4.4. The descriptive complexity of $\mathcal{O}(\mathbb{N}\langle\alpha\rangle)$ and $\mathcal{O}(D_\alpha)$

Here we establish a precise estimation of the descriptive complexity of the spaces $\mathcal{O}(\mathbb{N}\langle\alpha\rangle)$ and $\mathcal{O}(D_\alpha)$ in the hyperprojective hierarchy of qcb$_0$-spaces. Remember that D_α is the domain of a natural admissible representation of $\mathbb{N}\langle\alpha\rangle$ chosen in Section 2.5.

The lower bound in Lemma 4.5 for the complexity of $EQ(\pi_X)$ is in general far from optimal, as the following immediate corollary of Theorem 8.11 from [30] shows:

Proposition 4.6. *The sets $EQ(\pi_{\mathcal{N}})$ and $EQ(\gamma_{\mathcal{N}})$ are Wadge complete in $\mathbf{\Pi}^1_1(\mathcal{N})$.*

It turns out that it is possible to completely characterize the topological complexity of the spaces $\mathcal{O}(D_\alpha)$ and $\mathcal{O}(\mathbb{N}\langle\alpha\rangle)$. To show this, we first establish the following lemma. For a pointclass $\mathbf{\Gamma} \subseteq P(\mathcal{N})$ and a set $S \subseteq \mathcal{N}$, let $\mathbf{\Gamma} \leq_W S$ denote that $C \leq_W S$ for all $C \in \mathbf{\Gamma}$.

Lemma 4.7. (1) *Let X be a qcb$_0$-space, ν an admissible representation of $\mathcal{O}(X)$, and $f : X \to \mathcal{N}$ a continuous function. Then $\mathcal{N} \setminus rng(f) \leq_W EQ(\nu)$.*

(2) *Let X be a qcb$_0$-space and $\mathbf{\Gamma} \subseteq P(\mathcal{N})$ a pointclass such that any non-empty set in $\mathbf{\Gamma}$ is a continuous image of X. Then $\mathbf{\Gamma}_c \leq_W EQ(\nu)$ for every admissible representation ν of $\mathcal{O}(X)$.*

Proof. (1) Define a function $h : \mathcal{N} \to \mathbb{S}^X = \mathcal{O}(X)$ by: $h(p)(x) = \top$ iff $f(x) \neq p$. Since f and the function $neq : \mathcal{N} \times \mathcal{N} \to \mathbb{S}$ such that $neq(p, q) = \top$ iff $p \neq q$ are continuous, h is also continuous. For any $p \in \mathcal{N}$ we then have $p \in \mathcal{N} \setminus rng(f)$ iff $h(p) = X$. Let $q \in \mathcal{N}$ satisfy $\nu(q) = X$ and let g be the constant function on $\mathcal{N}$ sending all elements to q. Then $p \in \mathcal{N} \setminus rng(f)$ iff $h(p) = g(p)$, hence the continuous function $p \mapsto \langle h(p), neq(p, q)\rangle$ Wadge reduces $\mathcal{N} \setminus rng(f)$ to $EQ(\nu)$.

(2) We have to Wadge reduce any $S \in \mathbf{\Gamma}_c$ to $EQ(\nu)$. For $S = \mathcal{N}$ the assertion is trivial, so let $S \neq \mathcal{N}$. Then $\mathcal{N} \setminus S = rng(f)$ for some continuous function $f : X \to \mathcal{N}$. By (1), $S \leq_W EQ(\nu)$.

□

Theorem 4.8. *For any $1 \leq \alpha < \omega_1$, the spaces $\mathcal{O}(\mathbb{N}\langle\alpha\rangle)$ and $\mathcal{O}(D_\alpha)$ are in $\mathsf{QCB}_0(\mathbf{\Pi}^1_\alpha)$. Moreover, $\mathbf{\Pi}^1_\alpha(\mathcal{N}) \leq_W EQ(\nu)$ for every admissible representation ν of $\mathcal{O}(\mathbb{N}\langle\alpha\rangle)$ or of $\mathcal{O}(D_\alpha)$.*

Proof. From Propositions 2.9 and 2.10 we know that $\mathbb{N}\langle\alpha\rangle, D_\alpha \in \mathsf{QCB}_0(\mathbf{\Pi}^1_\beta)$, if $\alpha = \beta + 1$ is a successor ordinal, and $\mathbb{N}\langle\alpha\rangle, D_\alpha \in \mathsf{QCB}_0((\mathbf{\Pi}^1_{<\alpha})_\delta)$, if α is a limit ordinal. In both cases, Proposition 2.8 yields us $\mathcal{O}(\mathbb{N}\langle\alpha\rangle), \mathcal{O}(D_\alpha) \in \mathsf{QCB}_0(\mathbf{\Pi}^1_\alpha)$, because $\mathbb{S} \in \mathsf{QCB}_0(\mathbf{\Pi}^1_0)$, $\mathbf{\Pi}^1_\beta(\mathcal{N}) \subseteq \mathbf{\Sigma}^1_\alpha(\mathcal{N})$ and $(\mathbf{\Pi}^1_{<\alpha})_\delta(\mathcal{N}) \subseteq \mathbf{\Sigma}^1_\alpha(\mathcal{N})$ by Proposition 2.5.

Let now ν be an arbitrary admissible representation of $\mathcal{O}(\mathbb{N}\langle\alpha\rangle)$. By Proposition 2.11, for any non-empty set $S \in \mathbf{\Sigma}^1_\alpha(\mathcal{N})$ there is a continuous function f from $\mathbb{N}\langle\alpha\rangle$ onto S. Taking $X = \mathbb{N}\langle\alpha\rangle$ and $\mathbf{\Gamma} = \mathbf{\Sigma}^1_\alpha(\mathcal{N})$ in Lemma 4.7, we obtain $\mathbf{\Pi}^1_\alpha(\mathcal{N}) \leq_W EQ(\nu)$.

Finally, let ν be an arbitrary admissible representation of $\mathcal{O}(D_\alpha)$. By Proposition 2.11, for any non-empty set S from $\mathbf{\Sigma}^1_\alpha(\mathcal{N})$ there is a continuous surjection f from $\mathbb{N}\langle\alpha\rangle$ onto S. Then $f\delta_\alpha$ is a continuous surjection f from D_α onto S, so $\mathbf{\Pi}^1_\alpha(\mathcal{N}) \leq_W EQ(\nu)$ just as in the previous paragraph. □

Theorem 4.8 immediately implies the following

Corollary 4.9. *For any* $1 \leq \alpha < \omega_1$, $\mathcal{O}(\mathbb{N}\langle\alpha\rangle), \mathcal{O}(D_\alpha) \in \mathsf{QCB}_0(\mathbf{\Pi}^1_\alpha) \setminus \mathsf{QCB}_0(\mathbf{\Sigma}^1_\alpha)$.

5. Hyperspaces of compact sets

In this section we discuss hyperspaces on the collection of compact subsets.

5.1. Topologies on the compact sets

We dicuss two natural topologies on the set of compact subsets of X, known as the *Vietoris topology* and the *upper Vietoris topology*.

The upper Vietoris topology can fail to be T_0, unless one restricts oneself to saturated sets. Remember that a subset A of X is called *saturated* if, and only if, it is equal to its saturation. The *saturation* $\uparrow A$ of A is the intersection of all open sets containing A. So we will study the family of compact saturated subsets of X, which we henceforth denote by $\mathcal{K}(X)$. If X is T_1, then $\mathcal{K}(X)$ is indeed the family of all compact subsets, as in T_1-spaces the saturation of a set is the set itself.

The *Vietoris topology* on $\mathcal{K}(X)$ is generated by the subbasic open sets $\Box U := \{K \in \mathcal{K}(X) \mid K \subseteq U\}$ and $\Diamond U := \{K \in \mathcal{K}(X) \mid K \cap U \neq \emptyset\}$, where $U \in \mathcal{O}(X)$. The *upper Vietoris topology* only has the sets $\Box U$ for $U \in \mathcal{O}(X)$ as a subbasis. Obviously, the Vietoris topology refines the upper Vietoris topology.

We do not know whether any of these topologies are sequential in general. However, if X is countably based then both topologies are countably based and thus sequential. By $\mathcal{K}_{\mathrm{up}}(X)$ we denote the space of compact saturated subsets of X equipped with the sequentialisation of the upper Vietoris topology; $\mathcal{K}_{\mathrm{Viet}}$ is defined analogously. If X is a qcb$_0$-space, then both $\mathcal{K}_{\mathrm{up}}(X)$ and $\mathcal{K}_{\mathrm{Viet}}(X)$ are qcb$_0$-spaces as well (cf. Section 4.4.3 in [20]).

Remember that the compact-open topology on the function space Y^X is defined by the subbasis of open sets $\mathsf{C}(K, U) := \{f \in Y^X \mid f[K] \subseteq U\}$, where K is compact and $U \in \mathcal{O}(Y)$. Since $\mathsf{C}(\uparrow K, U) = \mathsf{C}(K, U)$, it suffices to consider only the *saturated* compact subsets K of X.

Below we will refer to the following continuity property of this construction.

Proposition 5.1. *For two sequential spaces X, Y, the map C is a sequentially continuous function from $\mathcal{K}_{\mathrm{up}}(X) \times \mathcal{O}(Y)$ to $\mathcal{O}(Y^X)$.*

Proof. Let $(K_n)_n$ converge to K_∞ in $\mathcal{K}_{\mathrm{up}}(X)$, let $(V_n)_n$ converge to V_∞ in $\mathcal{O}(Y)$, let $(f_n)_n$ converge to f_∞ in Y^X and let $f_\infty \in \mathsf{C}(K_\infty, V_\infty)$. Since $K_\infty \subseteq U_\infty := f_\infty^{-1}[V_\infty] \in \mathcal{O}(X)$, there is some n_1 such that $K_n \subseteq U_\infty$ for all $n \geq n_1$. From the fact that $(K_n)_n$ converges to K_∞ one can easily deduce that the set $L := K_\infty \cup \bigcup_{i \geq n_1} K_i$ is compact in X. Hence $M := f_\infty[L] \subseteq V_\infty$ is compact in Y. By Proposition 4.1(3) the set $W_m := V_\infty \cap \bigcap_{n \geq m} V_n$ is open in Y for all $m \in \omega$ and $V_\infty = \bigcup_{m \in \omega} W_m$. Therefore there is some $n_2 \in \omega$ with $M \subseteq W_0 \cup \ldots \cup W_{n_2}$, hence $M \subseteq W_{n_2}$ and $f_\infty \in \mathsf{C}(L, W_{n_2})$. Since the compact open topology induces the convergence relation on Y^X (see Section 2.3), there is some $n_3 \in \omega$ such that $f_n \in \mathsf{C}(L, W_{n_2})$ for all $n \geq n_3$. For all $n \geq \max\{n_1, n_2, n_3\}$ we have $K_n \subseteq L, W_{n_2} \subseteq V_n$ and thus $f_n \in \mathsf{C}(L, W_{n_2}) \subseteq \mathsf{C}(K_n, V_n)$. We conclude that $\big(\mathsf{C}(K_n, V_n)\big)_n$ converges to $\mathsf{C}(K_\infty, V_\infty)$ in $\mathcal{O}(Y^X)$. Hence C is sequentially continuous. □

5.2. The upper Vietoris topology for quasi-Polish spaces

It is well known that if X is a Polish space then $\mathcal{K}(X)$ with the Vietoris topology is also Polish (see e.g. Theorem 4.25 in [12]). Our next goal is to prove a similar result for the upper Vietoris topology.

