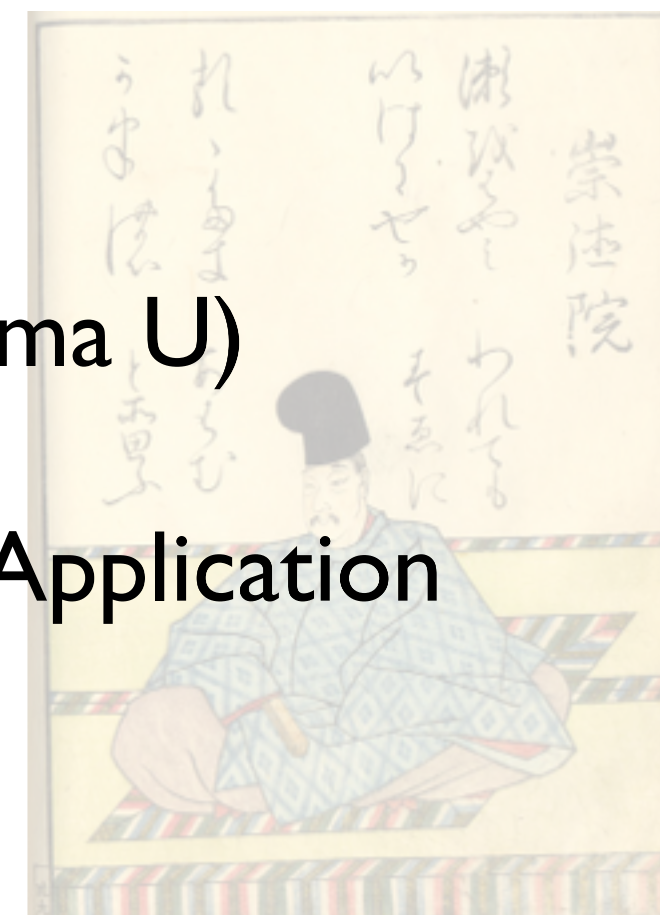
Compositional Z

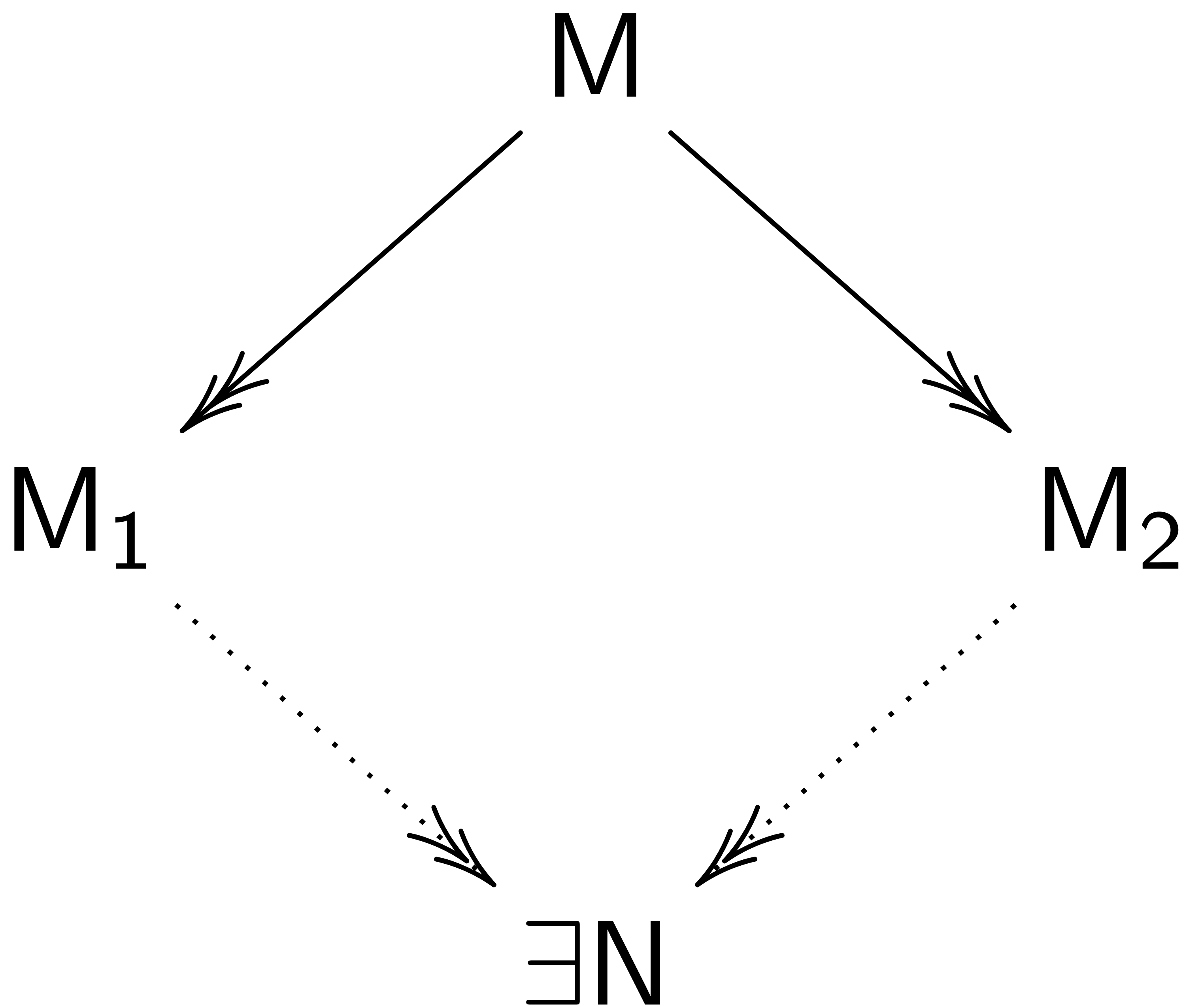
Confluence for Permutative Conversion

Koji Nakazawa (Nagoya U)
joint work with Ken-etsu Fujita (Gunma U)

Workshop on Mathematical Logic and its Application
2016.9 @ Kyoto

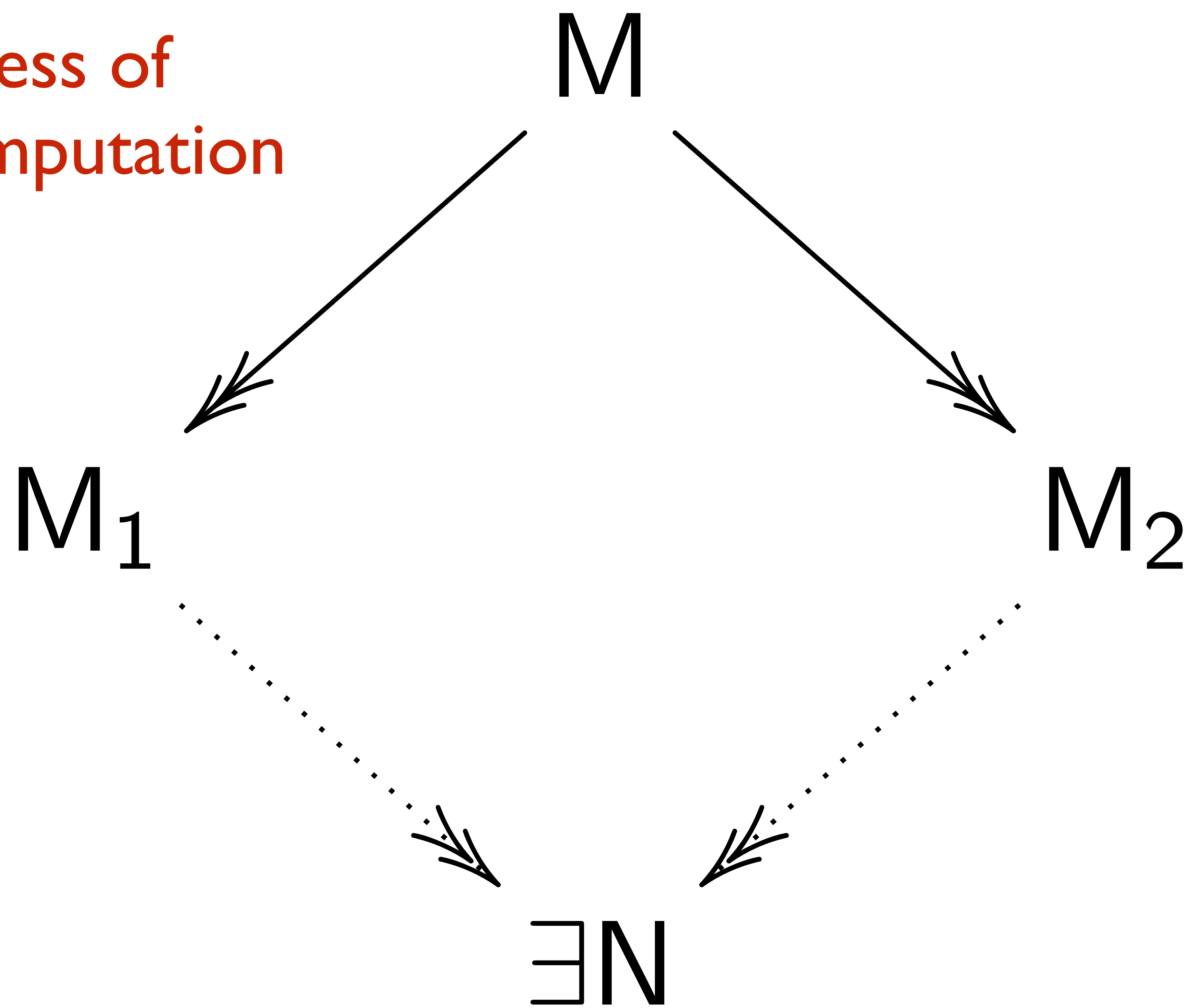


Confluence



Confluence

Uniqueness of
result of computation



Confluence

Uniqueness of
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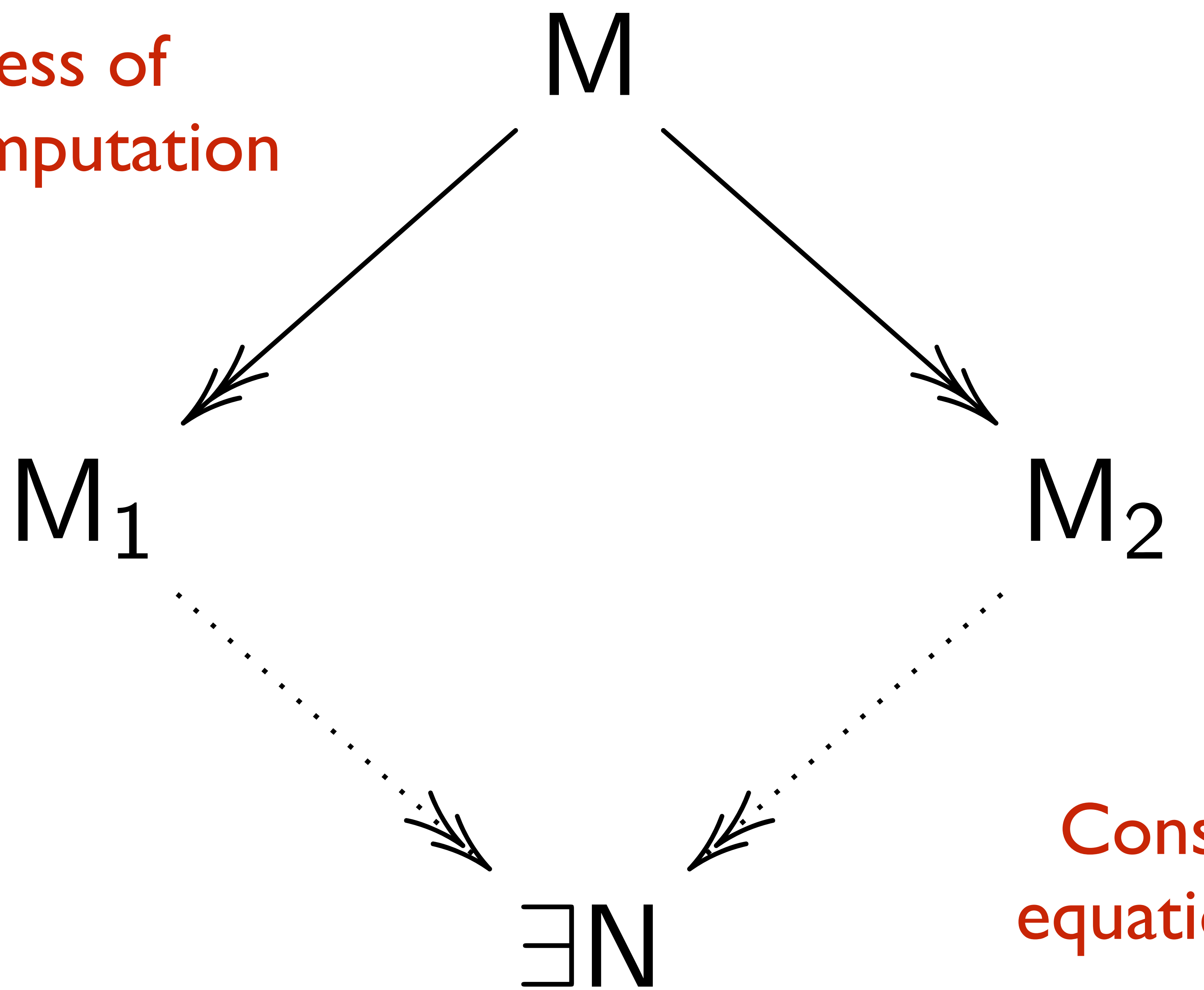


Consistency of
equational theory

This talk

- Brief history of confluence of λ-calculus
  - parallel reduction and Z theorem
- Compositional Z: a new confluence proof
  - simpler proof of λ + permutation rules
- Z property for Church-Rosser theorem
  - quantitative analysis

