Continuity in constructive analysis

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Aim:

Constructive analysis, with constructions $\sim$ good algorithms.


*The modulus of continuity $\omega$ is an indispensable part of the definition of a continuous function on a compact interval, although sometimes it is not mentioned explicitly. In the same way, the moduli of continuity of the restrictions of $f$ to each compact subinterval are indispensable parts of the definition of a continuous function $f$ on a general interval.*
A continuous function $f : (X, \rho, Q) \rightarrow (Y, \sigma, R)$ for separable metric spaces is given by

$$h: Q \rightarrow \mathbb{N} \rightarrow R$$

approximating map

plus $\alpha, \omega, \gamma, \delta$ depending on $w, r$ (center and radius of a ball):

$\alpha$: $Q \rightarrow \mathbb{Z}^+ \rightarrow \mathbb{Z}^+ \rightarrow \mathbb{N}$ such that $(h(u, n))_n$ (for $\rho(u, w) \leq \frac{1}{2r}$) is a Cauchy sequence with modulus $\alpha_{w, r}(p)$;

$a$ modulus $\omega$: $Q \rightarrow \mathbb{Z}^+ \rightarrow \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$ of (uniform) continuity, such that for $n \geq \alpha_{w, r}(p)$ and $\rho(u, w), \rho(v, w) \leq \frac{1}{2r}$

$$\rho(u, v) \leq \frac{2}{2\omega_{w, r}(p)} \rightarrow \sigma(h(u, n), h(v, n)) \leq \frac{1}{2p};$$

maps $\gamma$: $Q \rightarrow \mathbb{Z}^+ \rightarrow R$, $\delta$: $Q \rightarrow \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$ such that $\gamma(w, r)$ and $\delta(w, r)$ are center and radius of a ball containing all $h(u, n)$ (for $\rho(u, w) \leq \frac{1}{2r}$):

$$\rho(u, w) \leq \frac{1}{2r} \rightarrow \sigma(h(u, n), \gamma(w, r)) \leq \frac{1}{2\delta(w, r)}.$$  

$\alpha, \omega, \gamma, \delta$ are required to have monotonicity properties.

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Example: Inverse map \((0, \infty) \rightarrow \mathbb{R}\)

Let \(0 < c < d\), and \(q\) be minimal such that \(\frac{1}{2q} \leq c\). Then \(\text{inv}\) is given by

- the approximating map \(h(a, n) := \frac{1}{a}\)
- the Cauchy modulus \(\alpha(c, d, p) := 0\)
- the modulus \(\omega(c, d, p) := p + 2q + 1\) of uniform continuity, for

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|a - b| \leq \frac{1}{2p + 2q} \rightarrow \left| \frac{1}{a} - \frac{1}{b} \right| = \left| \frac{b - a}{ab} \right| \leq \frac{1}{2p},
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because \(ab \geq \frac{1}{2^{2q}}\)

- the center \(\gamma(c, d) := \frac{c}{c^2 - d^2}\) and radius \(\delta(c, d) := \frac{d}{c^2 - d^2}\) of a ball containing all \(\frac{1}{a}\) for \(|a - c| \leq d\).
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Application \( f(x) \) must (and can) be defined separately, since the approximating map operates on approximations only.

\( f(x) \) is independent from \( w, r \).

Application is compatible with equality on real numbers:

\[
x = y \rightarrow f(x) = f(y).
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\( f \) has \( \omega \) as a modulus of uniform continuity:

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|x - y| \leq \frac{1}{2\omega(p)} \rightarrow |f(x) - f(y)| \leq \frac{1}{2^p}.
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Composition can be defined.
Theorem. Every totally bounded set $A \subseteq \mathbb{R}$ has an infimum $y$.

Proof. Given $\varepsilon = \frac{1}{2^p}$, let $a_0 < a_1 < \cdots < a_{n-1}$ be an $\varepsilon$-net:
\[ \forall x \in A \exists i < n (|x - a_i| < \varepsilon) . \]
Let $b_p = \min \{ a_i \mid i < n \}$. $y := \lim p b_p$.

Corollary. $\inf_{x \in [a, b]} f(x)$ exists, for $f : [a, b] \rightarrow \mathbb{R}$ continuous.

Proof. Given $\varepsilon$, pick $a = a_0 < a_1 < \cdots < a_{n-1} = b$ s.t. $a_{i+1} - a_i < \omega(\varepsilon)$. Then $f(a_0), f(a_1), \ldots, f(a_{n-1})$ is an $\varepsilon$-net for $f$’s range.

Many $f(a_i)$ need to be computed.

Aim: Get $x$ with $f(x) = \inf_{y \in [a, b]} f(y)$ and a better algorithm, assuming convexity.
Algorithms in constructive proofs?

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$\forall x \in A \exists i < n(|x - a_i| < \varepsilon)$. Let $b_p = \min\{a_i | i < n\}$. $y := \lim_{p \to \infty} b_p$.

Corollary. $\inf_{x \in [a, b]} f(x)$ exists, for $f : [a, b] \to \mathbb{R}$ continuous.

Proof.
Given $\varepsilon$, pick $a = a_0 < a_1 < \cdots < a_{n-1} = b$ s.t. $a_{i+1} - a_i < \omega(\varepsilon)$. Then $f(a_0), f(a_1), \ldots, f(a_{n-1})$ is an $\varepsilon$-net for $f$’s range.