Lemma 5.2. *Every compact saturated subset K of a T_0-space X is equal to the saturation of a compact T_1-subspace of X.*

Proof. Let $\leq$ be the specialization order on X (i.e., $x \leq y$ iff x is in the closure of $\{y\}$), and let M be the subset of elements of K that are minimal with respect to the specialization order. Clearly M is a T_1-subspace of X. We show that K is the saturation of M, which easily implies that M is compact. Let $\{x_i \mid i \in I\}$ be a subset of K that is totally ordered with respect to the specialization order. Clearly $\bigcap_{i\in F} Cls\{x_i\}$ has non-empty intersection with K for every finite $F \subseteq I$. The compactness of K implies that $A = \bigcap_{i\in I} Cls\{x_i\}$ has non-empty intersection with K, hence any $y \in K \cap A$ is a lower bound for $\{x_i \mid i \in I\}$. It follows from Zorn's lemma that every $x \in K$ is greater than or equal to some $y \in M$. Therefore, any open subset of X that contains M contains all of K, which implies that K is the saturation of M. We remark that Zorn's lemma can be avoided if X is a qcb$_0$-space X; the Axiom of Dependent Choice is sufficient in this case. $\square$

This lemma is instrumental in showing:

Theorem 5.3. *Assume X and Y are countably based T_0-spaces. If X is a $\mathbf{\Pi}^0_2$-subset of Y, then $\mathcal{K}_{\mathrm{up}}(X)$ is homeomorphic to a $\mathbf{\Pi}^0_2$-subset of $\mathcal{K}_{\mathrm{up}}(Y)$.*

Proof. Fix a countable basis $\{B_0, B_1, \ldots\}$ for Y. Since $X \in \mathbf{\Pi}^0_2(Y)$, there exist open subsets U_i, V_i $(i \in \omega)$ of Y such that $x \in X$ iff $(\forall i \in \omega)[x \in U_i \implies x \in V_i]$. Let A be the subset of $\mathcal{K}(Y)$ defined as $K \in A$ iff for all finite $F \subseteq \omega$ and $i \in \omega$, if $K \in \Box(\bigcup_{j\in F} B_j \cup U_i)$ then $K \in \Box(\bigcup_{j\in F} B_j \cup V_i)$. Clearly $A \in \mathbf{\Pi}^0_2(\mathcal{K}_{\mathrm{up}}(Y))$.

Let $f\colon \mathcal{K}_{\mathrm{up}}(X) \to \mathcal{K}_{\mathrm{up}}(Y)$ map each $K \in \mathcal{K}_{\mathrm{up}}(X)$ to the saturation of K in Y. It is easy to see that f is a topological embedding. We claim that f is a homeomorphism from $\mathcal{K}_{\mathrm{up}}(X)$ to A.

First we show $range(f) \subseteq A$. Fix $K \in \mathcal{K}_{\mathrm{up}}(X)$. Since $f(K)$ is the saturation of K in Y, for all finite $F \subseteq \omega$ and $i \in \omega$, if $f(K) \subseteq U_i \cup \bigcup_{j\in F} B_j$ then $K \subseteq U_i \cup \bigcup_{j\in F} B_j$ hence $K \subseteq V_i \cup \bigcup_{j\in F} B_j$ because every element of K that is in U_i is also in V_i. It follows that $f(K) \subseteq V_i \cup \bigcup_{j\in F} B_j$, hence $f(K) \in A$.

Next we show $A \subseteq range(f)$. Fix $K \in A$. Using Lemma 5.2 we have that K is the saturation in Y of a T_1-subspace M of Y. Assume for a contradiction that $M \not\subseteq X$. Then there is $x \in M$ and $i \in \omega$ such that $x \in U_i \setminus V_i$. For every $y \in M \setminus \{x\}$ choose j_y such that $y \in B_{j_y}$ and $x \notin B_{j_y}$. The open set U_i together with the open sets B_{j_y} form a cover of the compact set M, hence there is finite $F \subseteq \omega$ such that $x \notin B_j$ for any $j \in F$ and $U_i \cup \bigcup_{j\in F} B_j$ covers M. Then $K \subseteq U_i \cup \bigcup_{j\in F} B_j$ but $x \in K$ is not in $V_i \cup \bigcup_{j\in F} B_j$, contradicting the assumption that $K \in A$. This proves that $M \subseteq X$, hence $K \in range(f)$.

Therefore, $A = range(f)$, hence $\mathcal{K}_{\mathrm{up}}(X)$ is homeomorphic to the $\mathbf{\Pi}^0_2$ subset A of $\mathcal{K}_{\mathrm{up}}(Y)$. $\square$

Corollary 5.4. *If X is quasi-Polish then $\mathcal{K}_{\mathrm{up}}(X)$ is quasi-Polish.*

Proof. Since X is quasi-Polish it is homeomorphic to a $\mathbf{\Pi}^0_2$-subspace of $P\omega$, hence $\mathcal{K}_{\mathrm{up}}(X)$ is homeomorphic to a $\mathbf{\Pi}^0_2$-subspace of $\mathcal{K}_{\mathrm{up}}(P\omega)$. The results of M. Smyth [31] show that for any ω-algebraic domain D, the set $\mathcal{K}(D)$ of compact saturated subsets is an ω-algebraic domain when ordered by reverse subset inclusion, and it is easily seen that the upper Vietoris topology and the Scott-topology coincide in this case (see Lemma 7.26 in [18]). Therefore, $\mathcal{K}_{\mathrm{up}}(P\omega)$ is quasi-Polish because every ω-algebraic domain with the Scott-topology is quasi-Polish, and it follows that $\mathcal{K}_{\mathrm{up}}(X)$ is quasi-Polish because it is homeomorphic to a $\mathbf{\Pi}^0_2$-subspace of $\mathcal{K}_{\mathrm{up}}(P\omega)$ by Theorem 5.3. $\square$

The following result is well known for the Vietoris topology, and here we show that it holds for the upper Vietoris topology as well.

Corollary 5.5. *Let $X \in \mathsf{CB}_0(\mathbf{\Pi}^1_1)$ be metrizable. Then $\mathcal{K}_{\mathrm{up}}(X) \in \mathsf{CB}_0(\mathbf{\Sigma}^1_1)$ if and only if X is Polish.*

Proof. We first show that $\mathcal{K}_{\mathrm{up}}(\mathbb{Q}) \notin \mathsf{CB}_0(\mathbf{\Sigma}^1_1)$. The identity function $f\colon \mathcal{K}_{\mathrm{Viet}}(\mathbb{R}) \to \mathcal{K}_{\mathrm{up}}(\mathbb{R})$ is continuous, hence if $\mathcal{K}_{\mathrm{up}}(\mathbb{Q})$ was an analytic subset of the quasi-Polish space $\mathcal{K}_{\mathrm{up}}(\mathbb{R})$ then $\mathcal{K}_{\mathrm{Viet}}(\mathbb{Q}) = f^{-1}(\mathcal{K}_{\mathrm{up}}(\mathbb{Q}))$ would be an analytic subset of the Polish space $\mathcal{K}_{\mathrm{Viet}}(\mathbb{R})$, contradicting the well-known fact that $\mathcal{K}_{\mathrm{Viet}}(\mathbb{Q})$ is co-analytic complete [12].

In general, if $X \in \mathsf{CB}_0(\mathbf{\Pi}^1_1)$ is metrizable space but not Polish, then using a theorem by Hurewicz (see Theorem 21.18 in [12]) we have that $\mathbb{Q}$ is homeomorphic to a closed subspace of X, hence $\mathcal{K}_{\mathrm{up}}(\mathbb{Q})$ is homeomorphic to a $\mathbf{\Pi}^0_2$-subspace of $\mathcal{K}_{\mathrm{up}}(X)$ by Theorem 5.3, which implies that $\mathcal{K}_{\mathrm{up}}(X)$ can not be homeomorphic to an analytic subset of $P\omega$. $\square$

6. Y-Based topological spaces

In this section we introduce and study the notion of a Y-based space (where Y is a topological space) which induces classifications of qcb$_0$-spaces alternative to the hyperprojective hierarchy of qcb$_0$-spaces.

6.1. Characterizing qcb$_0$-spaces

Recall that $\mathbb{S}$ is the Sierpinski space and $\mathcal{O}(X)$ is the hyperspace of open subsets of a space X topologised with the ω-Scott topology. If X is sequential (in particular a qcb$_0$-space), then $\mathcal{O}(X)$ is homeomorphic to $\mathbb{S}^X$ (see Section 4.1).

Definition 6.1. Let X and Y be topological spaces. A continuous function $\phi\colon Y \to \mathcal{O}(X)$ is a *Y-indexing* of a basis for X, if the range of ϕ is a basis for the topology on X. The space X is *Y-based* if there is a *Y-indexing* of a basis for X.

The introduced notions are purely topological and apply to arbitrary topological spaces. The following lemma shows some natural properties of these notions.

Lemma 6.2. (1) *Let X be Y-based and Y be a continuous image of a space Z. Then X is Z-based.*
(2) *Any topological subspace of a Y-based space is Y-based.*

Proof. (1) Let $\phi\colon Y \to \mathcal{O}(X)$ be a Y-indexing of a basis for X and ψ be a continuous surjection from Z onto Y. Then $\phi \circ \psi$ is a Z-indexing of a basis for X.
(2) Let ϕ be a Y-indexing of a basis for X and let Z be a topological subspace of X. Define $\psi\colon Y \to \mathcal{O}(Z)$ by $\psi(y) = Z \cap \phi(y)$. It is straightforward to check that ψ is a Y-indexing of a basis for Z.
□

The next proposition generalises the fact that any countably-based T_0-space embeds topologically into $P\omega$ which is homeomorphic to $\mathcal{O}(\omega)$.

Theorem 6.3. *Let X, Y be sequential T_0-spaces such that X is Y-based. Then X embeds topologically into $\mathcal{O}(Y)$.*

Proof. Let $\phi\colon Y \to \mathcal{O}(X)$ be a Y-indexing of a basis for X. By the cartesian closedness of Seq the function $\psi\colon X \to \mathbb{S}^Y$ defined by $\psi(x)(y) = \phi(y)(x)$ is continuous. Clearly, we have $y \in \psi(x)$ iff $x \in \phi(y)$.

To show that ψ is injective, let x and x' be distinct elements of X, and assume without loss of generality that there is open $U \subseteq X$ containing x, but not x'. By the definition of ϕ, there is some $y \in Y$ such that $x \in \phi(y) \subseteq U$, hence $y \in \psi(x)$ but $y \notin \psi(x')$. Therefore, $\psi(x) \neq \psi(x')$.

Finally, the set $\{y\}^{\subseteq} = \{V \in \mathcal{O}(Y) \mid y \in V\}$ is ω-Scott open. It is easy to see that $\psi[\phi(y)] = \{y\}^{\subseteq} \cap \mathit{range}(\psi)$. Since $\{\phi(y) \mid y \in Y\}$ is a basis for X, it follows that ψ is an open map onto $\mathit{range}(\psi)$. Therefore, ψ is a topological embedding of X into $\mathcal{O}(Y)$. □

Since X is $\mathcal{O}(X)$-based via the identity on $\mathcal{O}(X)$, we obtain:

Corollary 6.4. *Any sequential T_0-space X embeds topologically into $\mathcal{O}(\mathcal{O}(X))$.*

Furthermore, we obtain the following using Proposition 4.3.