History of Confluence of λ

λ_β

- Terms

$$M, N ::= x \mid \lambda x.M \mid MN$$

- Reduction rules

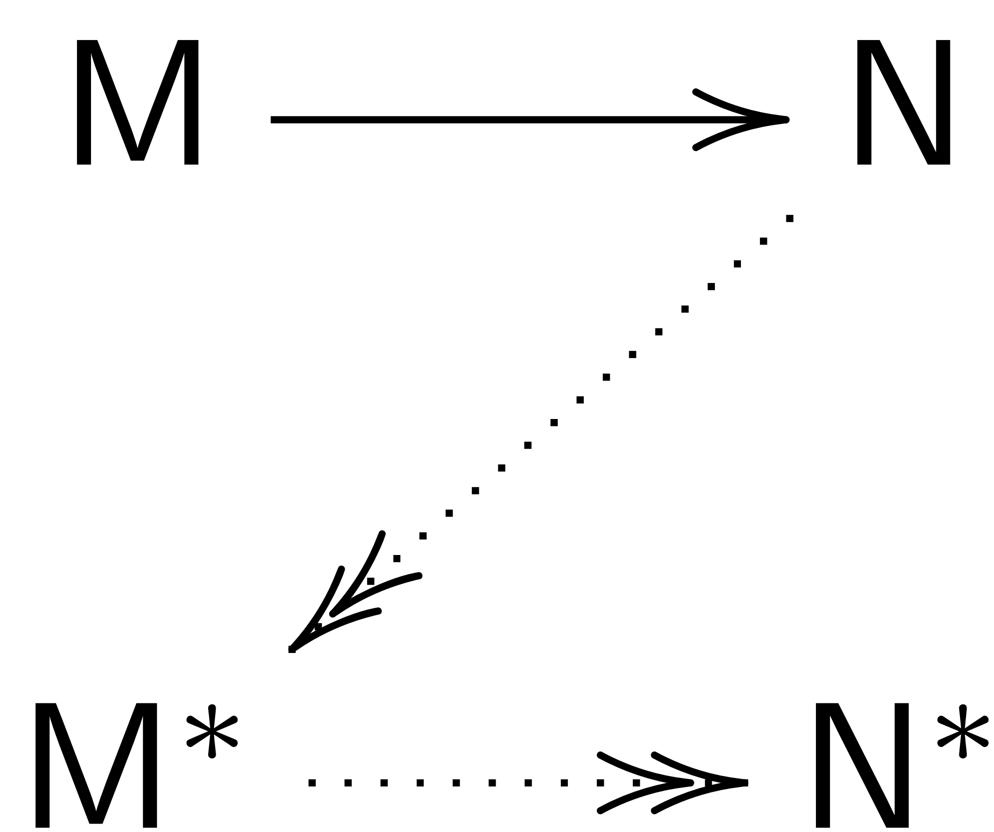
$$(\lambda x.M)N \quad \to_\beta \quad M[x := N]$$

History of Confluence of λ_β

- Church and Rosser (1936) "*Some Properties of Conversion*"
  - residuals of redexes
- Tait and Martin-Löf (19??)
  - parallel reduction
- Takahashi (1995) "*Parallel Reduction in λ-Calculus*"
  - maximum parallel reduction
- Dehornoy and van Oostrom (2008)
  - Z theorem

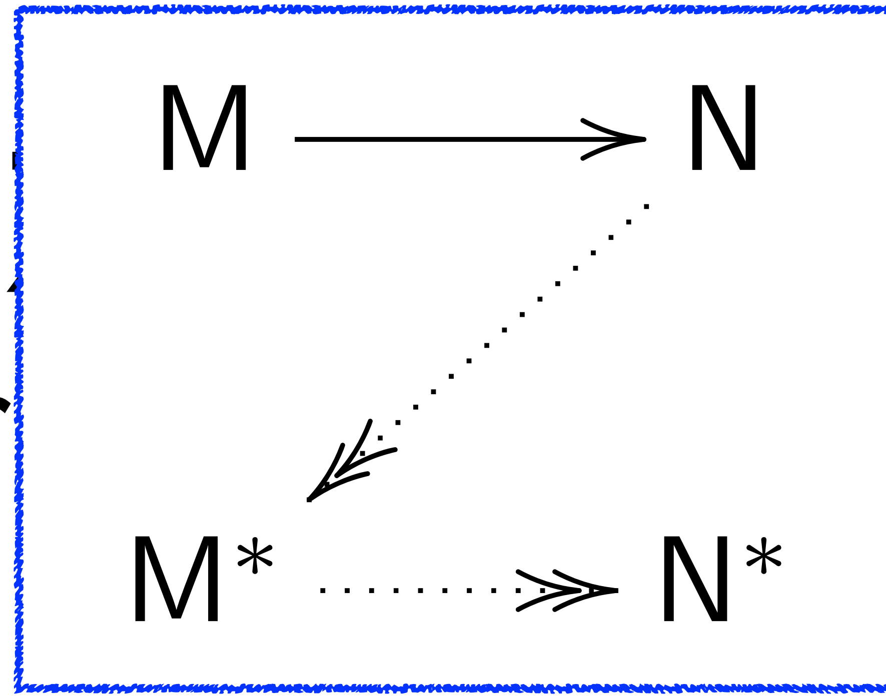
Z theorem
[Dehornoy&van Oostrom 2008]

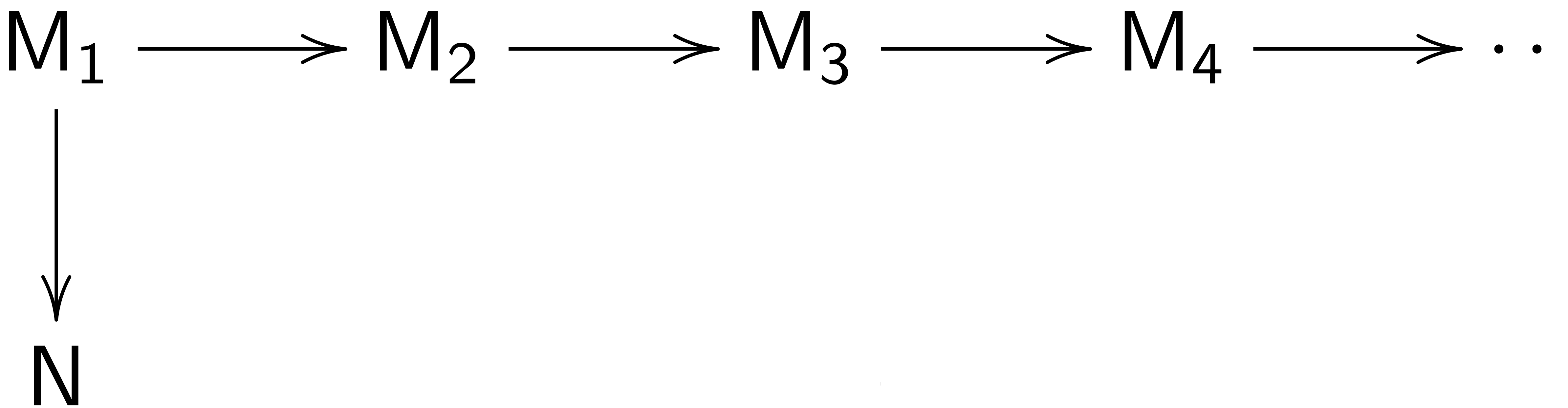
If we find a mapping $(\cdot)^*$ s.t.



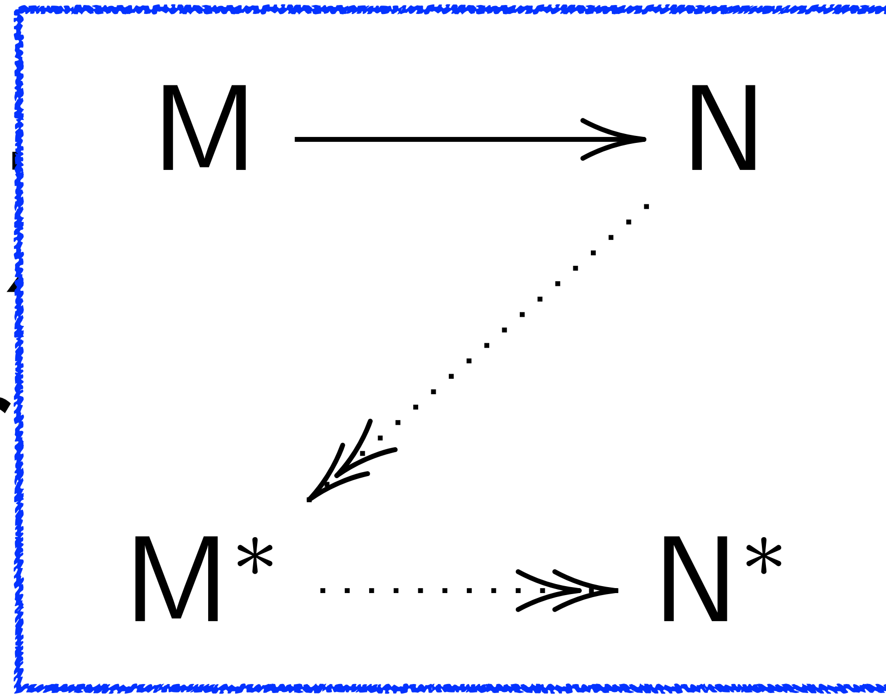
then the reduction system is confluent

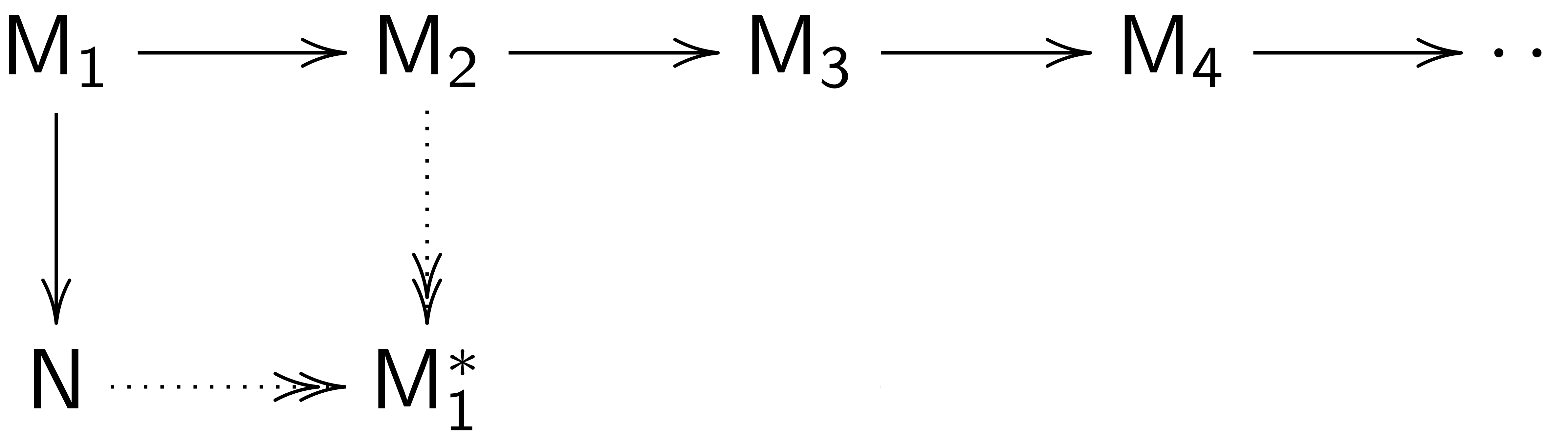
[Dehor … n 2008]



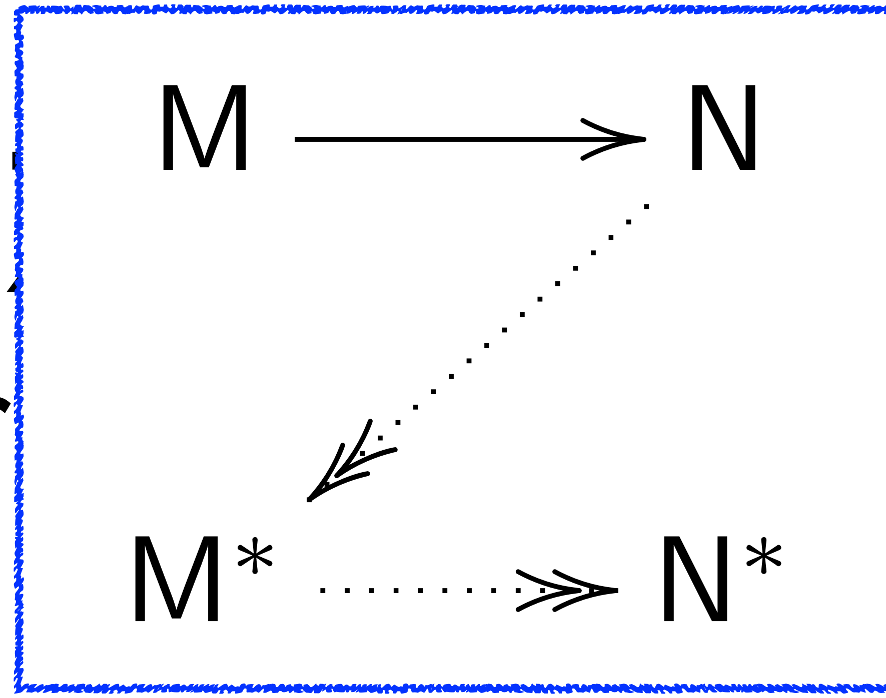


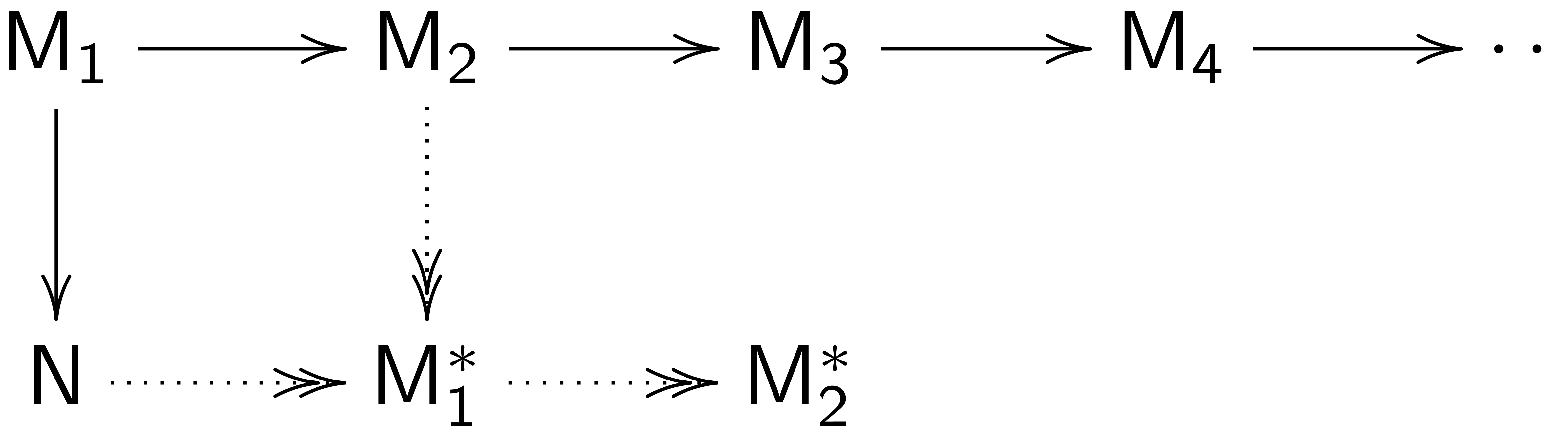
[Dehor n 2008]





[Dehor … n 2008]