Many $f(a_i)$ need to be computed.

Aim: Get $x$ with $f(x) = \inf_{y \in [a, b]} f(y)$ and a better algorithm, assuming convexity.
Theorem. Every totally bounded set $A \subseteq \mathbb{R}$ has an infimum $y$.

Proof.

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Many $f(a_i)$ need to be computed.

Aim: Get $x$ with $f(x) = \inf_{y \in [a, b]} f(y)$ and a better algorithm, assuming convexity.
Intermediate value theorem

Let $a < b$ be rationals. If $f : [a, b] \rightarrow \mathbb{R}$ is continuous with $f(a) \leq 0 \leq f(b)$, and with a uniform modulus of increase

$$\frac{1}{2^p} < d - c \rightarrow \frac{1}{2^p+q} < f(d) - f(c),$$

then we can find $x \in [a, b]$ such that $f(x) = 0$.

Proof (trisection method).

1. **Approximate Splitting Principle.** Let $x, y, z$ be given with $x < y$. Then $z \leq y$ or $x \leq z$.

2. **IVTaux.** Assume $a \leq c < d \leq b$, say $\frac{1}{2^p} < d - c$, and $f(c) \leq 0 \leq f(d)$. Construct $c_1, d_1$ with $d_1 - c_1 = \frac{2}{3}(d - c)$, such that $a \leq c \leq c_1 < d_1 \leq d \leq b$ and $f(c_1) \leq 0 \leq f(d_1)$.

3. **IVTcds.** Iterate the step $c, d \mapsto c_1, d_1$ in IVTaux.

Let $x = (c_n)_n$ and $y = (d_n)_n$ with the obvious modulus. As $f$ is continuous, $f(x) = 0 = f(y)$ for the real number $x = y$. 

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7 / 12
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\[7/12\]
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[k0]
left((cDC rat@@rat)(1@2)
  ([n1]
    (cId rat@@rat=>rat@@rat)
    ([cd3]
      [let cd4
        ((2#3)*left cd3+(1#3)*right cd3@
        (1#3)*left cd3+(2#3)*right cd3)
        [if (0<=(left cd4*left cd4-2+
            (right cd4*right cd4-2))/2)
          (left cd3@right cd4)
          (left cd4@right cd3)])])
  (IntToNat(2*k0)))

where cDC is a form of the recursion operator.
Kolmogorov 1932: “Zur Deutung der intuitionistischen Logik”

- View a formula $A$ as a computational problem, of type $\tau(A)$, the type of a potential solution or “realizer” of $A$.
- Example: $\forall_n \exists m > n \text{Prime}(m)$ has type $\mathbb{N} \rightarrow \mathbb{N}$.

Express this view as invariance under relizability axioms

$$\text{Inv}_A : A \leftrightarrow \exists z (z \text{ r } A).$$

Consequences are choice and independence of premise (Troelstra):

$$\forall x \exists y A(y) \rightarrow \exists f \forall x A(fx)$$
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Derivatives

Let \( f, g : I \to \mathbb{R} \) be continuous. \( g \) is called derivative of \( f \) with modulus \( \delta_f : \mathbb{Z}^+ \to \mathbb{N} \) of differentiability if for \( x, y \in I \) with \( x < y \),

\[
y \leq x + \frac{1}{2\delta_f(p)} \rightarrow |f(y) - f(x) - g(x)(y - x)| \leq \frac{1}{2^p}(y - x).
\]

A bound on the derivative of \( f \) serves as a Lipschitz constant of \( f \):

Lemma (BoundSlope)

Let \( f : I \to \mathbb{R} \) be continuous with derivative \( f' \). Assume that \( f' \) is bounded by \( M \) on \( I \). Then for \( x, y \in I \) with \( x < y \),

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Infimum of a convex function

Let \( f, f': [a, b] \to \mathbb{R} \) (\( a < b \)) be continuous and \( f' \) derivative of \( f \). Assume that \( f \) is strictly convex with witness \( q \), in the sense that \( f'(a) < 0 < f'(b) \) and

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- To obtain \( x \), apply the intermediate value theorem to \( f' \).
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Conclusion

Aim: constructive analysis, with constructions \(\sim\) good algorithms. Then extract these algorithms from proofs (realizability).

- Use order locatedness: given \(c < d\), for all \(u\)

\[
\forall v \in V (c \leq \rho(u, v)) \lor \exists v \in V (\rho(u, v) \leq d).
\]

- Avoid total boundedness (existence of \(\varepsilon\)-nets).

Generally

- View constructive analysis as an extension of classical analysis.
- Formalize proofs in TCF (based on the Scott-Ershov model of partial continuous functionals), extract algorithms (in Minlog).
- Data are important (real number, continuous function \ldots).
- Low type levels: continuous \(f : \mathbb{R} \to \mathbb{R}\) determined by its values on the rationals \(\mathbb{Q}\).
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- Formalize proofs in TCF (based on the Scott-Ershov model of partial continuous functionals), extract algorithms (in Minlog).
- Data are important (real number, continuous function \ldots).
- Low type levels: continuous \(f : \mathbb{R} \rightarrow \mathbb{R}\) determined by its values on the rationals \(\mathbb{Q}\).