Corollary 6.5. *If Y is a countably based locally compact Hausdorff space, then every Y-based sequential T_0-space is countably based.*

The next basic fact characterizes qcb$_0$-spaces in terms of the introduced notions.

Theorem 6.6. *For any sequential T_0-space X the following conditions are equivalent:*

(1) *X is Y-based for some $Y \subseteq \mathcal{N}$.*
(2) *X is Y-based for some zero-dimensional cb$_0$-space Y.*
(3) *X is Y-based for some qcb$_0$-space Y.*
(4) *X topologically embeds into $\mathcal{O}(Y)$ for some qcb$_0$-space Y.*
(5) *X is a qcb$_0$-space.*

Proof. By Proposition 2.1, (1) is equivalent to (2). Since any cb$_0$-space is a qcb$_0$-space, (2) implies (3). Theorem 6.3 yields (3) $\Longrightarrow$ (4). By Proposition 4.1, $\mathcal{O}(Y)$ is a qcb$_0$-space and consequently any topological subspace of $\mathcal{O}(Y)$ which happens to be sequential. This yields (4) $\Longrightarrow$ (5).
To show (5) $\Longrightarrow$ (1), let X be a qcb$_0$-space. By Proposition 4.1, $\mathcal{O}(X)$ is also qcb$_0$-space. By [20], there is an admissible representation $\phi\colon Z \to \mathcal{O}(X)$ of $\mathcal{O}(X)$, where $Z \subseteq \mathcal{N}$. Since ϕ is a surjection, it is trivially a Z-indexing of a basis for X. □

Corollary 6.7. *Every qcb$_0$-space topologically embeds into a space with a total admissible representation.*

Proof. By Section 4.3, $\mathcal{O}(Y)$ has a total admissible representation whenever Y is a countably based T_0-space. Every qcb$_0$-space X has an admissible representation $\xi\colon Z \to X$, where Z is a subspace of $\mathcal{N}$. By Theorem 6.3, X embeds into $\mathcal{O}(Z)$. ☐

Example 6.1. Given a topological space X and $Y \subseteq \mathcal{N}$, a countable pseudo-base [19] for X can be directly obtained from a Y-indexing $\phi\colon Y \to \mathcal{O}(X)$ of a basis for X in the following way. For each finite sequence $\sigma \in \omega^{<\omega}$, define

$$A_\sigma = \bigcap\{\phi(p) \mid p \in Y \text{ and } \sigma \sqsubseteq p\},$$

where $\sigma \sqsubseteq p$ if and only if σ is an initial prefix of p. Then $(A_\sigma)_{\sigma\in\omega^{<\omega}}$ is a countable pseudo-base for X. To see this, fix any $x \in X$ and open $U \subseteq X$ containing x. Assume $(x_i)_{i\in\omega}$ is a sequence of elements of X converging to x. Since ϕ is a Y-indexing of a basis for X, there is $p \in Y$ such that $x \in \phi(p) \subseteq U$. Since $\phi(p)$ is open, there is $j \in \omega$ such that $(x_i)_{i\geq j} \subseteq \phi(p)$. The set $K = \{x\} \cup \{x_i\}_{i\geq j}$ is a compact subset of X, hence $O_K = \{O \in \mathcal{O}(X) \mid K \subseteq O\}$ is a Scott-open subset of $\mathcal{O}(X)$. Clearly $\phi(p) \in O_K$, so the continuity of ϕ implies there is $\sigma \in \omega^{<\omega}$ such that $\sigma \sqsubseteq p$ and $\phi(q) \in O_K$ for every $q \in Y$ satisfying $\sigma \sqsubseteq q$. Therefore, $x \in K \subseteq A_\sigma \subseteq U$, and it follows that $(A_\sigma)_{\sigma\in\omega^{<\omega}}$ is a countable pseudo-base for X.

6.2. Classifying Y-based spaces

For any qcb$_0$-space Y, let $\mathit{Based}(Y)$ denote the class of Y-based qcb$_0$-spaces. For a class $\mathcal{S}$ of qcb$_0$-spaces, let $\mathit{Based}(\mathcal{S}) = \bigcup_{Y\in\mathcal{S}} \mathit{Based}(Y)$.

Theorem 6.6 induces some natural classifications of qcb$_0$-spaces. For example, one can relate to any family of pointclasses $\mathbf{\Gamma}$ the classes $\mathit{Based}(\mathbf{\Gamma}(\mathcal{N}))$ and $\mathit{Based}(\mathsf{QCB}_0(\mathbf{\Gamma}))$.

Proposition 6.8. *For any family of pointclasses $\mathbf{\Gamma}$, the classes $\mathit{Based}(\mathbf{\Gamma}(\mathcal{N}))$ and $\mathit{Based}(\mathsf{QCB}_0(\mathbf{\Gamma}))$ coincide.*

Proof. One direction is obvious, since $\mathbf{\Gamma}(\mathcal{N}) \subseteq \mathsf{QCB}_0(\mathbf{\Gamma})$. For the other direction, let $X \in \mathit{Based}(\mathsf{QCB}_0(\mathbf{\Gamma}))$, then X is Y-based for some $Y \in \mathsf{QCB}_0(\mathbf{\Gamma})$. Choose an admissible representation $\delta : D \to Y$ of Y such that $EQ(\delta) \in \mathbf{\Gamma}(\mathcal{N})$, so in particular $D \in \mathbf{\Gamma}(\mathcal{N})$. Since Y is a continuous image of D, $Y \in \mathit{Based}(\mathbf{\Gamma}(\mathcal{N}))$ by Lemma 6.2. ☐

Thus, the classical hierarchies of subsets of the Baire space induce the corresponding hierarchies of qcb$_0$-spaces, in particular the "hyperprojective base-hierarchy" $\mathit{Based}(\mathbf{\Sigma}^1_\alpha(\mathcal{N}))$; we simplify the notation to $\mathit{Based}(\mathbf{\Sigma}^1_\alpha)$ and relate this hierarchy to the admissible representations $\delta_\alpha\colon D_\alpha \to \mathbb{N}\langle\alpha\rangle$ of the continuous functionals of countable types (see Subsection 2.5).

Proposition 6.9. *For any $\alpha < \omega_1$, $\mathit{Based}(D_{\alpha+1}) = \mathit{Based}(\mathbf{\Pi}^1_\alpha) = \mathit{Based}(\mathbf{\Sigma}^1_{\alpha+1}) = \mathit{Based}(\mathbb{N}\langle\alpha+1\rangle)$. For any limit ordinal $\lambda < \omega_1$, $\mathit{Based}(D_\lambda) = \mathit{Based}((\mathbf{\Pi}^1_{<\lambda})_\delta) = \mathit{Based}(\mathbf{\Sigma}^1_\lambda) = \mathit{Based}(\mathbb{N}\langle\lambda\rangle)$.*

Proof. Since $D_{\alpha+1} \in \mathbf{\Pi}^1_\alpha(\mathcal{N})$ by Proposition 2.10, $\mathit{Based}(D_{\alpha+1}) \subseteq \mathit{Based}(\mathbf{\Pi}^1_\alpha)$. The inclusion $\mathit{Based}(\mathbf{\Pi}^1_\alpha) \subseteq \mathit{Based}(\mathbf{\Sigma}^1_{\alpha+1})$ is obvious. The inclusion $\mathit{Based}(\mathbf{\Sigma}^1_{\alpha+1}) \subseteq \mathit{Based}(\mathbb{N}\langle\alpha+1\rangle)$ follows from Lemma 6.2, because, by Proposition 2.11, any non-empty $\mathbf{\Sigma}^1_{\alpha+1}$-set is a continuous image of $\mathbb{N}\langle\alpha+1\rangle$. The inclusion $\mathit{Based}(\mathbb{N}\langle\alpha+1\rangle) \subseteq \mathit{Based}(D_{\alpha+1})$ follows again from Lemma 6.2, because $\mathbb{N}\langle\alpha+1\rangle$ is a continuous image of $D_{\alpha+1}$.

The second assertion is proved in the same way. ☐

By Theorem 6.3, any space from $\mathit{Based}(Y)$ topologically embeds into $\mathcal{O}(Y)$. A principal question is: for which qcb$_0$-spaces Y do we have that the space $\mathcal{O}(Y)$ is Y-based? Clearly, this is equivalent to saying that $\mathit{Based}(Y)$ is the class of spaces topologically embeddable into $\mathcal{O}(Y)$. Unfortunately, the assertion does not hold for all Y:

Example 6.2. The space $\mathcal{O}(\mathbb{Q})$ is not $\mathbb{Q}$-based. Suppose the contrary. Since $\mathbb{Q}$ is a continuous image of ω, $\mathcal{O}(\mathbb{Q})$ would be ω-based (i.e., countably based) by Lemma 6.2. But by Proposition 4.3 this would imply that $\mathbb{Q}$ is locally compact, a contradiction.

Nevertheless, the assertion $\mathcal{O}(Y) \in \mathit{Based}(Y)$ might hold for some natural spaces Y, in particular a positive answer to the following problem would clarify the nature of the hierarchy $\{\mathit{Based}(\mathbf{\Pi}^1_\alpha)\}$ considerably:

Problem 6.1. Does the assertion $\mathcal{O}(D_\alpha) \in Based(D_\alpha)$ hold for all $\alpha < \omega_1$?

If the answer is positive, $Based(D_\alpha)$ would coincide with the class of spaces topologically embeddable into $\mathcal{O}(D_\alpha)$. For $\alpha = 0$ the assertion holds because $\mathcal{O}(\omega)$ is homeomorphic to $P\omega$. For $\alpha = 1$ the assertion is also true, we will prove this in the next subsection. For $\alpha \geq 2$ we still do not know the answer. This is an obstacle to answering the principal question on the non-collapse of the introduced hierarchy $\{Based(D_\alpha)\}_{\alpha<\omega_1}$. By the non-collapse property we mean that the inclusion $Based(D_\alpha) \subseteq Based(D_\beta)$ is proper for each $\alpha < \beta < \omega_1$.

Although the non-collapse property is currently open, we can prove some slightly weaker version of this property. The next result (along with the assertion $\mathcal{O}(D_1) \in Based(D_1)$) implies, in particular, that $Based(D_0) \subsetneq Based(D_1)$.

Proposition 6.10. *For any $\alpha < \omega_1$, $\mathcal{O}(D_{\alpha+1}) \notin Based(D_\alpha)$. For any limit ordinal $\lambda < \omega_1$, $\mathcal{O}(D_\lambda) \notin Based(\bigoplus_{\alpha<\lambda} D_\alpha)$.*

Proof. For the first assertion, by Theorem 6.3 it suffices to show that $\mathcal{O}(D_{\alpha+1})$ does not embed topologically into $\mathcal{O}(D_\alpha)$. Suppose the contrary, then $EQ(\pi_{D_{\alpha+1}}) \leq_W EQ(\pi_{D_\alpha})$ by Lemma 3.3. Since $\mathcal{N} \setminus D_{\alpha+1} \leq_W EQ(\pi_{D_{\alpha+1}})$ by Lemma 4.5, $\mathcal{N} \setminus D_{\alpha+1} \leq_W EQ(\pi_{D_\alpha})$. Since $EQ(\pi_{D_\alpha}) \in \mathbf{\Pi}^1_\alpha(\mathcal{N})$ by Theorem 4.8, we have $\mathcal{N} \setminus D_{\alpha+1} \in \mathbf{\Pi}^1_\alpha(\mathcal{N})$. This contradicts Proposition 2.10.