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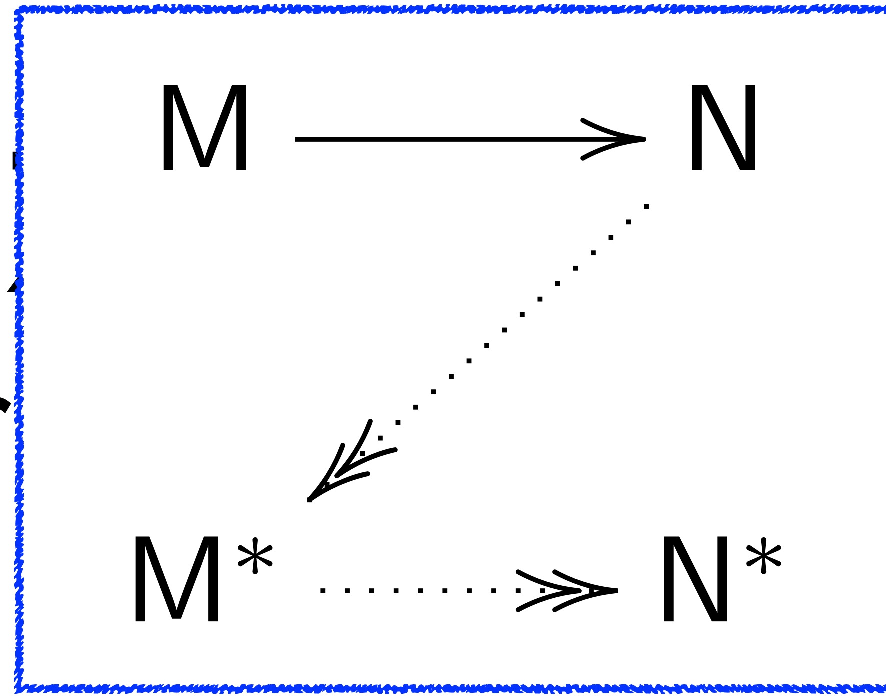

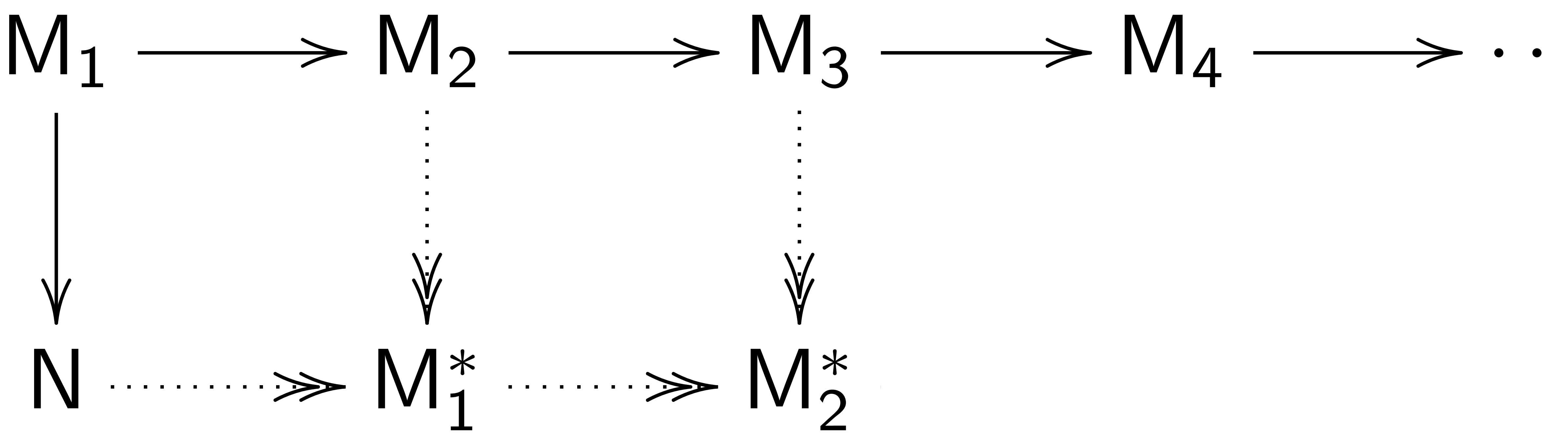

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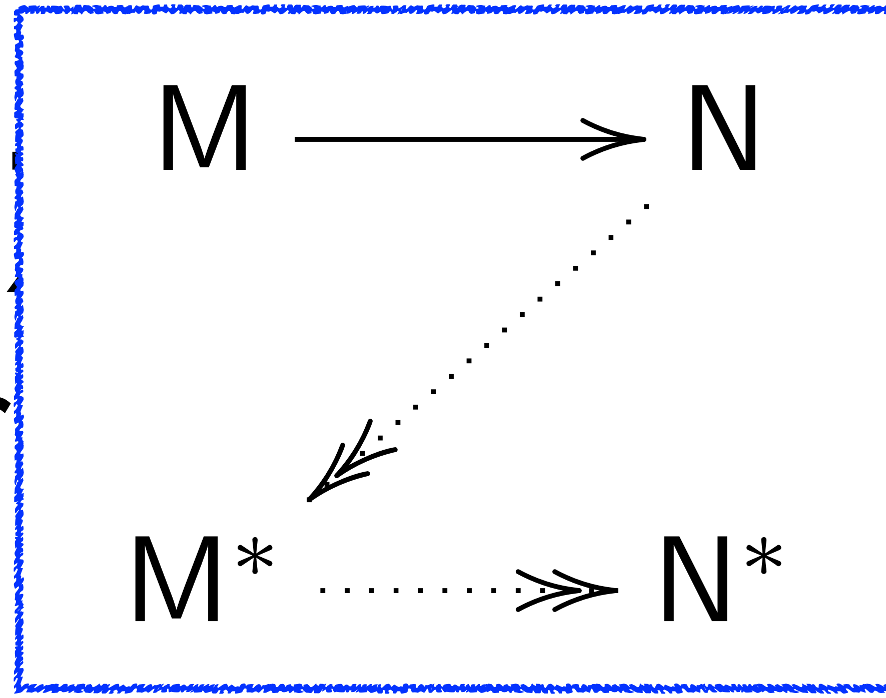

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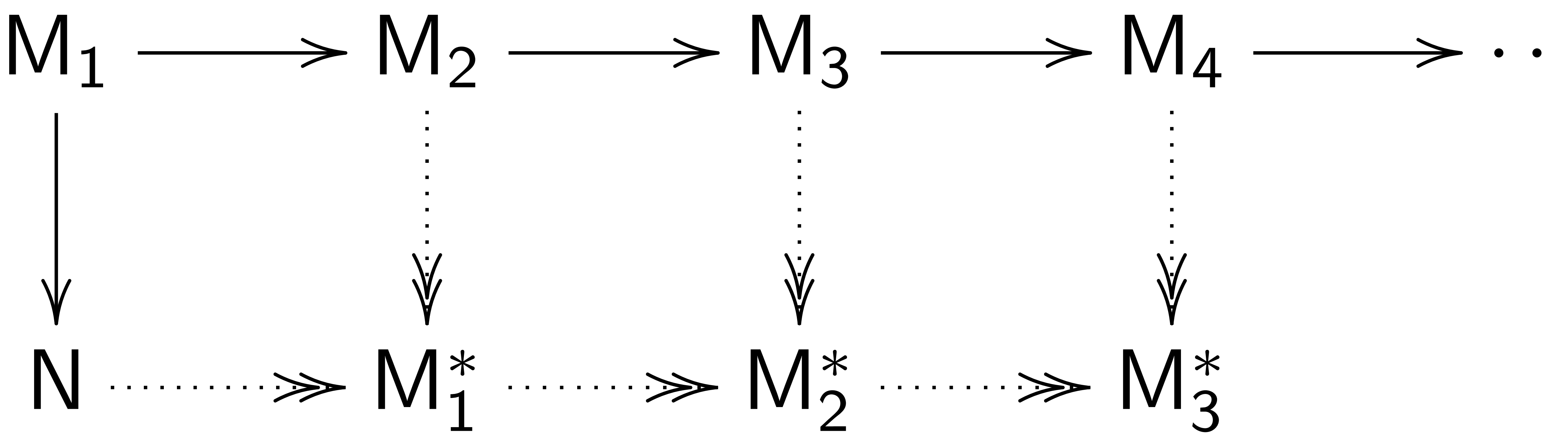

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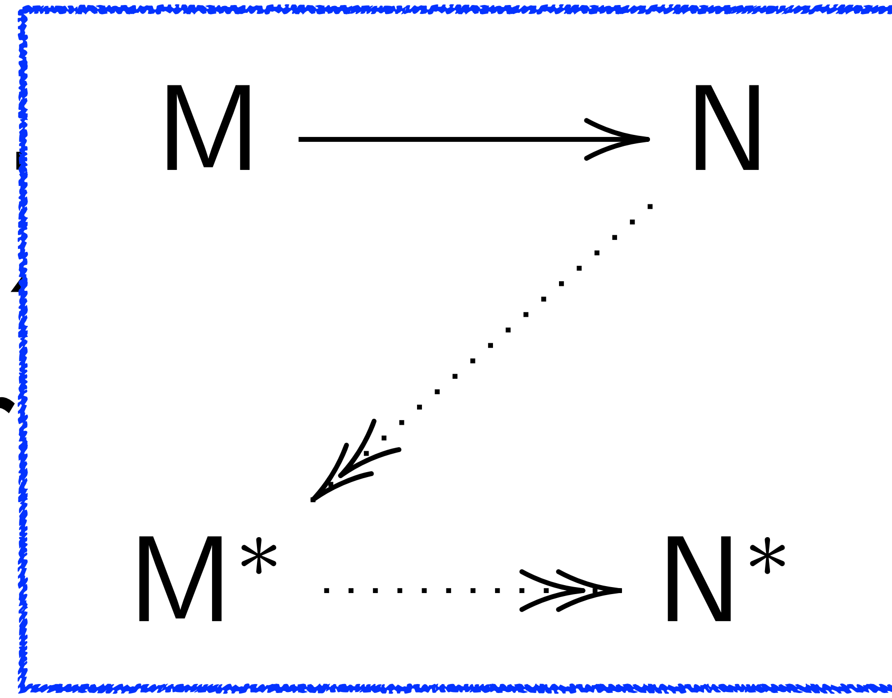

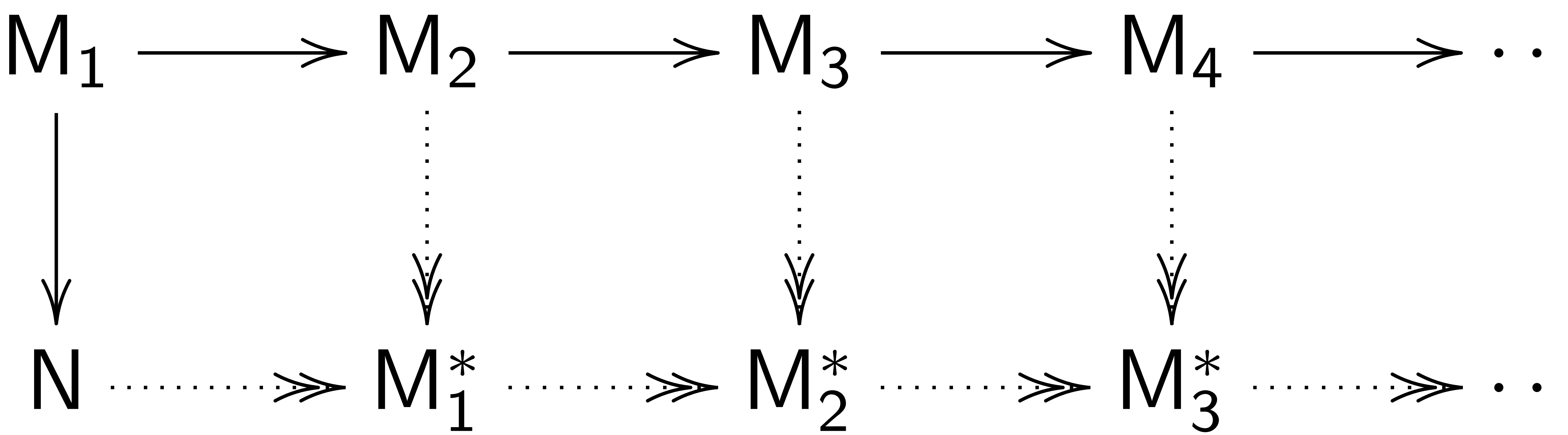

Confluence of λ_β by Z

Confluence of λβ by Z

- Takahashi's maximum parallel reduction is Z

$$
\begin{aligned}
x^* &= x \\
(\lambda x.M)^* &= \lambda x.M^* \\
((\lambda x.M)N)^* &= M^*[x := N^*] \\
(MN)^* &= M^*N^* \qquad (M \text{ is not abst.})
\end{aligned}
$$