Now we turn to the second assertion. Suppose for a contradiction that $\mathcal{O}(D_\lambda) \in Based(\bigoplus_{\alpha<\lambda} D_\alpha)$. Let $(\alpha_n)_n$ be an injective sequence consisting of all non-zero ordinals below λ. Define the subspace Y of $\mathcal{N}$ by $Y := \bigcup_n E_n$, where $E_n := \{q \in \mathcal{N} \mid q(0) = n, t(q) \in D_{\alpha_n}\}$ and $t(q) := (q(1), q(2), \cdots)$. Since Y is homeomorphic to $\bigoplus_{\alpha<\lambda} D_\alpha$, $\mathcal{O}(D_\lambda) \in Based(Y)$, hence $\mathcal{O}(D_\lambda)$ topologically embeds into $\mathcal{O}(Y)$. By Lemma 3.3, $EQ(\pi_{D_\lambda}) \leq_W EQ(\pi_Y)$. It suffices to show that $EQ(\pi_Y) \in (\mathbf{\Sigma}^1_{<\lambda})_\delta(\mathcal{N})$, because this implies $EQ(\pi_Y) \in \mathbf{\Sigma}^1_\lambda(\mathcal{N})$ by Proposition 2.5, hence also $EQ(\pi_{D_\lambda}) \in \mathbf{\Sigma}^1_\lambda(\mathcal{N})$ which contradicts Corollary 4.9.

Using the notation of Section 4.3, we have

$$\pi_Y(p) = Y \cap \pi_\mathcal{N}(p) = \bigcup_n (E_n \cap \pi_\mathcal{N}(p)) = \bigcup_n \pi_{E_n}(p).$$

Since $E_0, E_1, \cdots$ are pairwise disjoint, $\pi_Y(p) = \pi_Y(q)$ iff $\forall n(\pi_{E_n}(p) = \pi_{E_n}(q))$. In other words, $EQ(\pi_Y) = \bigcap_n EQ(\pi_{E_n})$. Since $E_n \in \mathbf{\Pi}^1_{\alpha_n}(\mathcal{N})$ for each $n < \omega$, $EQ(\pi_{E_n}) \in \mathbf{\Pi}^1_{\alpha_n+1}(\mathcal{N}) \subseteq \mathbf{\Sigma}^1_{<\lambda}(\mathcal{N})$ by Proposition 2.5, hence $EQ(\pi_Y) \in (\mathbf{\Sigma}^1_{<\lambda})_\delta(\mathcal{N})$. This completes the proof. □

To deduce from the last proposition the announced weak version of the non-collapse property, we also need the following relation between the hyperprojective hierarchy of qcb$_0$-spaces and the hierarchy $\{Based(D_\alpha)\}$ which is interesting in its own right:

Proposition 6.11. *For any $\alpha < \omega_1$, $\mathsf{QCB}_0(\mathbf{\Pi}^1_\alpha) \subseteq Based(D_{\alpha+2}) = Based(\mathbb{N}\langle\alpha + 2\rangle) = Based(\mathbf{\Pi}^1_{\alpha+1})$.*

Proof. Let $X \in \mathsf{QCB}_0(\mathbf{\Pi}^1_\alpha)$. By Proposition 2.8, $\mathcal{O}(X) \cong \mathbb{S}^X \in \mathsf{QCB}_0(\mathbf{\Pi}^1_{\alpha+1})$. Let $\beta : B \to \mathcal{O}(X)$ be an admissible representation of $\mathcal{O}(X)$ with $B, EQ(\beta) \in \mathbf{\Pi}^1_{\alpha+1}(\mathcal{N})$. Since $B \in \mathbf{\Sigma}^1_{\alpha+2}(\mathcal{N})$ and $B \neq \emptyset$, by Proposition 2.11 there is a continuous surjection f from $D_{\alpha+2}$ onto B. Then βf is a continuous surjection f from $D_{\alpha+2}$ onto $\mathcal{O}(X)$. Therefore, $X \in Based(D_{\alpha+2}) = Based(\mathbb{N}\langle\alpha + 2\rangle) = Based(\mathbf{\Pi}^1_{\alpha+1})$ by Proposition 6.9. □

Conversely, we have $Based(\mathbf{\Pi}^1_\alpha) \nsubseteq \mathsf{QCB}_0(\mathbf{\Pi}^1_\beta)$ for all ordinals $\alpha, \beta < \omega_1$, because $D_{\beta+2} \in Based(\omega) \setminus \mathsf{QCB}_0(\mathbf{\Pi}^1_\beta)$ by Proposition 2.10.

The second item of the next corollary is the announced weak version of the non-collapse property.

Corollary 6.12. (1) *For any $\alpha < \omega_1$, $\mathcal{O}(D_\alpha), \mathcal{O}(\mathbb{N}\langle\alpha\rangle) \in Based(D_{\alpha+2}) = Based(\mathbb{N}\langle\alpha + 2\rangle)$.*

(2) *For any $\alpha < \omega_1$, the inclusion $Based(D_\alpha) \subset Based(D_{\alpha+3})$ is proper.*

Proof. (1) By Theorem 4.8, $\mathcal{O}(D_\alpha), \mathcal{O}(\mathbb{N}\langle\alpha\rangle) \in \mathsf{QCB}_0(\mathbf{\Pi}^1_\alpha)$, hence the assertion follows from Proposition 6.11.

(2) By Proposition 6.10 we have $\mathcal{O}(D_{\alpha+1}) \notin Based(D_\alpha)$. By (1), $\mathcal{O}(D_{\alpha+1}) \in Based(D_{\alpha+3})$, hence $Based(D_{\alpha+3}) \nsubseteq Based(D_\alpha)$.

□

Problem 6.2. For which $\alpha < \omega_1$ can the inclusion from Proposition 6.11 be improved to $\mathsf{QCB}_0(\mathbf{\Pi}^1_\alpha) \subseteq$ *Based*$(D_{\alpha+1})$ or even to $\mathsf{QCB}_0(\mathbf{\Pi}^1_\alpha) \subseteq$ *Based*(D_α)?

Remark 6.1. One can try to weaken the notion of Y-based space in order to obtain the desired property that $\mathcal{O}(Y)$ is Y-based (in the weakened sense) for any qcb$_0$-space Y. E.g., one could say that X is weakly Y-subbased if there is a continuous function $\phi : Y \to \mathcal{O}(X)$ such that $\phi(Y)$ is a subbase of X. For this modification, we would obtain essentially the same results as above. Nevertheless, a deeper modification (considered in Section 7 below) will be sufficient to settle the analogues of the open questions above for the sequential embeddings in place of topological embeddings.

6.3. $\mathcal{N}$-Based spaces

Here we obtain some additional information on the class *Based*$(\mathcal{N})$ = *Based*(D_1) of $\mathcal{N}$-based qcb$_0$-spaces. This class seems to be important since it includes natural non-countably based spaces that are relatively simple.

First we state an interesting property of quasi-Polish spaces.

Proposition 6.13. *If X is quasi-Polish then $\mathcal{O}(X)$ is $\mathcal{N}$-based.*

Proof. By Corollary 5.4, the space $\mathcal{K}(X)$ of saturated compact subsets of X with the upper Vietoris topology is quasi-Polish, hence there is a total admissible representation $\nu : \mathcal{N} \to \mathcal{K}(X)$ of $\mathcal{K}(X)$. Define $\tilde{\nu} : \mathcal{N} \to \mathcal{O}\mathcal{O}(X)$ by $\tilde{\nu}(p) = \{U \in \mathcal{O}(X) \mid \nu(p) \subseteq U\}$. By Proposition 5.1, $\tilde{\nu}$ is continuous. Since X is consonant by Proposition 4.2, $\tilde{\nu}(\mathcal{N})$ is a base of $\mathcal{O}(X)$. Hence, $\mathcal{O}(X)$ is $\mathcal{N}$-based. □

For metrizable spaces $X \in \mathsf{CB}_0(\mathbf{\Pi}^1_1)$ we have the following complete characterization of when $\mathcal{O}(X)$ is $\mathcal{N}$-based.

Proposition 6.14. *Let $X \in \mathsf{CB}_0(\mathbf{\Pi}^1_1)$ be metrizable. Then $\mathcal{O}(X)$ is $\mathcal{N}$-based if and only if X is Polish.*

Proof. It only remains to show that if $X \in \mathsf{CB}_0(\mathbf{\Pi}^1_1)$ is not Polish, then $\mathcal{O}(X)$ is not $\mathcal{N}$-based. Note that for any such space X, Corollary 5.5 implies that $\mathcal{K}_{\text{up}}(X) \notin \mathsf{CB}_0(\mathbf{\Sigma}^1_1)$.

Assume for a contradiction that $\phi\colon \mathcal{N} \to \mathcal{O}(\mathcal{O}(X))$ is a $\mathcal{N}$-indexing of a basis for $\mathcal{O}(X)$. Then the function $\phi^\omega : \mathcal{N}^\omega \to \mathcal{O}(\mathcal{O}(X))$ mapping $p \in \mathcal{N}^\omega$ to $\bigcup_{i\in\omega} \phi(f(i))$ is continuous and it is a surjection because $\mathcal{O}(X)$ is hereditarily Lindelöf. The subset A of $\mathcal{N}$ that gets mapped by ϕ^ω to $\emptyset$ is closed, hence there is a continuous $f\colon \mathcal{N} \to \mathcal{N}$ such that *range*$(f) = \mathcal{N} \setminus A$. Clearly $g := \phi^\omega \circ f$ is a continuous surjection from $\mathcal{N}$ onto $\mathcal{O}(\mathcal{O}(X)) \setminus \{\emptyset\}$.

From Theorem 3.1 in [23] we know that $r\colon H \mapsto \bigcap(H)$ is a continuous retraction from $\mathcal{O}(\mathcal{O}(X)) \setminus \{\emptyset\}$ to $\mathcal{K}_{\text{up}}(X)$. Then the composition $r \circ g$ is a continuous surjection from $\mathcal{N}$ to $\mathcal{K}_{\text{up}}(X)$, contradicting $\mathcal{K}_{\text{up}}(X) \notin \mathsf{CB}_0(\mathbf{\Sigma}^1_1)$. □

Corollary 6.15. *A qcb$_0$-space is $\mathcal{N}$-based if, and only if, it embeds topologically in $\mathcal{O}(\mathcal{N})$. Furthermore, Based$(\omega) \subsetneqq$ Based$(\mathcal{N})$ and Based$(D_0) \subsetneqq$ Based(D_1).*

Proof. Theorem 6.3 implies the only-if-part. The if-part follows from the fact that $\mathcal{O}(\mathcal{N})$ is $\mathcal{N}$-based by Proposition 6.13. Since the space $\mathcal{O}(\mathcal{N})$ is not countably based by Proposition 4.6, it is in *Based*$(\mathcal{N}) \setminus$ *Based*(ω) = *Based*$(D_1) \setminus$ *Based*(D_0). □

The last corollary shows that the role of $\mathcal{O}(\mathcal{N})$ in the class *Based*$(\mathcal{N})$ is in a sense similar to the role of $\mathcal{O}(\omega) = P\omega$ in the class of countably based spaces. It seems instructive to continue this analogy and investigate, for instance, the analogue of Proposition 2.7(2) for the class *Based*$(\mathcal{N})$. Probably, this is more complicated than for cb$_0$ spaces, because in [25] the injectivity property of $P\omega$ was of principal importance while $\mathcal{O}(\mathcal{N})$ is not injective. Indeed, it follows from Theorem 2.12 in [24] (see also Theorem 3.8 in Chapter II of [9]) that the space $\mathcal{O}(X)$ is injective iff the complete lattice $(\mathcal{O}(X); \subseteq)$ is continuous iff the space X is core-compact. In particular, the space $\mathcal{O}(\mathcal{N})$ is not injective.

Example 6.3. Another example of a $\mathcal{N}$-based space is the Gruenhage-Streicher space X, which was shown in [10] to be a qcb$_0$-space whose sobrification is not sequential. The underlying set of X is $\omega \times \omega$ and a basis for the topology of X is given by the collection of all sets of the form $U(x,\xi) := \{x\} \cup \{(i,j) \in \omega^2 \mid i > x_1 \text{ and } j \geq \xi(i)\}$, where $x = (x_1, x_2) \in X$ and $\xi\colon \omega \to \omega$. Therefore, the function $\phi\colon \omega^2 \times \mathcal{N} \to \mathcal{O}(X)$, defined as $\phi(x,\xi) = U(x,\xi)$, will be a $\omega^2 \times \mathcal{N}$-indexing of a basis for X provided we can show that ϕ is sequentially continuous. Towards this end, we fix a sequence $(\xi_n)_n$ converging to ξ_∞ in $\mathcal{N}$. For each $i \in \omega$ there are only finitely many $n \in \omega$ such that $\xi_\infty(i) \neq \xi_n(i)$, hence for each $k \in \omega$ there is a function $f_k\colon \omega \to \omega$ such that $f_k(i) = \max\{\xi_n(i) \mid n \geq k\}$. Then for any $x \in X$ we have that the infinite intersection $\bigcap_{n\geq k} \phi(x,\xi_n)$ is equal to $U(x,f_k)$, hence is an open subset of X. Furthermore, it is easily verified that $\phi(x,\xi_\infty) \subseteq \bigcup_{k\in\omega} U(x,f_k)$. It follows from Proposition 4.1(3) that $(\phi(x,\xi_n))_n$ converges to $\phi(x,\xi_\infty)$ in $\mathcal{O}(X)$, which completes the proof that ϕ is sequentially continuous. We conclude that X is $\mathcal{N}$-based because $\omega^2 \times \mathcal{N}$ is a continuous image of $\mathcal{N}$.

7. Sequentially Y-based spaces

In this section we consider some modifications of the notion of Y-based spaces from the previous section which are more suitable to the nature of sequential spaces (in particular, qcb$_0$-spaces). This will be sufficient to settle the analogues of the open questions in Subsection 6.2 for the sequential embeddings in place of topological embeddings.

7.1. Basic facts

One could define several modifications of the notion of Y-based space. For instance, for qcb$_0$-spaces X, P we could say that a function $\phi\colon P \to \mathcal{O}(X)$ is a *P-indexed sequential basis* for X, if ϕ is continuous and range(ϕ) is a subbasis for a topology τ on X such that the sequentialisation of τ is the Scott topology in $\mathcal{O}(X)$. Under this definition, some interesting facts may be established, e.g., one can show that for any $\alpha < \omega_1$ the space $\mathbb{N}\langle\alpha + 1\rangle$ has an $\mathbb{N}\langle\alpha\rangle$-indexed sequential basis (see Corollary 7.10).

In this paper, we also consider the following deeper modification:

Definition 7.1. Let X, P be sequential spaces.

(1) We call a collection $\mathcal{B}$ of open subsets of X *a sequential basis* for X, if $\mathcal{B}$ is a subbase of a topology τ on the set X such that the sequentialisation of τ is equal to $\mathcal{O}(X)$.
(2) A function $\phi\colon P \to \mathcal{O}(X)$ is called a P-indexed sequential basis for X, if ϕ is continuous and its range $rng(\phi)$ is a sequential basis for X.
(3) For a function $\phi\colon P \to \mathcal{O}(X)$, we define $\mathcal{B}_\phi$ to consist of all intersections of the form $\bigcap_{n\leq\infty} \phi(p_n)$, where $(p_n)_n$ converges to p_∞ in P.
(4) A function $\phi\colon P \to \mathcal{O}(X)$ is called a *P-indexed generating system for* X, if ϕ is continuous and $\mathcal{B}_\phi$ is a sequential basis for X.
(5) X is called *sequentially P-based*, if there is a P-indexed generating system for X.

Note that by Proposition 4.1(3) the elements of $\mathcal{B}_\phi$ are open in X, if ϕ is continuous, because $(\phi(p_n))_n$ converges to $\phi(p_\infty)$ in $\mathcal{O}(X)$.

Now we study for which spaces P the existence of a P-indexed generating system implies the existence of a P-indexed sequential basis.

Lemma 7.2. *Let P be a sequential space such that there is a continuous surjection from P onto $P^{\mathbb{N}_\infty}$. Then any sequential space X is sequentially P-based if, and only if, X has a P-indexed sequential basis.*

Proof. From right to left, the assertion is trivial, so let X be sequentially P-based. Let $\phi\colon P \to \mathcal{O}(X)$ be an P-indexed generating system for X and let $S\colon P \to P^{\mathbb{N}_\infty}$ be a continuous surjection. We define $\phi'\colon P \to \mathcal{O}(X)$ by

$$\phi'(p) := \bigcap_{n\leq\infty} \phi(S(p)(n)).$$

Then ϕ' is continuous by the cartesian closedness of Seq and by Proposition 4.1(3). Since the range of S contains exactly all convergent sequences $(p_n)_{n\leq\infty}$ of P, we have $rng(\phi') = B_\phi$. Thus ϕ' is a continuous P-indexing of a sequential basis for X. □

Lemma 2.13 shows that the spaces $\mathbb{N}\langle\alpha\rangle$ and $\mathbb{N}\langle<\lambda\rangle$ fulfill the requirement of Lemma 7.2. We obtain:

Corollary 7.3. (1) *For any $\alpha < \omega_1$, a sequential space X is sequentially $\mathbb{N}\langle\alpha\rangle$-based if, and only if, there is an $\mathbb{N}\langle\alpha\rangle$-indexed sequential basis for X.*
(2) *For any limit ordinal $\lambda < \omega_1$, a sequential space X is sequentially $\mathbb{N}\langle<\lambda\rangle$-based if, and only if, there is an $\mathbb{N}\langle<\lambda\rangle$-indexed sequential basis for X.*

Although Definition 7.1 is very technical, it is justified by several nice properties the main of which is the following theorem:

Theorem 7.4. *Let X and P be sequential T_0-spaces. Then X is sequentially P-based if, and only if, X embeds sequentially into $\mathcal{O}(P)$.*

Proof. Let X be sequentially P-based via the P-indexed generating system $\phi\colon P \to \mathcal{O}(X)$. We define the function $e\colon X \to \mathcal{O}(P)$ by $e(x) := \{p \in P \mid x \in \phi(p)\}$. Identifying opens of X and P with their characteristic functions in $\mathbb{S}^X$ or $\mathbb{S}^P$ (see Subsection 4.1), we can view e as a function to $\mathbb{S}^P$ given by $e(x)(p) = \phi(p)(x)$. Hence e is continuous by the cartesian closedness of Seq.

To show the injectivity of e, let x and z be distinct elements of X. Since X is T_0, we can assume without loss of generality that there is open $U \subseteq X$ containing x, but not z. Then the constant sequence $(z)_n$ does not converge to x in X. Therefore there is some set $B \in \mathcal{B}_\phi$ containing x, but not z. Furthermore there is a convergent sequence $(p_n)_{n\leq\infty}$ of P with $B = \bigcap_{n\leq\infty} \phi(p_n)$. This implies that there is some $n_0 \in \mathbb{N}_\infty$ with $p_{n_0} \in e(x) \setminus e(z)$. Hence e is injective.

Now let $(x_n)_n$ be a sequence that does not converge to x_∞ in X. Then there exists an open set $B \in \mathcal{B}_\phi$ with $x_\infty \in B$ and $x_n \notin B$ for infinitely many $n \in \omega$. Hence there is a convergent sequence $(p_m)_{m\leq\infty}$ with $B = \bigcap_{m\leq\infty} \phi(p_m)$. We choose a strictly increasing function $\varphi\colon \omega \to \omega$ with $x_{\varphi(n)} \notin B$ and a sequence $(m_n)_n$ in $\mathbb{N}_\infty$ such that $x_{\varphi(n)} \notin \phi(p_{m_n})$ for all $n \in \omega$. If there is some $k \in \mathbb{N}_\infty$ such that k occurs infinitely often in $(m_n)_n$, then we set $m_\infty := k$ and choose a strictly increasing function $\psi\colon \omega \to \omega$ with $m_{\psi(n)} = k = m_\infty$. Otherwise $(m_n)_n$ converges to ∞ in $\mathbb{N}_\infty$; in this case we choose $m_\infty := \infty$ and let ψ be the identity on $\mathbb{N}$. In both cases we have $p_{m_{\psi(n)}} \notin e(x_{\varphi\psi(n)})$ for all $n \in \omega$, but $p_{m_\infty} \in e(x_\infty)$. Since $(p_{m_{\psi(n)}})_n$ converges to p_{m_∞} in P, this implies that $(e(x_{\varphi\psi(n)}))_n$ does not converge to $e(x_\infty)$ in $\mathcal{O}(P)$. By Definition 3.1, e is a sequential embedding of X into $\mathcal{O}(P)$.

For the other direction, assume that e is a sequential embedding of X into $\mathcal{O}(P)$. We define $\phi\colon P \to \mathcal{O}(X)$ by $\phi(p) := \{x \in X \mid p \in e(x)\}$. An analogous argument as above yields that ϕ is continuous.
Let $(x_n)_n$ be a sequence that does not converge to x_∞ in X. By assumption, $(cf(e(x_n)))_n$ does not converge to $cf(e(x_\infty))$ in $\mathbb{S}^P$. So there is a convergent sequence $(p_n)_{n\leq\infty}$ in P such that $(cf(e(x_n))(p_n))_n$ does not converge to $cf(e(x_\infty))(x_\infty)$ in $\mathbb{S}$. Hence $p_\infty \in e(x_\infty)$ and there is some strictly increasing $\varphi\colon \omega \to \omega$ with $p_{\varphi(n)} \notin e(x_{\varphi(n)})$ for all n. Moreover, as $e(x_\infty)$ is open, there is some n_0 with $p_{\varphi(n)} \in e(x_\infty)$ for all $n \geq n_0$. As $(p_{\varphi(n)})_{n\geq n_0}$ converges to p_∞ in P, the set $B := \phi(p_\infty) \cap \bigcap_{n\geq n_0} \phi(p_{\varphi(n)})$ is an element of $\mathcal{B}_\phi$ by Proposition 4.1(3). By the construction we have $x_\infty \in B$ and $x_{\varphi(n)} \notin B$ for all $n \geq n_0$. Hence $(x_n)_n$ does not converge to x_∞ w.r.t. to the topology induced on X by $\mathcal{B}_\phi$ as a subbase, thus it does not converge in X. Therefore X is sequentially P-based. □

Theorem 7.4 solves in the positive the "sequential analogue" of the question "is $\mathcal{O}(P) \in \mathit{Based}(P)$ for each P?" discussed in the previous section. It has several nice corollaries including the following:

Corollary 7.5. *Let X, P, S be sequential T_0-spaces.*

(1) *The space $\mathcal{O}(P)$ is sequentially P-based.*
(2) *Let X be sequentially P-based and P be a quotient of Z. Then X is sequentially Z-based.*

Proof. (1) Immediate by Theorem 7.4.
(2) By Lemma 3.2, the space $\mathcal{O}(P) \cong \mathbb{S}^P$ sequentially embeds into $\mathcal{O}(Z) \cong \mathbb{S}^Z$. By Theorem 7.4, X sequentially embeds into $\mathcal{O}(P)$, hence also in $\mathcal{O}(Z)$. By Theorem 7.4, X is sequentially Z-based.