Confluence of λβ by Z

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$$

- Key lemmas

$$M \to^* M^* \qquad \text{(i)}$$

$$M^*[x := N^*] \to^* (M[x := N])^* \qquad \text{(ii)}$$

Confluence of λ_β by Z

- Proof of the base case

$$(\lambda x.M)N \longrightarrow M[x := N]$$

- Key lemmas

$$M \to^* M^* \qquad \text{(i)}$$

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Confluence of λβ by Z

- Proof of the base case

$$
\begin{array}{ccc}
(\lambda x.M)N & \longrightarrow & M[x := N] \\
\downarrow & & \downarrow \\
M^*[x := N^*] & & (M[x := N])^*
\end{array}
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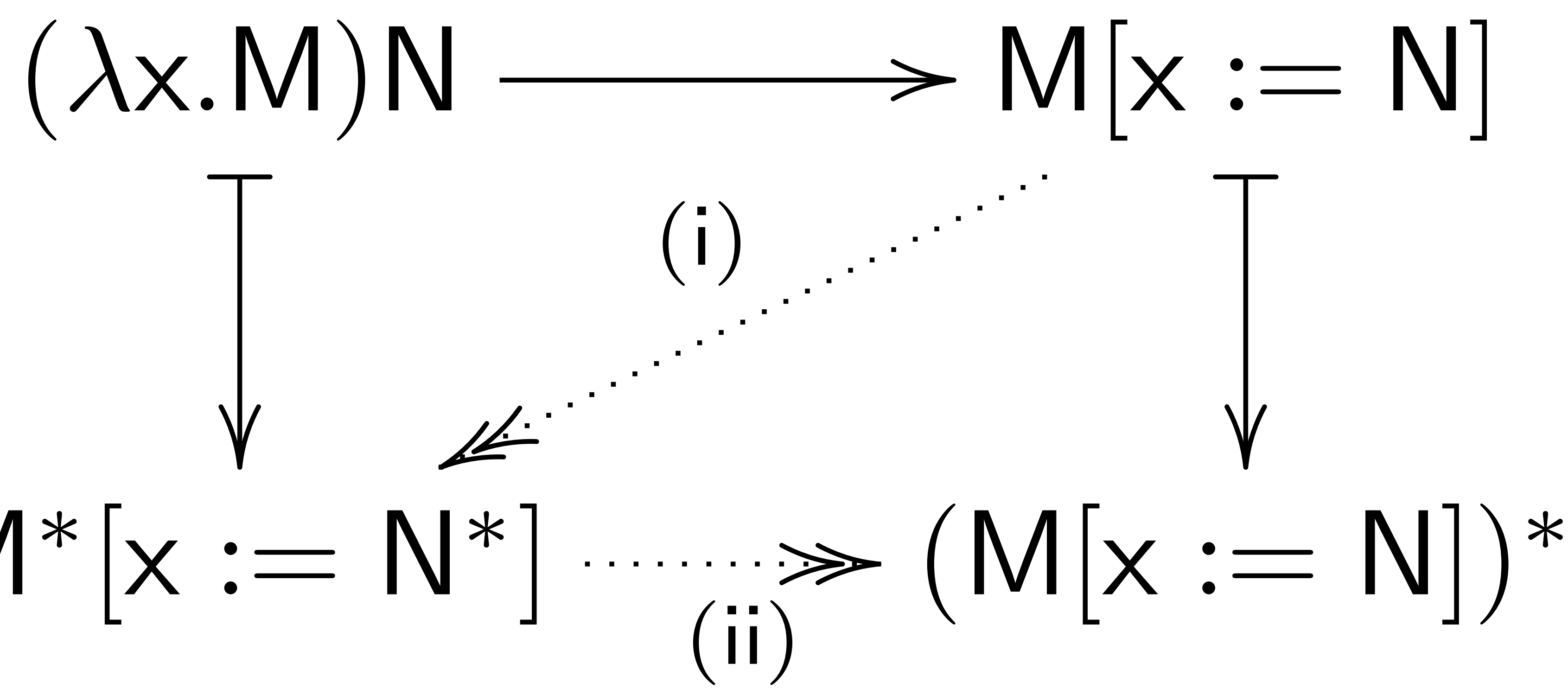
Confluence of λ_β by Z

- Proof of the base case



- Key lemmas

$$M \to^* M^* \qquad \text{(i)}$$

$$M^*[x := N^*] \to^* (M[x := N])^* \qquad \text{(ii)}$$

Z for Permutative Conversion

Permutative conversion

- for natural deduction with $\vee$ and $\exists$ [Prawitz 1965]
- exchanges order of elimination rules
  - for normal proofs to have good properties such as the subformula property
- makes confluence proofs much harder [Ando 2003]

Exchanging E-Rules

$$\dfrac{\dfrac{\begin{matrix}\vdots\, P \\ \Gamma \vdash A_1 \vee A_2\end{matrix} \quad \begin{matrix}\vdots\, Q_1 \\ \Gamma, A_1 \vdash B \to C\end{matrix} \quad \begin{matrix}\vdots\, Q_2 \\ \Gamma, A_2 \vdash B \to C\end{matrix}}{\Gamma \vdash B \to C}\,(E_\vee) \qquad \begin{matrix}\vdots\, R \\ \Gamma \vdash B\end{matrix}}{\Gamma \vdash C}\,(E_\to)$$

$$\Big\downarrow\ \pi$$

$$\dfrac{\begin{matrix}\vdots\, P \\ \Gamma \vdash A_1 \vee A_2\end{matrix} \quad \dfrac{\begin{matrix}\vdots\, Q_1 \\ \Gamma, A_1 \vdash B \to C\end{matrix} \quad \begin{matrix}\vdots\, R \\ \Gamma \vdash B\end{matrix}}{\Gamma, A_1 \vdash C}\,(E_\to) \quad \dfrac{\begin{matrix}\vdots\, Q_2 \\ \Gamma, A_2 \vdash B \to C\end{matrix} \quad \begin{matrix}\vdots\, R \\ \Gamma \vdash B\end{matrix}}{\Gamma, A_1 \vdash C}\,(E_\to)}{\Gamma \vdash C}\,(E_\vee)$$

Exchangi

(case P with $x_1 \to Q_1$ | $x_2 \to Q_2$)R

$=$

$P[x_1.Q_1, x_2.Q_2]R$

$$\dfrac{\dfrac{\dfrac{\vdots\ P}{\Gamma \vdash A_1 \vee A_2} \quad \dfrac{\vdots\ Q_1}{\Gamma, A_1 \vdash B \to C} \quad \dfrac{\vdots\ Q_2}{\Gamma, A_2 \vdash B \to C}}{\Gamma \vdash B \to C}\,(E_\vee) \qquad \dfrac{\vdots\ R}{\Gamma \vdash B}}{\Gamma \vdash C}\,(E_\to)$$

$\downarrow\ \pi$

$$\dfrac{\dfrac{\vdots\ P}{\Gamma \vdash A_1 \vee A_2} \quad \dfrac{\dfrac{\vdots\ Q_1}{\Gamma, A_1 \vdash B \to C} \quad \dfrac{\vdots\ R}{\Gamma \vdash B}}{\Gamma, A_1 \vdash C}\,(E_\to) \quad \dfrac{\dfrac{\vdots\ Q_2}{\Gamma, A_2 \vdash B \to C} \quad \dfrac{\vdots\ R}{\Gamma \vdash B}}{\Gamma, A_1 \vdash C}\,(E_\to)}{\Gamma \vdash C}\,(E_\vee)$$

π

$P[x_1.Q_1R, x_2.Q_2R]$

λ$_{\beta\pi}$

- Terms and eliminators

$$M, N ::= x \mid \lambda x.M \mid \iota_1 M \mid \iota_2 M \mid Me$$
$$e ::= M \mid [x_1.N_1, x_2.N_2]$$

- Reduction rules

$$\begin{aligned}
(\lambda x.M)N &\rightarrow_\beta M[x := N] \\
(\iota_i M)[x_1.N_1, x_2.N_2] &\rightarrow_\beta N_i[x_i := M] \\
M[x_1.N_1, x_2.N_2]e &\rightarrow_\pi M[x_1.N_1 e, x_2.N_2 e]
\end{aligned}$$

$\lambda_{\beta\pi}$

uniform representation of elimination for → and ∨

- Terms and eliminators

$$M, N ::= x \mid \lambda x.M \mid \iota_1 M \mid \iota_2 M \mid Me$$

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- Reduction rules

$$\begin{array}{rcl} (\lambda x.M)N & \to_\beta & M[x := N] \\ (\iota_i M)[x_1.N_1, x_2.N_2] & \to_\beta & N_i[x_i := M] \\ M[x_1.N_1, x_2.N_2]e & \to_\pi & M[x_1.N_1 e, x_2.N_2 e] \end{array}$$

$\lambda_{\beta\pi}$

uniform representation of elimination for → and ∨

- Terms and eliminators

$$M, N ::= x \mid \lambda x.M \mid \iota_1 M \mid \iota_2 M \mid Me$$

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- Reduction rules

$$(\lambda x.M)N \rightarrow_\beta M[x := N]$$

$$(\iota_i M)[x_1.N_1, x_2.N_2] \rightarrow_\beta N_i[x_i := M]$$

$$M[x_1.N_1, x_2.N_2]e \rightarrow_\pi M[x_1.N_1 e, x_2.N_2 e]$$

left associative
$(M[x_1.N_1,x_2.N_2])e$

permutative conversion

$\lambda_{\beta\pi}$, for simplicity

- Terms and eliminators

$$M, N ::= x \mid \lambda x.M \mid \iota M \mid Me$$

$$e ::= M \mid [x.N]$$

- Reduction rules

$$
\begin{aligned}
(\lambda x.M)N &\quad \to_\beta \quad M[x := N] \\
(\iota M)[x.N] &\quad \to_\beta \quad N[x := M] \\
M[x.N]e &\quad \to_\pi \quad M[x.Ne]
\end{aligned}
$$

Where are difficulties?

- Parallel reduction for π-reduction
- Maximum complete development for the combination of β- and π-reductions

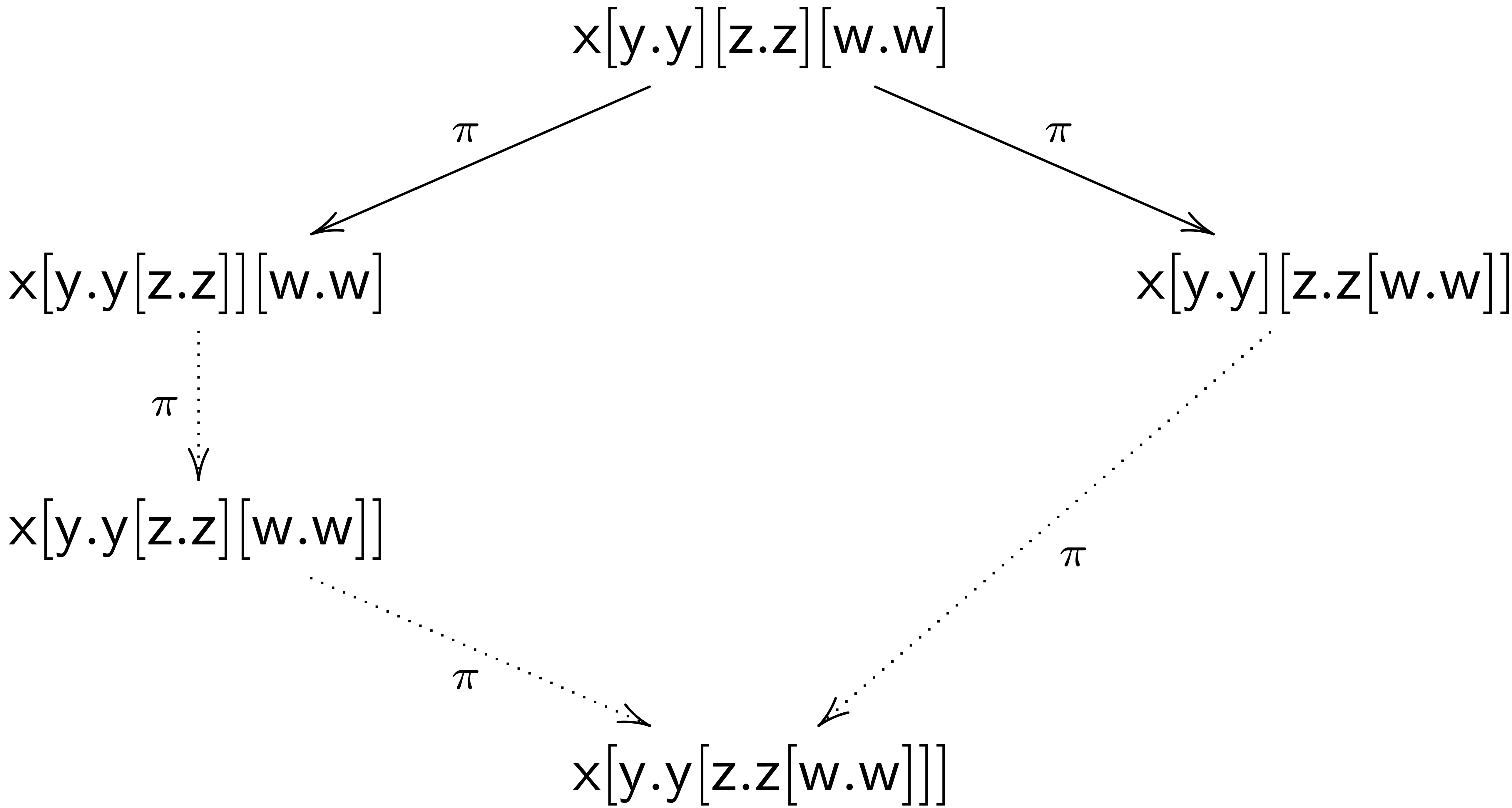
Parallel reduction for π?

x[y.y][z.z][w.w]

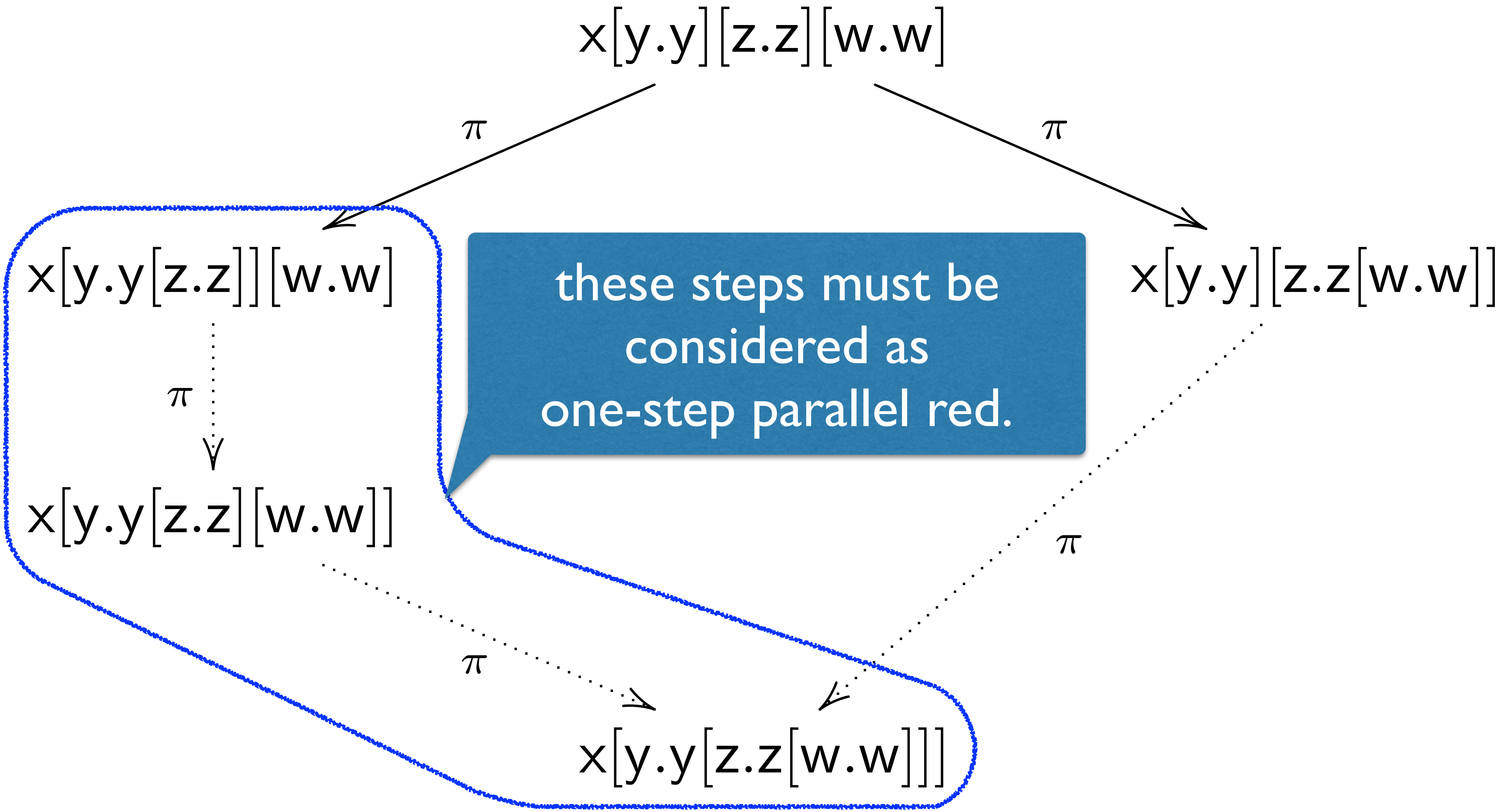
$\swarrow_{\pi}$ $\searrow_{\pi}$

x[y.y[z.z]][w.w] x[y.y][z.z[w.w]]