□

For any sequential space P we of course have $\mathit{Based}(P) \subseteq \mathit{SBased}(P)$. An interesting question is "for which P this inclusion is proper?" It is proper at least for some P. From Corollary 7.5 and Example 6.2 it follows that the space $\mathcal{O}(\mathbb{Q})$ is sequentially $\mathbb{Q}$-based, but not $\mathbb{Q}$-based. This example can be improved to the following:

Proposition 7.6. *The space $\mathcal{O}(\mathbb{Q})$ is sequentially $\mathcal{N}$-based, but not $\mathcal{N}$-based.*

Proof. $\mathcal{O}(\mathbb{Q})$ is sequentially $\mathbb{Q}$-based by the first part of Corollary 7.5. Since $\mathbb{Q}$ is a quotient of $\mathcal{N}$ [16], $\mathcal{O}(\mathbb{Q})$ is sequentially $\mathcal{N}$-based by the second part of Corollary 7.5. On the other hand, $\mathcal{O}(\mathbb{Q})$ is not $\mathcal{N}$-based by Proposition 6.14. □

We now show how to construct generating systems for countable products and function spaces (formed in Seq).

Proposition 7.7. *Let X_i, P_i be sequential T_0-spaces such that X_i is sequentially P_i-based for $i \in \omega$. Then the sequential product $\prod_{i\in\omega} X_i$ is sequentially $(\bigoplus_{i\in\omega} P_i)$-based.*

Proof. Let $\phi_i \colon P_i \to \mathcal{O}(X_i)$ be a P_i-indexed generating system for X_i. We define $\phi_\infty \colon \bigoplus_{k\in\omega} P_k \to \mathcal{O}(\prod\limits_{i\in\omega} X_i)$ by

$$\phi_\infty(k,p) := \big\{x \in \prod_{i\in\omega} X_i \,\big|\, x(k) \in \phi_k(p)\big\}$$

for all $k \in \omega$, $p \in P_k$. Since Seq is cartesian closed and has all countable limits and colimits, ϕ_∞ is continuous. Now let $(x_n)_n$ be a sequence that does not converge to x_∞ in $\prod_{i\in\omega} X_i$. Then there is some $k \in \omega$ and some open set $B \in \mathcal{B}_{\phi_k}$ such that $(x_n(k))_n$ does not converge to $x_\infty(k)$ in X_k, $x_\infty(k) \in B$ and $x_n(k) \notin B$ for infinitely many n. Choose some convergent sequence $(p_m)_{m\le\infty}$ in P_k with $B = \bigcap_{m\le\infty} \phi_k(p_m)$. Then $(k, p_m)_{m\le\infty}$ is a convergent sequence of the coproduct $\bigoplus_{i\in\omega} P_i$. Thus the set $B' := \bigcap_{m\le\infty} \phi_\infty(\bar{k}, p_m)$ is an element of $\mathcal{B}_{\phi_\infty}$. The construction yields $x_\infty \in B'$ and $x_n \notin B'$ for infinitely many n. So $\mathcal{B}_{\phi_\infty}$ is a sequential basis for $\prod_{i\in\mathbb{N}} X_i$. We conclude that the product $\prod_{i\in\omega} X_i$ formed in Seq is sequentially $(\bigoplus_{k\in\omega} P_k)$-based. □

Proposition 7.8. *Let X, Y, P be sequential T_0-spaces such that Y is sequentially P-based. Then Y^X is sequentially $(P \times X)$-based.*

Proof. By Theorem 7.4, there exists a sequential embedding $e \colon Y \to \mathbb{S}^P$ of Y into $\mathbb{S}^P$. We define a function $E \colon Y^X \to \mathbb{S}^{P\times X}$ by $E(f)(p,x) := e(f(x))(p)$. By cartesian closedness of Seq the function E is continuous. As e is injective, E is injective as well. Now let $(f_n)_{n\le\infty}$ be a sequence of functions in Y^X such that $(E(f_n))_n$ converges to $E(f_\infty)$ in $\mathbb{S}^{P\times X}$. To show that $(f_n)_n$ converges continuously to f_∞, let $(x_n)_n$ converge to x_∞ in X. If $(p_n)_n$ converges to p_∞ in P, then $\big(e(f_n(x_n))(p_n)\big)_n = \big(E(f_n)(p_n, x_n)\big)_n$ converges to $E(f_\infty)(p_\infty, x_\infty) = e(f_\infty(x_\infty))(p_\infty)$, hence $(e(f_n(x_n))_n$ converges to $e(f_\infty(x_\infty))$ in $\mathbb{S}^P$. Since e embeds Y sequentially into $\mathbb{S}^X$, $(f_n(x_n))_n$ converges to $f_\infty(x_\infty)$ in Y. Thus $(f_n)_n$ converges to f_∞ in Y^X. We conclude that Y^X sequentially embeds into $\mathcal{O}(P \times X)$. By Theorem 7.4, Y^X is sequentially $(P \times X)$-based. □

We can slightly improve this proposition to the following corollary.

Corollary 7.9. *Let X, Y, P, S be sequential T_0-spaces such that Y is sequentially P-based and X is a quotient of S. Then Y^X is sequentially $(P \times S)$-based.*

Proof. By Proposition 7.8, Y^X is sequentially $(P \times X)$-based. Since in Seq the product of two quotient maps is a quotient map (see [27]), $P \times X$ is a quotient of $P \times S$. By Corollary 7.5, Y^X is also sequentially $(P \times S)$-based. □

From Proposition 7.8 we obtain the following nice property of the spaces of functionals:

Corollary 7.10. *For any $\alpha < \omega_1$, the space $\mathbb{N}\langle\alpha + 1\rangle$ is sequentially $\mathbb{N}\langle\alpha\rangle$-based. For any limit ordinal $\lambda < \omega_1$, the space $\mathbb{N}\langle\lambda\rangle$ is sequentially $\mathbb{N}\langle{<}\lambda\rangle$-based.*

Proof. By Proposition 7.8, $\mathbb{N}\langle\alpha + 1\rangle$ is sequentially $(\omega \times \mathbb{N}\langle\alpha\rangle)$-based. Since $\omega \times \mathbb{N}\langle\alpha\rangle \cong \mathbb{N}\langle\alpha\rangle$ by Proposition 2.12, $\mathbb{N}\langle\alpha + 1\rangle$ is sequentially $\mathbb{N}\langle\alpha\rangle$-based. The same argument proves also the second assertion because $\omega^{\mathbb{N}\langle<\lambda\rangle} \cong \mathbb{N}\langle\lambda\rangle$ and $\omega \times \mathbb{N}\langle{<}\lambda\rangle \cong \mathbb{N}\langle{<}\lambda\rangle$. □

7.2. Classifying sequentially Y-based spaces

For any qcb$_0$-space Y, let $SBased(Y)$ denote the class of sequentially Y-based qcb$_0$-spaces. For a class $\mathcal{S}$ of qcb$_0$-spaces Y, let $SBased(\mathcal{S}) = \bigcup_{Y \in \mathcal{S}} SBased(Y)$. Obviously, $Based(Y) \subseteq SBased(Y)$ for each qcb$_0$-space Y.

Theorem 7.4 induces some natural classifications of qcb$_0$-spaces. For example, one can relate to any family of pointclasses $\mathbf{\Gamma}$ the classes $SBased(\mathbf{\Gamma}(\mathcal{N}))$ and $SBased(\mathsf{QCB}_0(\mathbf{\Gamma}))$.

The next assertion is proved just as its analogue Proposition 6.8, using Corollary 7.5.

Proposition 7.11. *For any family of pointclasses $\mathbf{\Gamma}$, the classes $SBased(\mathbf{\Gamma}(\mathcal{N}))$ and $SBased(\mathsf{QCB}_0(\mathbf{\Gamma}))$ coincide.*

Thus, the classical hierarchies of subsets of the Baire space induce the corresponding hierarchies of qcb$_0$-spaces, in particular the "hyperprojective sequential-based-hierarchy" $SBased(\mathbf{\Sigma}^1_\alpha(\mathcal{N}))$; we simplify the notation to $SBased(\mathbf{\Sigma}^1_\alpha)$ and relate this hierarchy to the admissible representations $\delta_\alpha : D_\alpha \to \mathbb{N}\langle\alpha\rangle$ of the continuous functionals from Subsection 2.5.

The next assertion is proved just as its analogue Proposition 6.9, using Corollary 7.5 (in fact, one needs the additional observation that any non-empty $\mathbf{\Sigma}^1_\alpha$-set is not only a continuous image, but even a quotient of $\mathbb{N}\langle\alpha\rangle$; this follows from the proof of Theorem 7.2 in [25]).

Proposition 7.12. *For any $\alpha < \omega_1$, $SBased(D_{\alpha+1}) = SBased(\mathbf{\Pi}^1_\alpha) = SBased(\mathbf{\Sigma}^1_{\alpha+1}) = SBased(\mathbb{N}\langle\alpha+1\rangle)$. For any limit ordinal $\lambda < \omega_1$, $SBased(D_\lambda) = SBased((\mathbf{\Pi}^1_{<\lambda})_\delta) = SBased(\mathbf{\Sigma}^1_\lambda) = SBased(\mathbb{N}\langle\lambda\rangle)$.*

Next we solve the principal question on the non-collapse property of the hierarchy $\{SBased(D_\alpha)\}_{\alpha<\omega_1}$. Remember that the corresponding result for the hierarchy $\{Based(D_\alpha)\}$ remained open.

Proposition 7.13. *The hierarchy $\{SBased(D_\alpha)\}$ does not collapse, i.e. $SBased(D_\alpha) \subsetneqq SBased(D_\beta)$ for all $\alpha < \beta < \omega_1$. More precisely, $SBased(D_\alpha) \subsetneqq SBased(D_{\alpha+1})$ for each $\alpha < \omega_1$ and $SBased(\bigoplus_{\alpha<\lambda} D_\alpha) \subsetneqq SBased(D_\lambda)$ for each limit ordinal $\lambda < \omega_1$.*

Proof. By Theorem 7.4 and Corollary 7.5, it suffices to show that $\mathcal{O}(D_{\alpha+1})$ does not sequentially embed into $\mathcal{O}(D_\alpha)$ and $\mathcal{O}(D_\lambda)$ does not sequentially embed into $SBased(\bigoplus_{\alpha<\lambda} D_\alpha)$. This is checked just in the same way as in the proof of Proposition 6.10. ☐

The next fact shows that the class $SBased(\mathcal{N})$ is rather rich.

Proposition 7.14. *Let X be a qcb$_0$-space having a total admissible representation $\xi\colon \mathcal{N} \to X$. Then $\mathcal{O}(X)$ embeds sequentially into $\mathcal{O}(\mathcal{N})$.*

Proof. By Lemma 3.2, $\xi^{-1} : \mathcal{O}(X) \to \mathcal{O}(\mathcal{N})$ is a sequential embedding. ☐

Problem 7.1. As we know from Example 6.2 and Theorem 7.4, $Based(\mathbb{Q}) \subsetneqq SBased(\mathbb{Q})$ and $Based(\mathcal{N}) \subsetneqq SBased(\mathcal{N})$. We would like to know for which sequential spaces X $Based(X) \subsetneqq SBased(X)$. In particular, we guess that $Based(D_\alpha) \subsetneqq SBased(D_\alpha)$ for all non-zero ordinals $\alpha < \omega_1$ and $SBased(\bigoplus_{\alpha<\lambda} D_\alpha) \subsetneqq SBased(\bigoplus_{\alpha<\lambda} D_\alpha)$ for all limit ordinals $\lambda < \omega_1$. Good possible witnesses seem to be the spaces $\mathbb{N}\langle\alpha+1\rangle$ and $\mathbb{N}\langle\lambda\rangle$ respectively (see Corollary 7.10).

7.3. Functionals of countable types in the hierarchy $\{SBased(D_\alpha)\}$

As we know from Proposition 7.13, the spaces $\mathcal{O}(D_\alpha)$ are natural witnesses for the non-collapse property of the hierarchy $\{SBased(D_\alpha)\}$. Here we show that the spaces $\mathbb{N}\langle\alpha\rangle$ provide other natural witnesses for this property.

Given a qcb$_0$-space X, we let 0_X denote the constantly zero function $\lambda x \in X.0$ in ω^X.

Lemma 7.15. *Let X be a qcb$_0$-space, $Y \subseteq \mathcal{N}$, $f\colon X \to Y$ a continuous function, and $A = Y \setminus f(X)$ the complement of the range of f. Then there is a continuous function $g\colon Y \to \left(\omega^X\right)^\omega$ such that $g(y)$ is a sequence in ω^X converging to 0_X if and only if $y \in A$.*

Proof. For $y \in Y$ and $n \in \omega$ we let $\uparrow y[n]$ denote the clopen subset of Y of elements that agree with y in the first n places. Define $g\colon Y \to \left(\omega^X\right)^\omega$ as $g(y)(n)(x) = 0$ if $f(x) \notin\uparrow y[n]$ and $g(y)(n)(x) = 1$, otherwise. The continuity of g follows from the continuity of f and the fact that $\uparrow y[n]$ is clopen.

If $y \notin A$, then there is $x \in X$ such that $f(x) = y$. It follows that $g(y)(n)(x) = 1$ for all $n \in \omega$, hence $g(y)$ does not converge to 0_X.

Conversely, let $y \in A$ be given and assume $(x_n)_{n\in\omega}$ converges to x_∞ in X. Clearly, $f(x_\infty) \neq y$ hence there is $n_0 \in \omega$ such that $f(x_\infty) \notin\uparrow y[n_0]$. The continuity of f and the convergence of $(x_n)_n$ to x_∞ imply there is $n_1 \in \omega$ such that $f(x_n) \notin\uparrow y[n_0]$ for all $n \geq n_1$, hence $g(y)(n)(x_n) = 0$ for all $n \geq n_0 + n_1$. It follows that $g(y)$ converges to 0_X. □

The next result shows that Corollary 7.10 is in a sense optimal.

Theorem 7.16. (1) *For any* $\alpha < \omega_1$, $\mathbb{N}\langle \alpha + 2\rangle \notin \mathit{SBased}(\mathbb{N}\langle\alpha\rangle)$.
(2) *For any limit ordinal* $\lambda < \omega_1$, $\mathbb{N}\langle\lambda + 1\rangle \notin \mathit{SBased}(\mathbb{N}\langle<\lambda\rangle)$.
(3) *For any limit ordinal* $\lambda < \omega_1$, $\mathbb{N}\langle\lambda\rangle \notin \bigcup_{\alpha<\lambda} \mathit{SBased}(\mathbb{N}\langle\alpha\rangle)$.

Proof. (1) Fix a set $A \in \mathbf{\Pi}^1_{\alpha+1}(\mathcal{N}) \setminus \mathbf{\Sigma}^1_{\alpha+1}(\mathcal{N})$. By Proposition 2.11, there is a continuous function $f\colon \mathbb{N}\langle\alpha + 1\rangle \to \mathcal{N}$ such that A equals the complement of the range of f. From Lemma 7.15 (with $\mathbb{N}\langle\alpha + 1\rangle$ in place of X) we obtain a continuous $g\colon \mathcal{N} \to (\mathbb{N}\langle\alpha + 2\rangle)^\omega$ such that $g(y)$ is a sequence in $\mathbb{N}\langle\alpha + 2\rangle$ converging to $0_{\mathbb{N}\langle\alpha+1\rangle}$ if and only if $y \in A$.

Now assume for a contradiction that $\mathbb{N}\langle\alpha + 2\rangle \in \mathit{SBased}(\mathbb{N}\langle\alpha\rangle)$. Since $\mathbb{N}\langle\alpha\rangle$ is a quotient of D_α, Lemma 7.3 implies there is continuous $\phi\colon D_\alpha \to \mathcal{O}(\mathbb{N}\langle\alpha + 2\rangle)$ such that the range of ϕ is a sequential basis for $\mathbb{N}\langle\alpha + 2\rangle$. It follows that

$$y \in A \iff \forall x \in \mathcal{N}. \left[\left(x \in D_\alpha \wedge 0_{\mathbb{N}\langle\alpha+1\rangle} \in \phi(x)\right) \implies \forall^\infty_n . g(y)(n) \in \phi(x)\right].$$

First note that the set of open subsets of $\mathbb{N}\langle\alpha + 2\rangle$ containing $0_{\mathbb{N}\langle\alpha+1\rangle}$ is open in $\mathcal{O}(\mathbb{N}\langle\alpha + 2\rangle)$, hence the continuity of ϕ implies that $U_0 := \{x \in D_\alpha \mid 0_{\mathbb{N}\langle\alpha+1\rangle} \in \phi(x)\}$ is an open subset of D_α, hence $U_0 \in \mathbf{\Sigma}^1_\alpha(\mathcal{N})$ by Proposition 2.10. Furthermore, for each $n \in \omega$, the function $h_n\colon D_\alpha \times \mathcal{N} \to \mathbb{S}$ defined as $h_n(x, y) = \top \iff g(y)(n) \in \phi(x)$ is continuous by cartesian closedness. It follows that

$$B_0 := \{(x, y) \in D_\alpha \times \mathcal{N} \mid \forall^\infty_n . g(y)(n) \in \phi(x)\} = \bigcup_{k\in\omega} \bigcap_{n\geq k} h_n^{-1}(\top)$$

is in $\mathbf{\Sigma}^0_2(D_\alpha \times \mathcal{N})$. Let B be a $\mathbf{\Sigma}^0_2$ subset of $\mathcal{N} \times \mathcal{N}$ such that $B_0 = B \cap (D_\alpha \times \mathcal{N})$. Then

$$y \in A \iff \forall x \in \mathcal{N}. [x \notin U_0 \vee (x, y) \in B],$$

hence $A \in \mathbf{\Pi}^1_\alpha(\mathcal{N})$, contradicting our choice of A.

(2) Fix a set $A \in \mathbf{\Pi}^1_\lambda(\mathcal{N}) \setminus \mathbf{\Sigma}^1_\lambda(\mathcal{N})$. As in (1) one can show that there exists a continuous $g\colon \mathcal{N} \to (\mathbb{N}\langle\lambda + 1\rangle)^\omega$ such that $g(y)$ is a sequence in $\mathbb{N}\langle\lambda + 1\rangle$ converging to $0_{\mathbb{N}\langle\lambda\rangle}$ if and only if $y \in A$.

Now assume for a contradiction that $\mathbb{N}\langle\lambda + 1\rangle \in \mathit{SBased}(\mathbb{N}\langle<\lambda\rangle)$. As $\mathbb{N}\langle<\lambda\rangle \cong \bigoplus_{\alpha<\lambda} \mathbb{N}\langle\alpha_k\rangle$ and $\mathbb{N}\langle\alpha\rangle$ is a quotient of D_α for each $\alpha < \lambda$, Lemma 7.3 implies there is a continuous $\phi\colon \bigoplus_k D_\alpha \to \mathcal{O}(\mathbb{N}\langle\lambda + 1\rangle)$ such that the range of ϕ is a sequential basis for $\mathbb{N}\langle\lambda + 1\rangle$. It follows that

$$y \in A \iff \forall\alpha < \lambda. \forall x \in D_\alpha. \left[\left(0_{\mathbb{N}\langle\lambda\rangle} \in \phi(x)\right) \implies \forall^\infty_n . g(y)(n) \in \phi(x)\right]$$

Then $A = \bigcap_{\alpha<\lambda} C_\alpha$, where

$$C_\alpha = \{y \mid \forall x \in D_\alpha \left(0_{\mathbb{N}\langle\lambda\rangle} \in \phi(x)\right) \implies \forall^\infty_n . g(y)(n) \in \phi(x)\}.$$

Since, by the same argument as in (1), $C_\alpha \in \mathbf{\Pi}^1_{<\lambda}(\mathcal{N})$ for each $\alpha < \lambda$, $A \in (\mathbf{\Pi}^1_{<\lambda})_\delta(\mathcal{N}) \subseteq \mathbf{\Sigma}^1_\lambda$. A contradiction.

(3) Suppose the contrary: $\mathbb{N}\langle\lambda\rangle$ is sequentially $\mathbb{N}\langle\alpha\rangle$-based for some $\alpha < \lambda$. By Theorem 7.4, $\mathbb{N}\langle\lambda\rangle$ embeds sequentially in $\mathcal{O}(\mathbb{N}\langle\alpha\rangle)$. Since $\alpha + 2 < \lambda$, $\mathbb{N}\langle\alpha + 2\rangle$ is a retract of $\mathbb{N}\langle\lambda\rangle$ by Proposition 2.12. Then $\mathbb{N}\langle\alpha + 2\rangle$

embeds topologically in $\mathbb{N}\langle\lambda\rangle$, hence $\mathbb{N}\langle\alpha+2\rangle$ embeds sequentially in $\mathbb{N}\langle\lambda\rangle$, hence $\mathbb{N}\langle\alpha+2\rangle$ embeds sequentially in $\mathcal{O}(\mathbb{N}\langle\alpha\rangle)$, hence $\mathbb{N}\langle\alpha+2\rangle$ is sequentially $\mathbb{N}\langle\alpha\rangle$-based. Contradiction with (1).

□

The next immediate corollary of Theorem 7.16 and Corollary 7.10 shows that the spaces $\mathbb{N}\langle\alpha\rangle$ witness the non-collapse property of the hierarchy $\{SBased(D_\alpha)\}$.