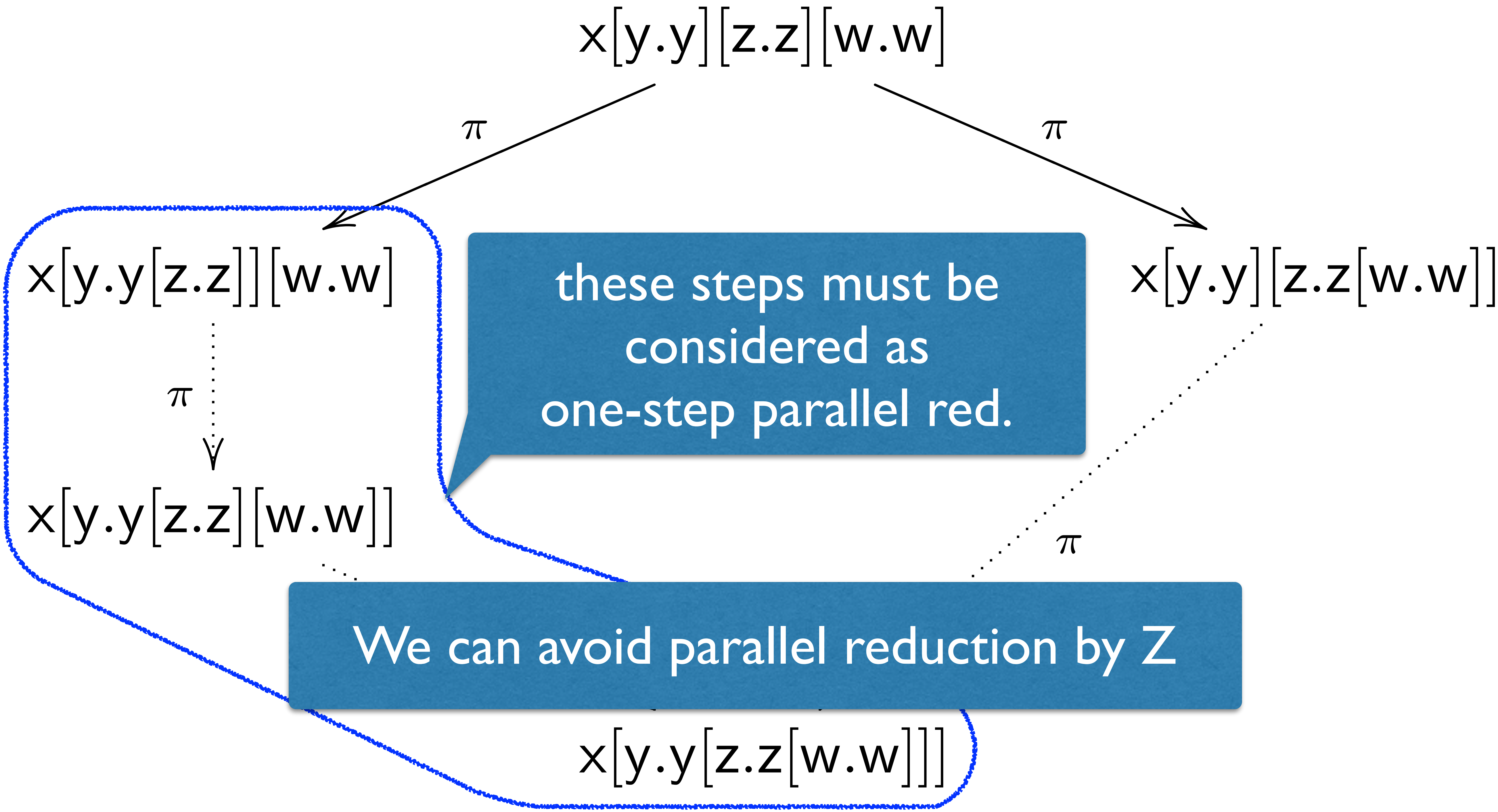
Parallel reduction for π?

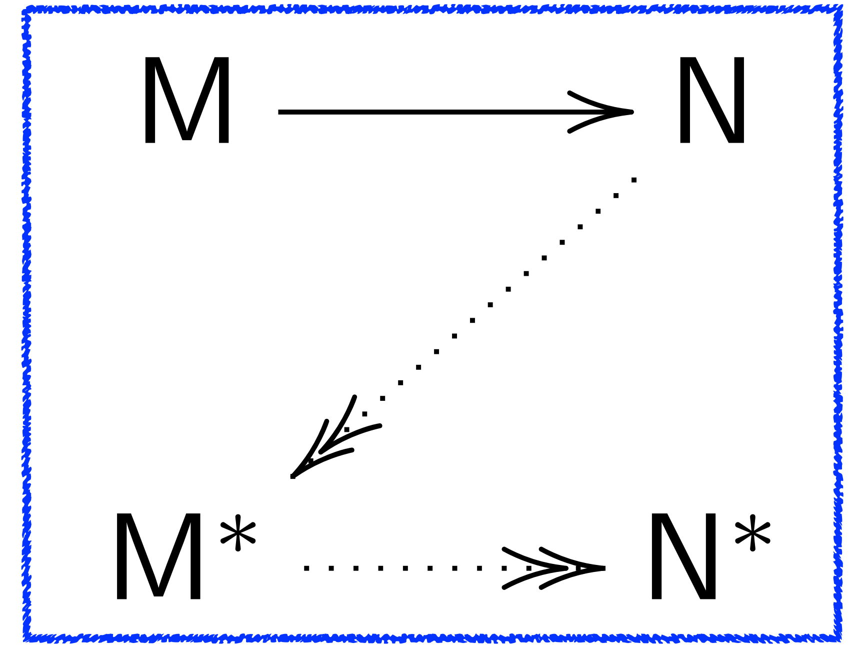


Parallel reduction for π?

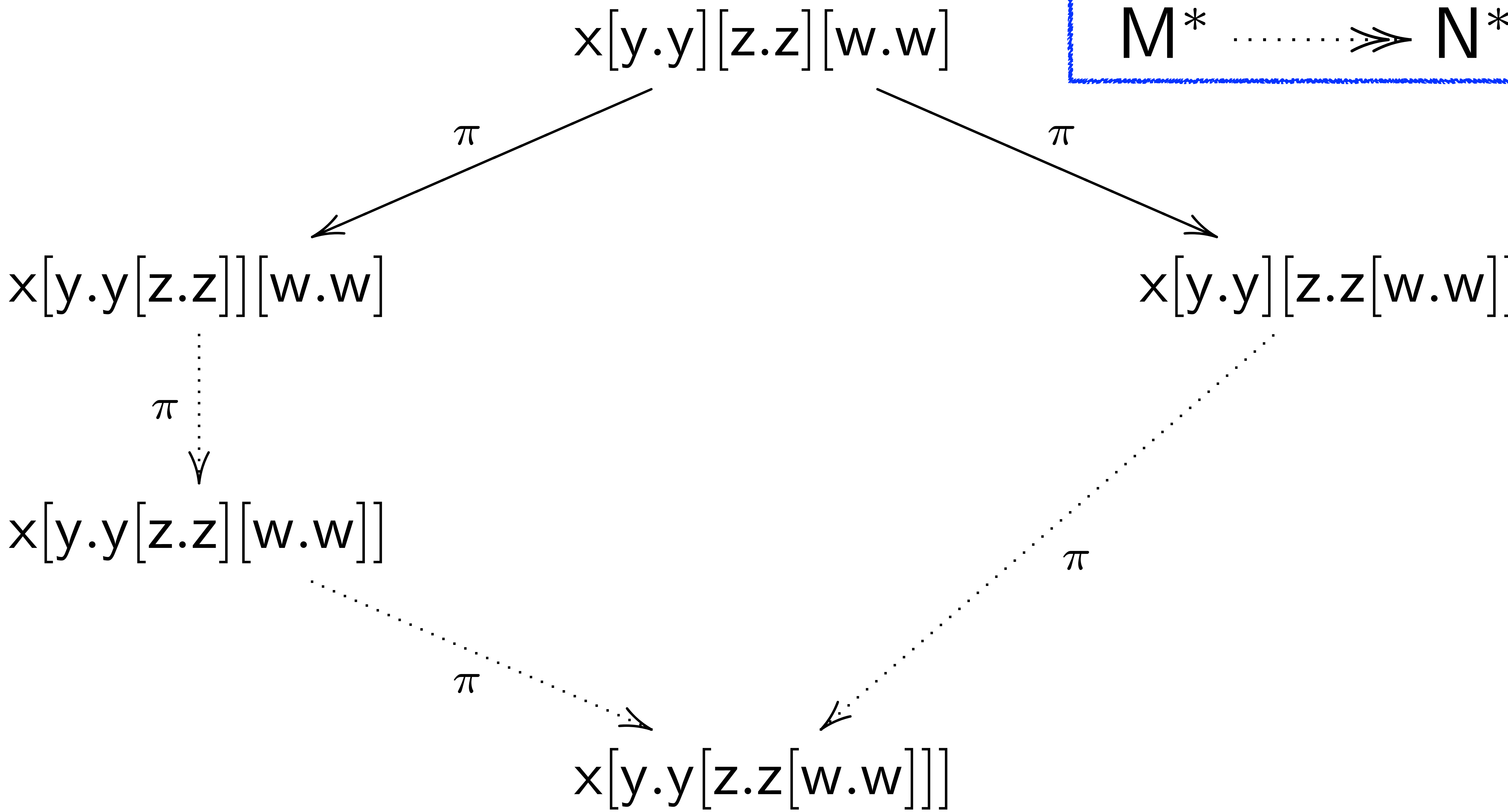


Parallel reduction for π?

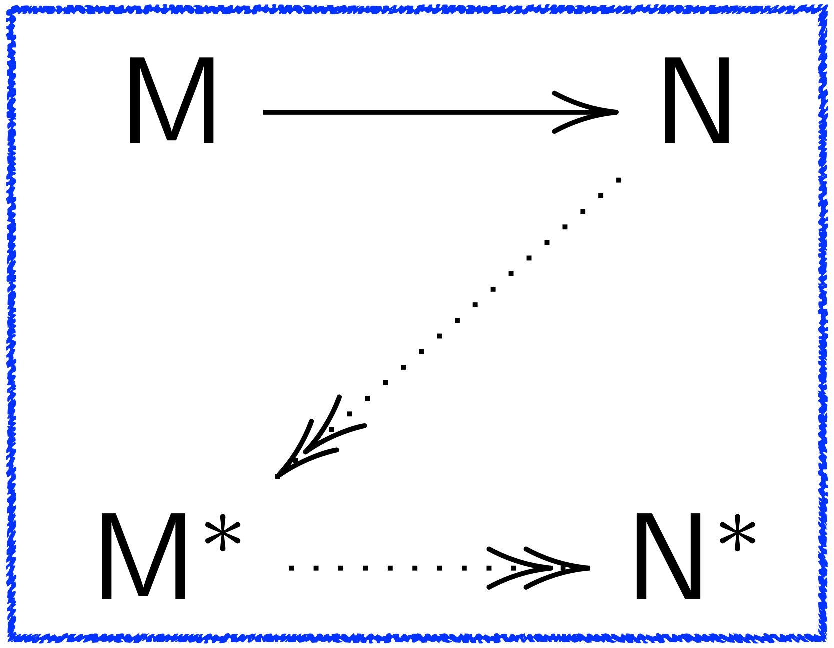


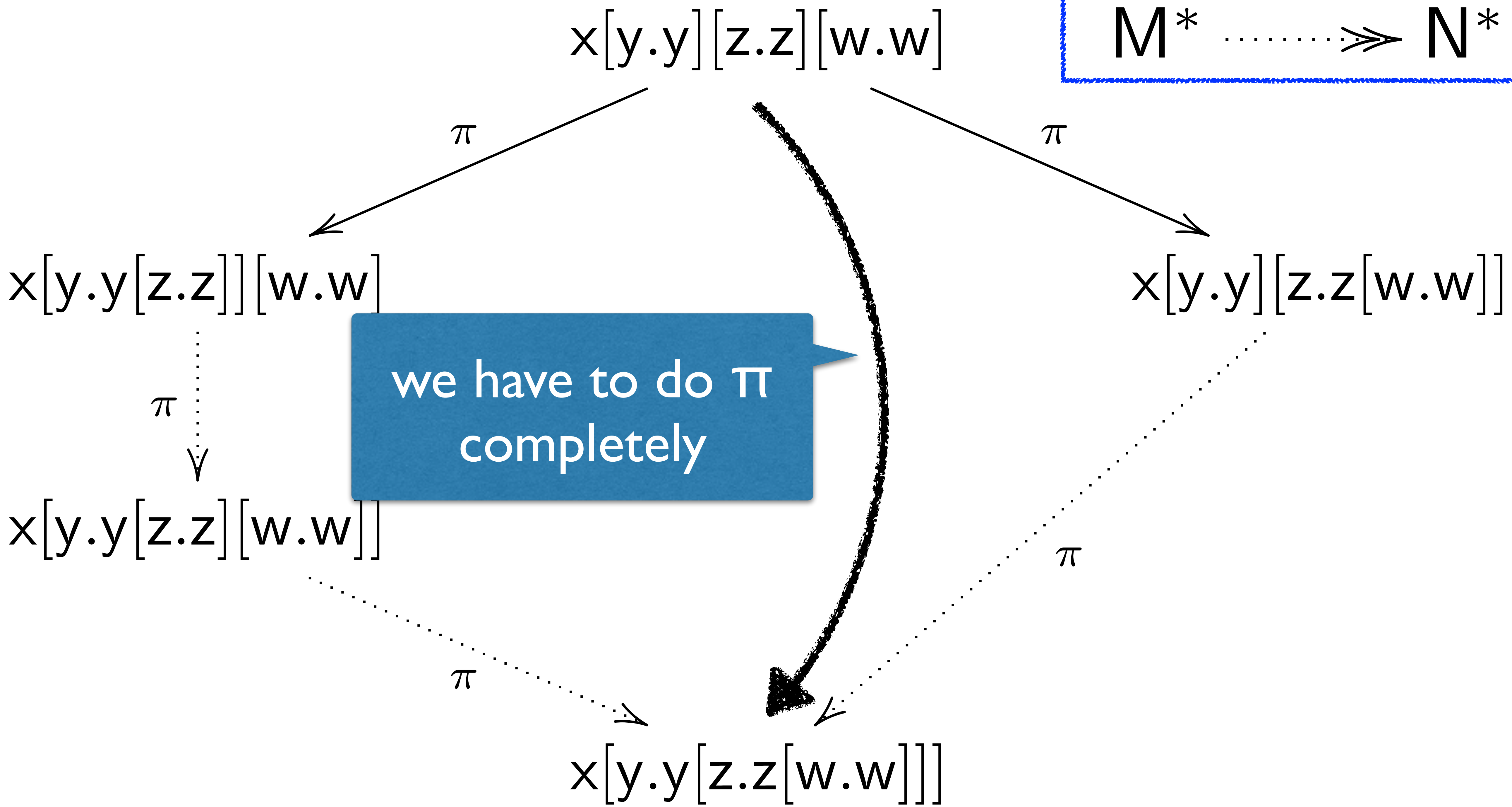


Z for π?