Corollary 7.17. (1) *For any $\alpha < \omega_1$, $\mathbb{N}\langle\alpha+2\rangle \in SBased(\mathbb{N}\langle\alpha+1\rangle) \setminus SBased(\mathbb{N}\langle\alpha\rangle)$.*
(2) *For any limit ordinal $\lambda < \omega_1$, $\mathbb{N}\langle\lambda+1\rangle \in SBased(\mathbb{N}\langle\lambda\rangle) \setminus SBased(\mathbb{N}\langle{<}\lambda\rangle)$.*
(3) *For any limit ordinal $\lambda < \omega_1$, $\mathbb{N}\langle\lambda\rangle \in SBased(\bigoplus_{\alpha<\lambda}\mathbb{N}\langle\alpha\rangle) \setminus \bigcup_{\alpha<\lambda} SBased(\mathbb{N}\langle\alpha\rangle)$.*

We also can deduce the following interesting corollary about the continuous functionals.

Corollary 7.18. *For all $\alpha < \beta < \omega_1$, $\mathbb{N}\langle\beta\rangle$ does not sequentially embed into $\mathbb{N}\langle\alpha\rangle$.*

Proof. First we check by induction on α that $\mathbb{N}\langle\alpha+1\rangle$ does not be sequentially embed into $\mathbb{N}\langle\alpha\rangle$. For $\alpha = 0$ this is obvious. Let $\alpha = \gamma + 1$ be successor and suppose the contrary, so $\mathbb{N}\langle\gamma+2\rangle$ sequentially embeds into $\mathbb{N}\langle\gamma+1\rangle$. Since $\mathbb{N}\langle\gamma+1\rangle$ is sequentially $\mathbb{N}\langle\gamma\rangle$-based by Corollary 7.10, it sequentially embeds in $\mathcal{O}(\mathbb{N}\langle\gamma\rangle)$ by Theorem 7.4. Then $\mathbb{N}\langle\gamma+2\rangle$ sequentially embeds in $\mathcal{O}(\mathbb{N}\langle\gamma\rangle)$, hence $\mathbb{N}\langle\gamma+2\rangle$ is sequentially $\mathbb{N}\langle\gamma\rangle$-based by Theorem 7.4. Contradiction with Theorem 7.16. Let $\alpha = \lambda$ be limit and suppose the contrary, so $\mathbb{N}\langle\lambda+1\rangle$ sequentially embeds into $\mathbb{N}\langle\lambda\rangle$. Since $\mathbb{N}\langle\lambda\rangle$ is sequentially $\mathbb{N}\langle{<}\lambda\rangle$-based by Corollary 7.10, it sequentially embeds in $\mathcal{O}(\mathbb{N}\langle{<}\lambda\rangle)$ by Theorem 7.4. Then $\mathbb{N}\langle\lambda+1\rangle$ sequentially embeds in $\mathcal{O}(\mathbb{N}\langle{<}\lambda\rangle)$, hence $\mathbb{N}\langle\lambda+1\rangle$ is sequentially $\mathbb{N}\langle{<}\lambda\rangle$-based by Theorem 7.4. Contradiction with Theorem 7.16.

It remains to consider the case $\alpha + 1 < \beta$. Suppose for a contradiction that $\mathbb{N}\langle\beta\rangle$ sequentially embeds into $\mathbb{N}\langle\alpha\rangle$. Since $\mathbb{N}\langle\alpha+1\rangle$ embeds sequentially in $\mathbb{N}\langle\beta\rangle$ by the proof of item (3) in Theorem 7.16, $\mathbb{N}\langle\alpha+1\rangle$ embeds sequentially in $\mathbb{N}\langle\alpha\rangle$. Contradiction with the previous paragraph. □

8. On universal spaces

In this section we discuss which classes of qcb$_0$-spaces have and which do not have a universal space. This is of interest because universal spaces are noticeable in several branches of set-theoretic topology.

We start with introducing the main notions of this section.

Definition 8.1. (1) Let $\mathcal{S}$ be a class of topological spaces. A space X is *universal in $\mathcal{S}$*, if $X \in \mathcal{S}$ and any space from $\mathcal{S}$ embeds topologically in X.
(2) Let $\mathcal{S}$ be a class of sequential spaces. A space X is *sequentially universal in $\mathcal{S}$*, if $X \in \mathcal{S}$ and any space from $\mathcal{S}$ embeds sequentially in X.

The first notion above is well-known in topology. For instance, $P\omega$ is universal in the class of cb$_0$-spaces (Proposition 2.2), $\mathcal{N}$ is universal in the class of zero-dimensional cb$_0$-spaces (Proposition 2.1), $[0;1]^\omega$ is universal in the class of separable metrizable spaces [12], while the class of all topological spaces has no universal space. The second notion is a "sequential version" of the first one which is natural when dealing with sequential (in particular, qcb$_0$-) spaces.

Since $P\omega$ is universal in the class of cb$_0$-spaces and Y-based spaces are designed as a natural generalization of countably based spaces, it is natural to ask for which $Y \subseteq \mathcal{N}$ the class of Y-based spaces has a universal space. At least, from Theorem 6.3 we immediately obtain the following:

Corollary 8.2. *Let $Y \subseteq \mathcal{N}$ be such that the space $\mathcal{O}(Y)$ is Y-based. Then $\mathcal{O}(Y)$ is universal in the class of Y-based topological spaces. In particular, the space $\mathcal{O}(\mathcal{N})$ is universal in the class of $\mathcal{N}$-based spaces.*

For the sequential version, we similarly derive from Theorem 7.4 the following result:

Corollary 8.3. For any qcb$_0$-space Y, the qcb$_0$-space $\mathcal{O}(Y)$ is sequentially universal in $SBased(Y)$.

It is still open whether or not $Based(D_\alpha)$ contains a universal space when $\alpha > 1$. However, we see that each level of the hierarchy $\{SBased(D_\alpha)\}$ contains a sequentially universal space $\mathcal{O}(D_\alpha)$ with a total admissible representation. The same applies to the hierarchies of cb$_0$-spaces in [25] (obviously, $P\omega$ is a universal space in $\mathsf{CB}_0(\mathbf{\Gamma})$ for each family of pointclasses $\mathbf{\Gamma}$ that contains $\mathbf{\Pi}^0_2$).

For the hierarchies of qcb$_0$-spaces in [25, 26] the situation is more complicated. Currently we do not know which of the classes $\mathsf{QCB}_0(\mathbf{\Gamma})$, where $\mathbf{\Gamma}$ is a level of the Borel or hyperprojective hierarchy, have a universal (or a sequentially universal) space. Nevertheless, we can show that the class of all qcb$_0$-spaces, as well as some natural pointclasses related to the hyperprojective hierarchy qcb$_0$-spaces, do not have universal spaces. Recall from [25, 26] that $\mathsf{QCB}_0(\mathbf{P}) := \bigcup_{n<\omega} \mathsf{QCB}_0(\mathbf{\Sigma}^1_n)$ and $\mathsf{QCB}_0(\mathbf{HP}) := \bigcup_{\alpha<\omega_1} \mathsf{QCB}_0(\mathbf{\Sigma}^1_\alpha)$ denote the classes of projective and of hyperprojective qcb$_0$-spaces, respectively.

Theorem 8.4. (1) *There is no universal (nor a sequentially universal) qcb$_0$-space.*
(2) *For any limit ordinal $\lambda < \omega_1$, there is no universal (nor a sequentially universal) space in* $\mathsf{QCB}_0(\mathbf{\Sigma}^1_{<\lambda})$.
(3) *There is no universal (nor a sequentially universal) space in* $\mathsf{QCB}_0(\mathbf{P})$ *(nor in* $\mathsf{QCB}_0(\mathbf{HP})$*).*

Proof. (1) Suppose for a contradiction that X is a topologically (or sequentially) universal qcb$_0$-space. By Theorem 6.6, X topologically embeds into $\mathcal{O}(Y)$ for some $Y \subseteq \mathcal{N}$, hence $\mathcal{O}(Y)$ is also a universal qcb$_0$-space. Since there are hypercontinuum many subsets of $\mathcal{N}$ and at most continuum many of them are Wadge reducible to $EQ(\pi_Y)$, there is $Z \subseteq \mathcal{N}$ such that $\mathcal{N} \setminus Z \not\leq_W EQ(\pi_Y)$. By Lemma 4.5(2), $\mathcal{N} \setminus Z \leq_W EQ(\pi_Z)$, hence $EQ(\pi_Z) \not\leq_W EQ(\pi_Y)$. By Lemma 3.3, there is no continuous injection of $\mathcal{O}(Z)$ into $\mathcal{O}(Y)$, hence the qcb$_0$-space $\mathcal{O}(Z)$ does not embed (topologically or sequentially) into $\mathcal{O}(Y)$. This contradicts the universality of $\mathcal{O}(Y)$.

(2) Suppose for a contradiction that X is a topologically (or sequentially) universal space in $\mathsf{QCB}_0(\mathbf{\Sigma}^1_{<\lambda})$, so there is an admissible representation $\delta : Y \to X$ of X such that $EQ(\delta) \in \mathbf{\Sigma}^1_{<\lambda}(\mathcal{N})$. As above, the space $\mathcal{O}(Y)$ is also universal in $\mathsf{QCB}_0(\mathbf{\Sigma}^1_{<\lambda})$. Choose $Z \in \mathbf{\Sigma}^1_{<\lambda}(\mathcal{N})$ such that $\mathcal{N} \setminus Z \not\leq_W EQ(\pi_Y)$ (e.g. let Z be Wadge complete in $\mathbf{\Sigma}^1_{\alpha+1}$ where $\alpha < \lambda$ satisfies $EQ(\pi_Y) \in \mathbf{\Sigma}^1_\alpha(\mathcal{N})$). Repeating the argument from the previous paragraph, we obtain a contradiction.

Item (3) follows from item (2) because $\mathsf{QCB}_0(\mathbf{P}) = \mathsf{QCB}_0(\mathbf{\Sigma}^1_{<\omega})$ and $\mathsf{QCB}_0(\mathbf{HP})$ is the union of the classes $\mathsf{QCB}_0(\mathbf{\Sigma}^1_{<\lambda})$ where λ ranges over the limit countable ordinals.

□

9. Conclusion

We introduced and studied some hierarchies of qcb$_0$-spaces which classify spaces by the complexity of their bases. These hierarchies complement hierarchies from [25] and suggest some approaches to the problem of better understanding the non-countably based qcb$_0$-spaces. The new hierarchies are divided into two classes. The first one is based on the purely topological notion of a Y-based space, this simple notion leads to complications in the study of the related hierarchies and principal open questions. The second is based on a more complicated and less intuitive notion of a sequentially Y-based space, but leads to an elegant theory and to solutions of some principal questions.

We expect that notions and results of this paper will be of use in further understanding non-countably based qcb$_0$-spaces. For example, the introduced hierarchies provide useful methods for determining whether or not a given qcb$_0$-space can be topologically or sequentially embedded into another one. In particular, we used these methods to show that if $\alpha < \beta$ then $\mathbb{N}\langle\beta\rangle$ can not be sequentially embedded into $\mathbb{N}\langle\alpha\rangle$. Furthermore, we were able to solve (in the negative) the principal question on the existence of a universal qcb$_0$-space. This is a new indication that qcb$_0$-spaces are much harder than cb$_0$-spaces.

Corollary 6.7 shows that every qcb$_0$-space can be topologically embedded into one with a total admissible representation, which might be interpreted as a kind of "completion" of the space. This observation can be refined further if sequential embeddings are permitted, in which case we have shown that every space in $SBased(D_\alpha)$ can be sequentially embedded into some space in $SBased(D_\alpha)$ with a total admissible representation (see the comments after Corollary 8.3).

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