Z for π?


M
N
M*
N*


x[y.y][z.z][w.w]
π
π
x[y.y[z.z]][w.w]
x[y.y][z.z[w.w]]
π
x[y.y[z.z][w.w]]
we have to do π completely
π
π
x[y.y[z.z[w.w]]]

Z for βπ?

- A naïve definition

$$
\begin{aligned}
x^* &= x \\
(\lambda x.M)^* &= \lambda x.M^* \\
(\iota M)^* &= \iota M^* \\
((\lambda x.M)N)^* &= M^*[x := N^*] \\
((\iota M)[x.N])^* &= N^*[x := M^*] \\
(Me)^* &= M^* @ e^* \qquad \text{(otherwise)} \\
\\
(M[x.N])@e &= M[x.N@e] \\
M@e &= Me \qquad \text{(otherwise)}
\end{aligned}
$$

Z for βπ?

- A naïve definition

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\begin{aligned}
x^* &= x \\
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\end{aligned}
$$

$$
\begin{aligned}
(M[x.N])@e &= M[x.N@e] \\
M@e &= Me \qquad \text{(otherwise)}
\end{aligned}
$$

is not Z

Z for [

$$
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\end{aligned}
$$

- Monotonicity fails

$$(\iota(x[y.y]))[z.z]w \rightarrow_\pi (\iota(x[y.y]))[z.zw]$$

Z for

$$
\begin{aligned}
x^* &= x \\
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\end{aligned}
$$

- Monotonicity fails

$$(\iota(x[y.y]))[z.z]w \to_\pi (\iota(x[y.y]))[z.zw]$$

$$
\begin{aligned}
&((\iota(x[y.y]))[z.z]w)^* \\
=\ &((\iota(x[y.y]))[z.z])^* @ w \\
=\ &(x[y.y]) @ w \\
=\ &x[y.yw]
\end{aligned}
\qquad
\begin{aligned}
&((\iota(x[y.y]))[z.zw])^* \\
=\ &(zw)^*[z := x[y.y]] \\
=\ &x[y.y]w
\end{aligned}
$$

$$
\begin{aligned}
x^* &= x \\
(\lambda x.M)^* &= \lambda x.M^* \\
(\iota M)^* &= \iota M^* \\
((\lambda x.M)N)^* &= M^*[x := N^*] \\
((\iota M)[x.N])^* &= N^*[x := M^*] \\
(Me)^* &= M^* @ e^* \qquad \text{(otherwise)}
\end{aligned}
$$

permutation is applied to the result of β

- Monotonicity fails

$$(\iota(x[y.y]))[z.z]w \rightarrow_\pi (\iota(x[y.y]))[z.zw]$$

$$
\begin{aligned}
&((\iota(x[y.y]))[z.z]w)^* \\
&= ((\iota(x[y.y]))[z.z])^* @ w \\
&= (x[y.y]) @ w \\
&= x[y.yw]
\end{aligned}
\qquad\qquad
\begin{aligned}
&((\iota(x[y.y]))[z.zw])^* \\
&= (zw)^*[z := x[y.y]] \\
&= x[y.y]w
\end{aligned}
$$

$$
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((\lambda x.M)N)^* &= M^*[x := N^*] \\
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$$(\iota(x[y.y]))[z.z]w \to_\pi (\iota(x[y.y]))[z.zw]$$

$$
\begin{aligned}
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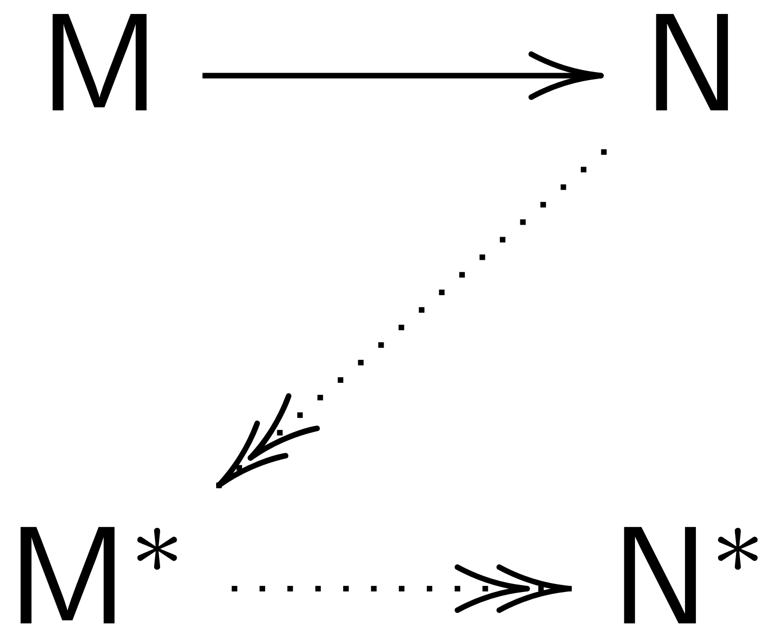
&= ((\iota(x[y.y]))[z.z])^* @ w \\
&= (x[y.y]) @ w \\
&= x[y.yw]
\end{aligned}
\qquad
\begin{aligned}
&((\iota(x[y.y]))[z.zw])^* \\
&= (zw)^*[z := x[y.y]] \\
&= x[y.y]w
\end{aligned}
$$

- We want to consider functions for π and β separately (and adapt Z to their composition)

Compositional Z

Z and weak Z

$(\cdot)^*$ is Z for $\rightarrow$ iff



Z and weak Z

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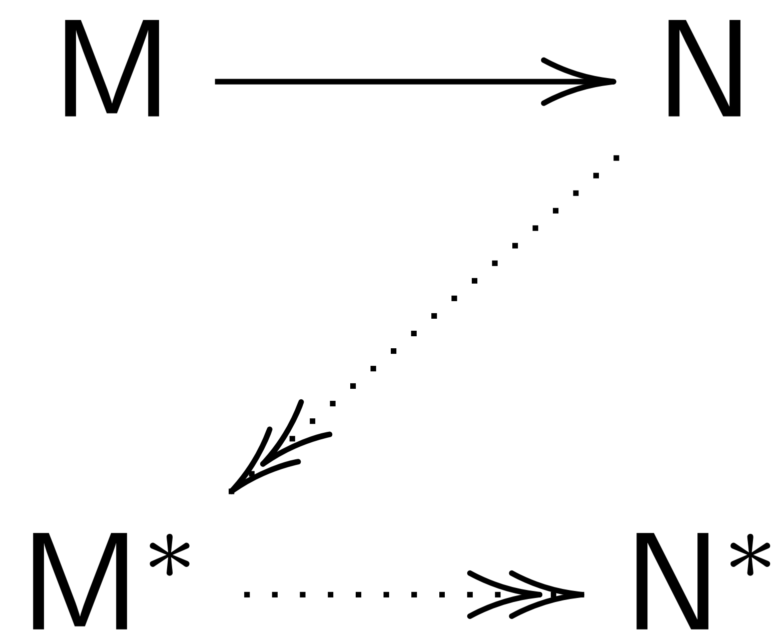
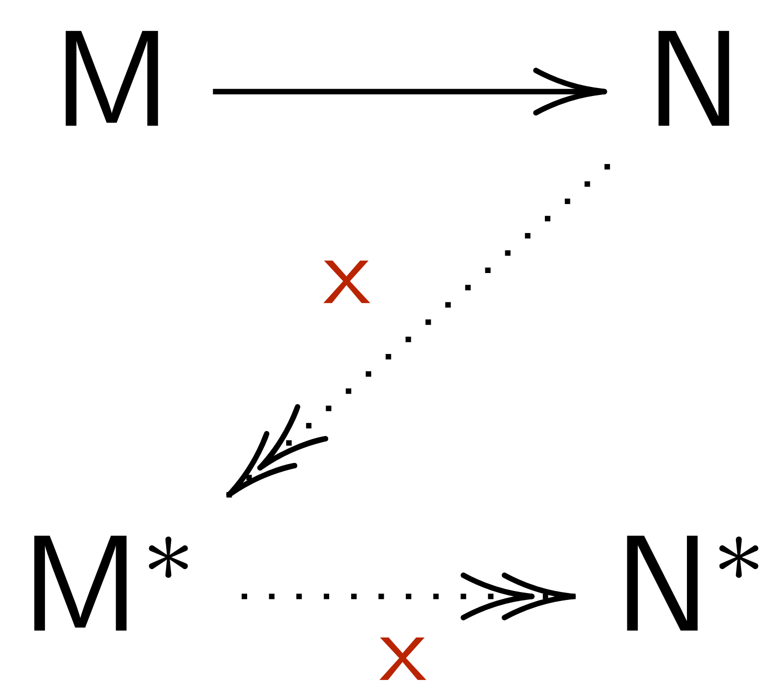


$(\cdot)^*$ is weakly Z for $\to$ by $\to_x$ iff



Compositional Z
[N&Fujita'15]

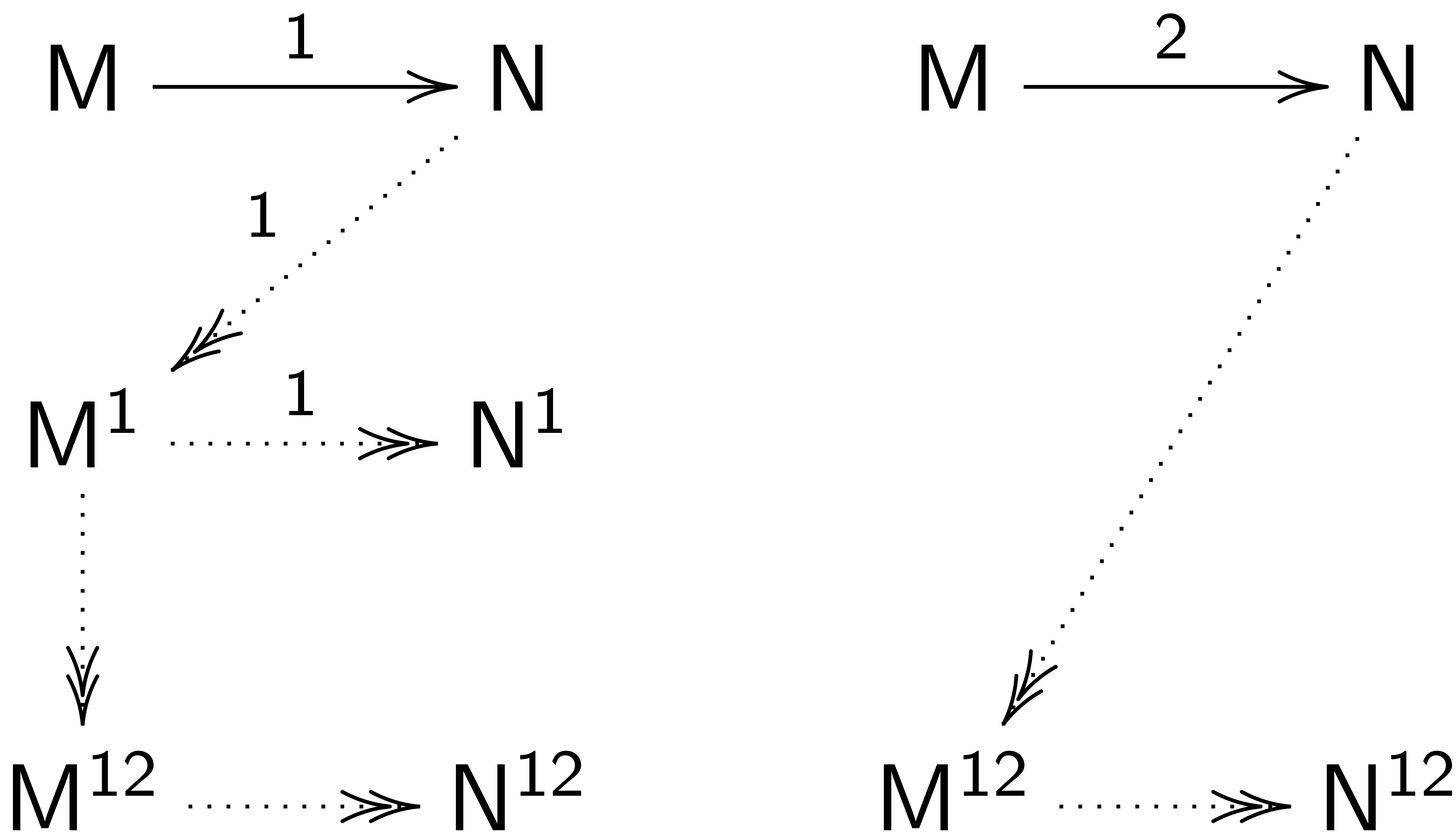
- Let $\to = \to_1 \cup \to_2$

  If mappings $(\cdot)^1$ and $(\cdot)^2$ satisfying following,

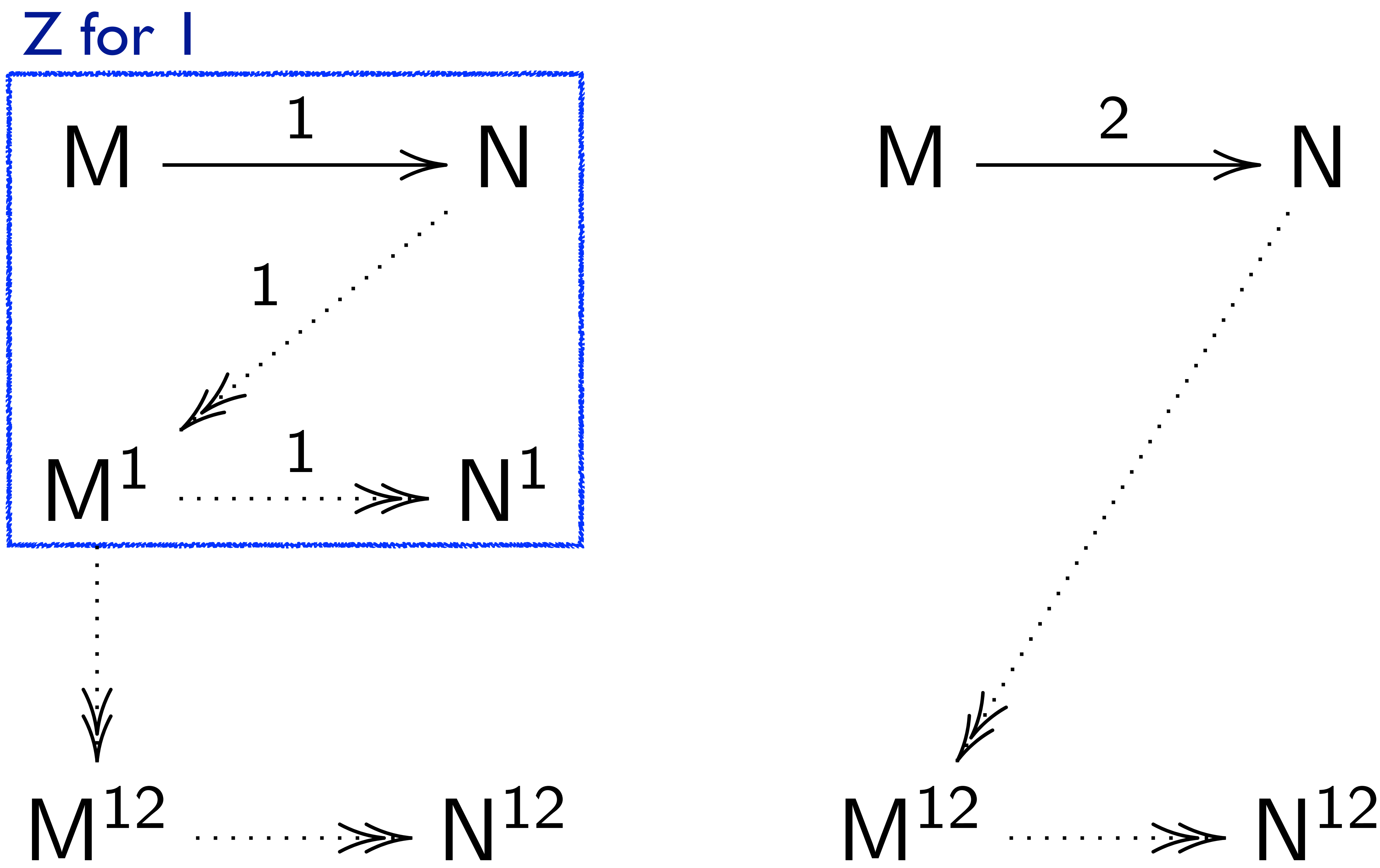
  - $(\cdot)^1$ is Z for $\to_1$
  - if $M \to_1 N$, then $M^2 \to^* N^2$
  - $M^1 \to^* M^{12}$ holds for any M
  - $(\cdot)^{12}$ is weakly Z for $\to_2$ by $\to$

then the composition $(\cdot)^{12}$ is Z for $\to$

Compositional Z

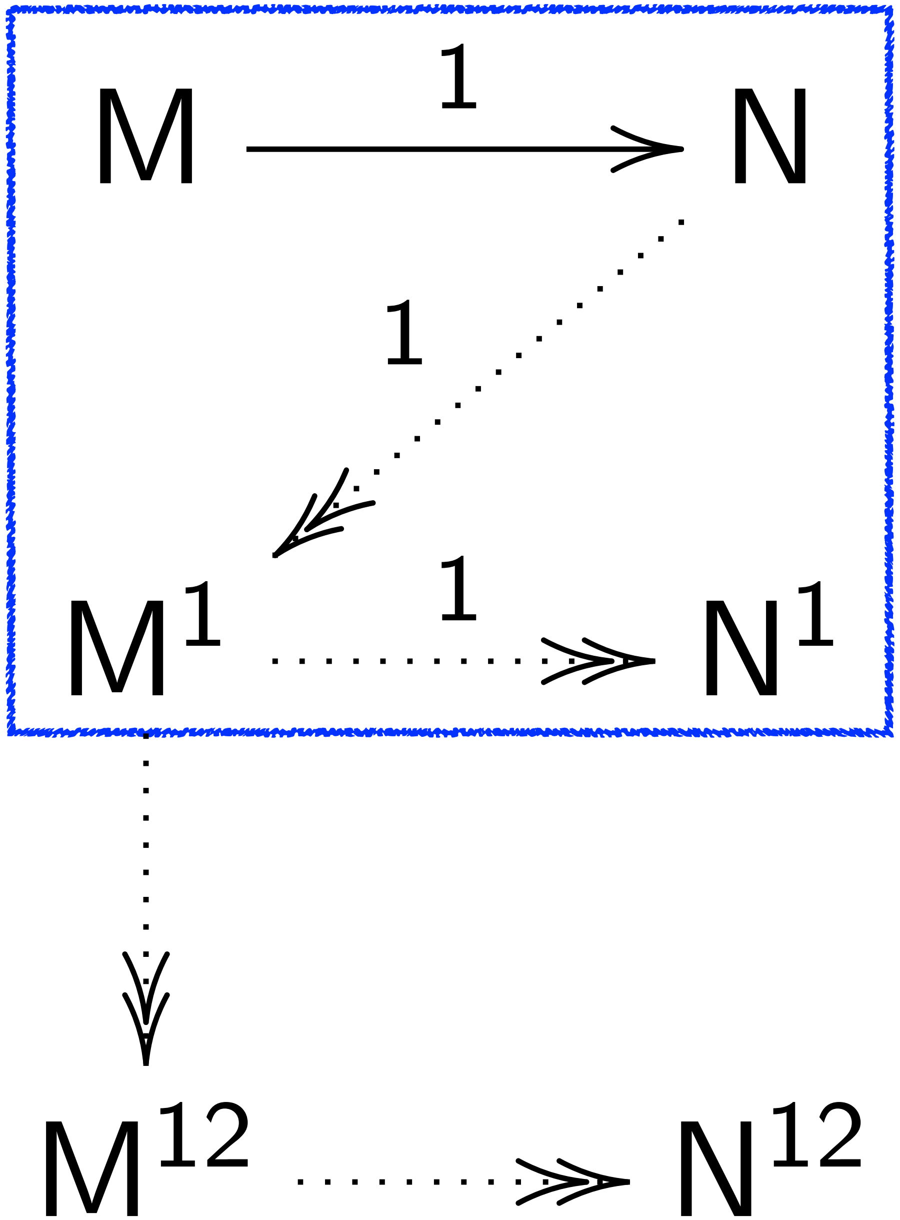


Compositional Z

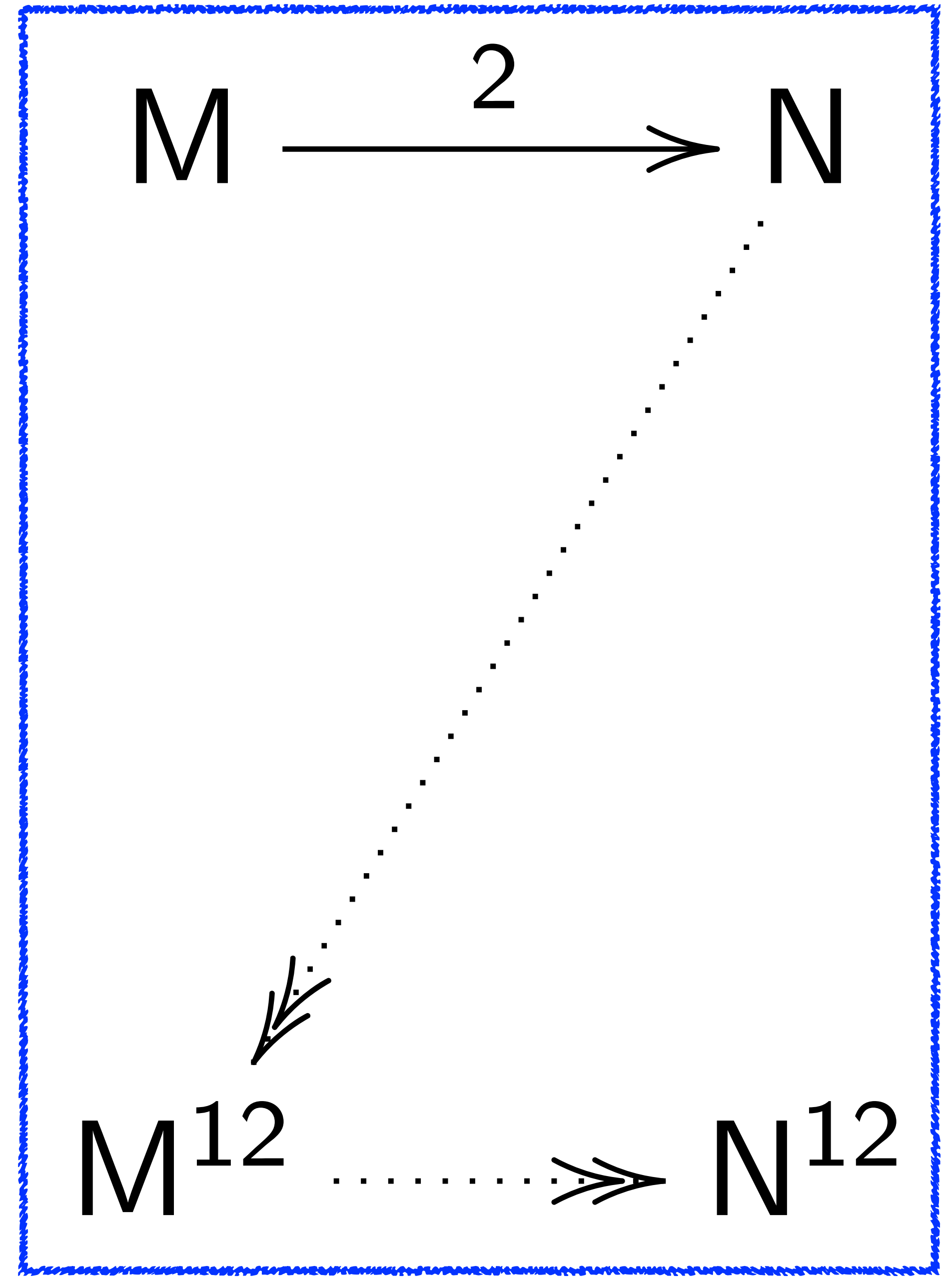


Compositional Z

Z for 1



weak Z for 2



Confluence of βπ by compositional Z

$$
\begin{aligned}
x^P &= x \\
(\lambda x.M)^P &= \lambda x.M^P \\
(\iota M)^P &= \iota M^P \\
(Me)^P &= M^P @ e^P
\end{aligned}
\qquad
\begin{aligned}
x^B &= x \\
(\lambda x.M)^B &= \lambda x.M^B \\
(\iota M)^B &= \iota M^B \\
((\lambda x.M)N)^B &= M^B[x := N^B] \\
((\iota M)[x.N])^B &= N^B[x := M^B] \\
(Me)^B &= M^B e^B \quad \text{(otherwise)}
\end{aligned}
$$

The mappings $(\cdot)^P$ and $(\cdot)^B$ satisfies the conditions of the compositional Z for $\to_\pi$ and $\to_\beta$

Applications

λ with permutative conversion π and β

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λμ with permutative conversion	πμ and β

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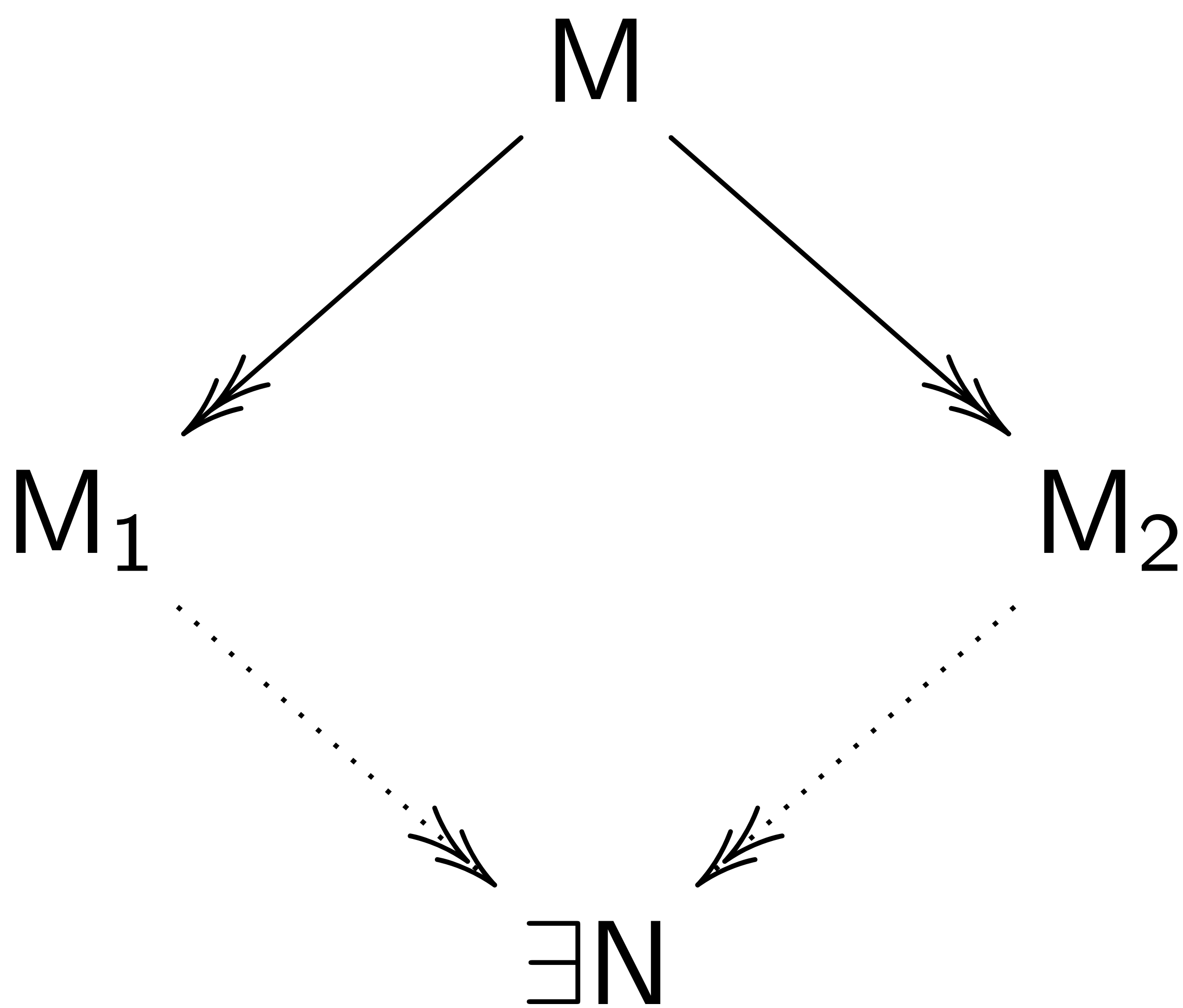
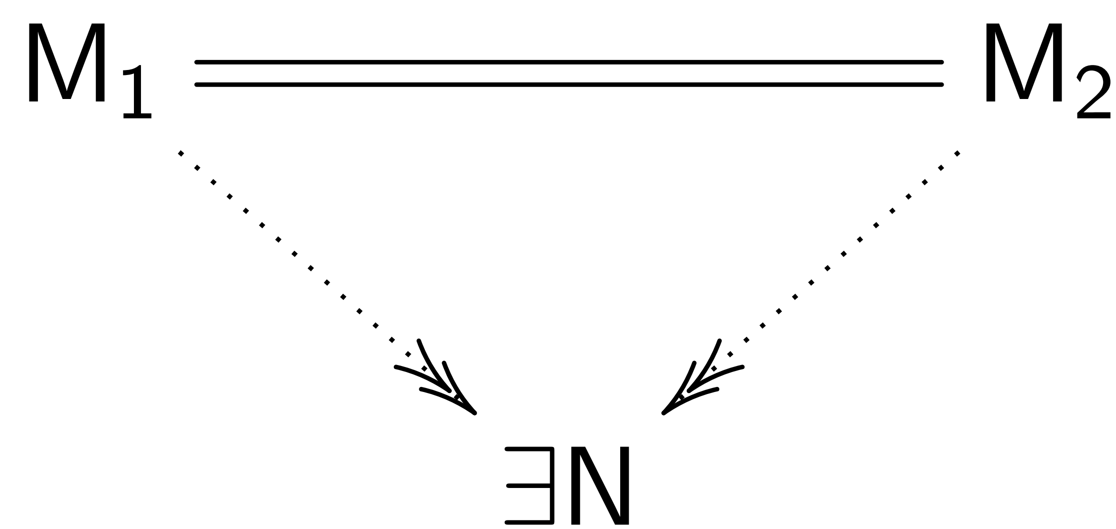
Compositional Z enables us to prove confluence by dividing reduction system into two parts

Church-Rosser via Z

Confluence vs Church-Rosser



Confluence



Church-Rosser (CR)

Confluence vs Church-Rosser

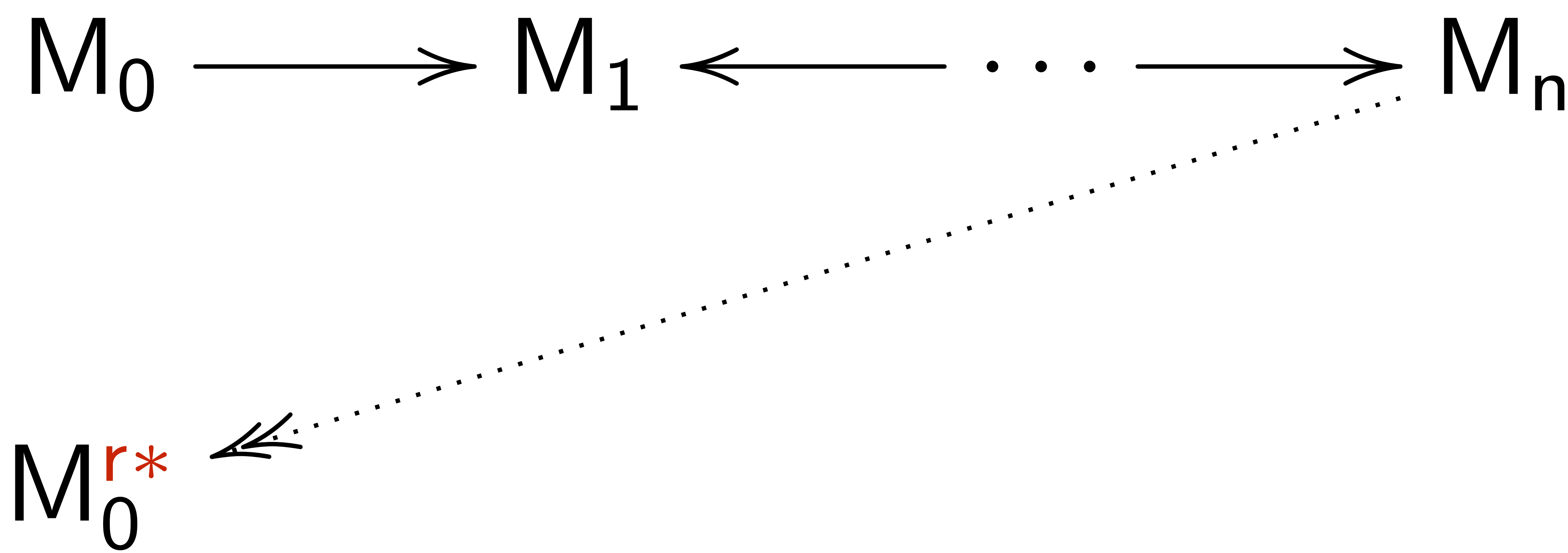
- In many textbooks, CR is shown as a corollary of confluence
  - In fact, they are equivalent in almost all of rewriting systems
- How can we prove CR directly?

Church-Rosser via Z

- Suppose
  - $(A, \rightarrow)$: an ARS
  - M^*: a Z function on A, and M^{n*} = n-fold of M^*
- Cross-Point Theorem [Fujita 2016]
  - a constructive proof of CR

For $M =_A N$, we can find a common reduct decided by the numbers of $\rightarrow$ and $\leftarrow$ in the conversion sequence

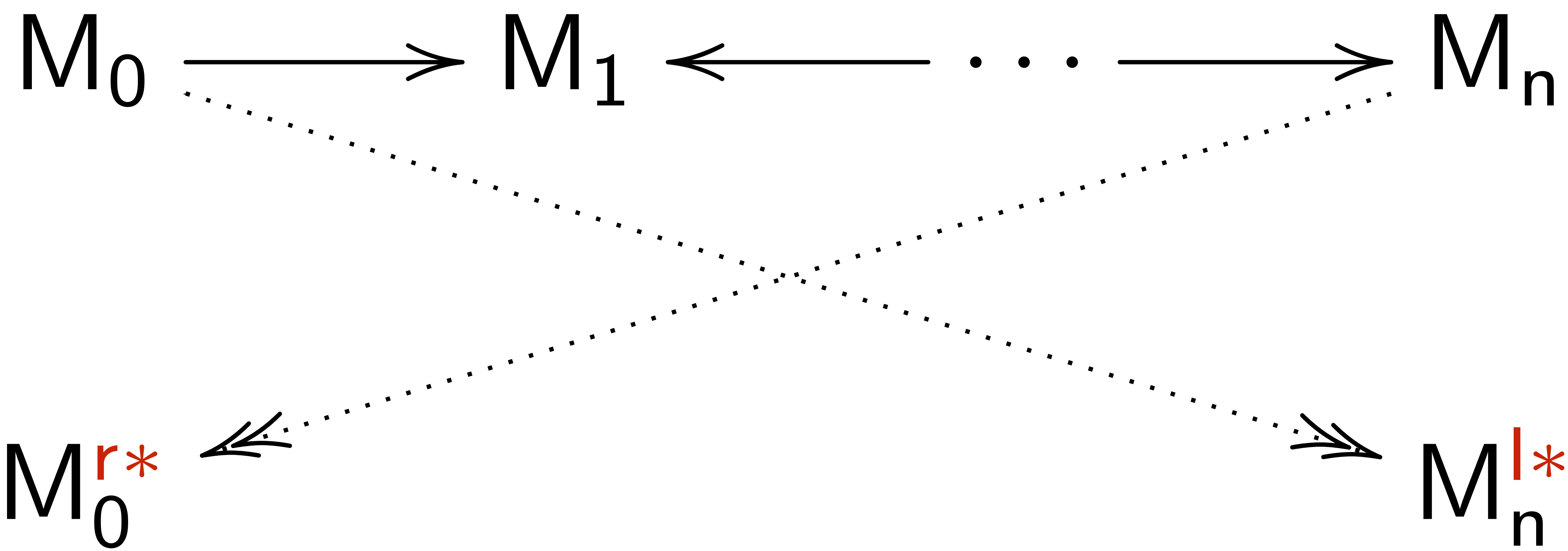
Main Lemma



r = # of → in $M_0 = M_n$

l = # of ← in $M_0 = M_n$

Main Lemma



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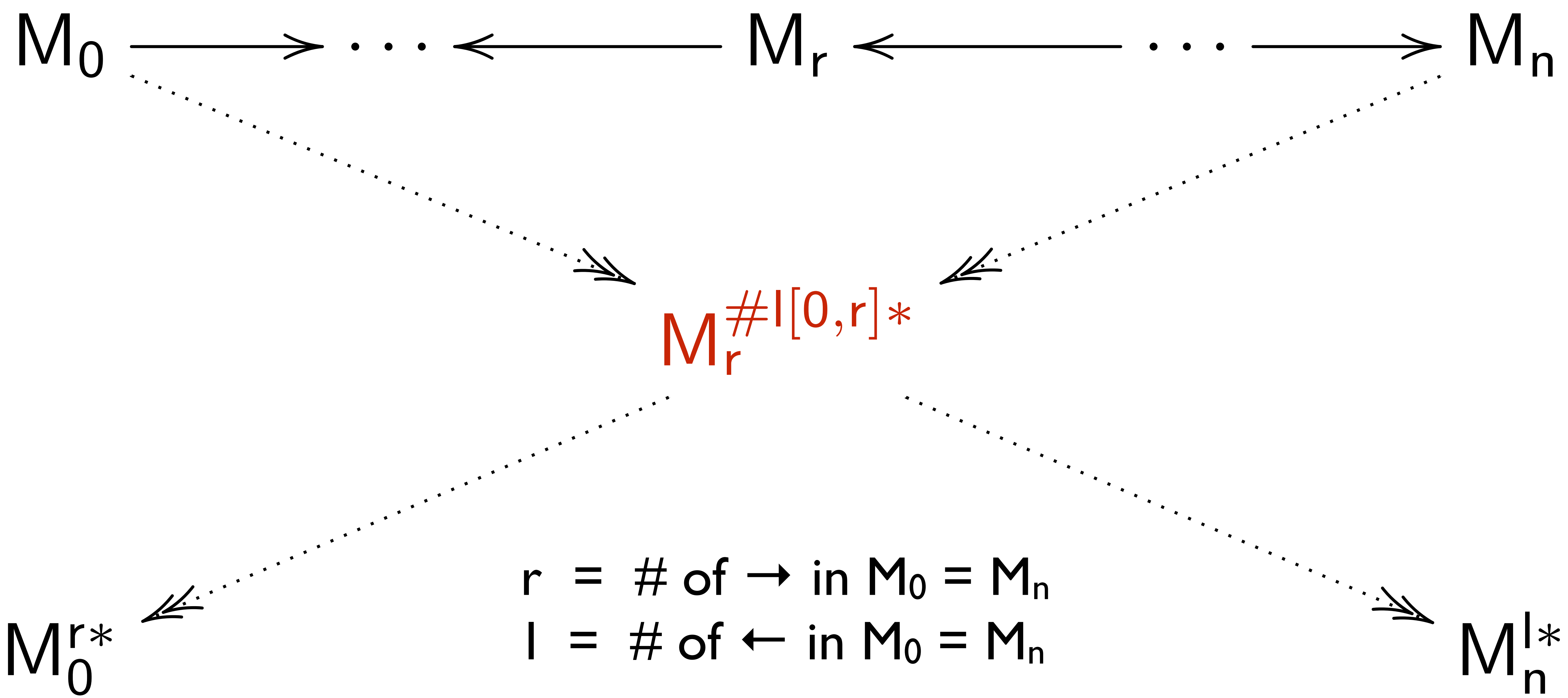
Cross-point theorem [Fujita'16]



r = # of → in $M_0 = M_n$

l = # of ← in $M_0 = M_n$

#l[0,r] = # of ← in $M_0 \ldots M_r$

= # of → in $M_r \ldots M_n$ = #r[r,n]

Quantitative analysis via Z [Fujita'16]

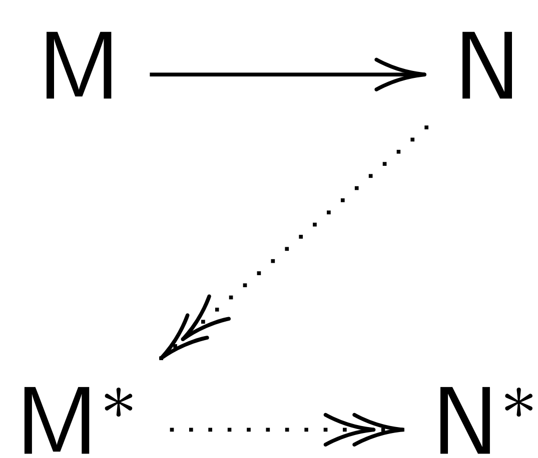
- From a bound of steps in Z property, we can give a bound of steps in CR

- Main lemma: $M =_A N \quad \Rightarrow \quad N \to^{\mathrm{Main}(M=N)} M^{r*}$

  - where Main(M=N) is defined from Rev, Mon and maximum term size in M=N

$$M \to N \quad \Rightarrow \quad N \to^{\mathrm{Rev}(|M|)} M^*$$

$$M \to^n N \quad \Rightarrow \quad M^* \to^{\mathrm{Mon}(|M|,n)} N^*$$



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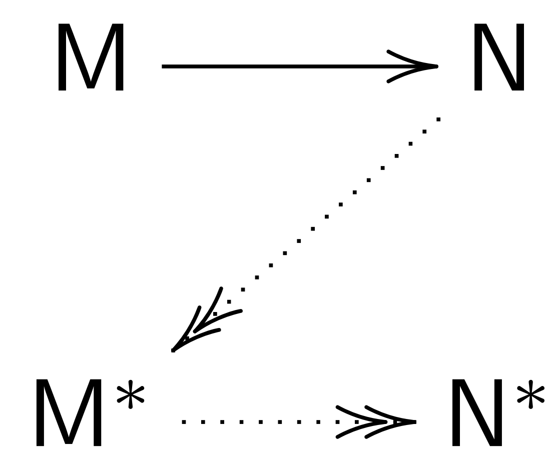

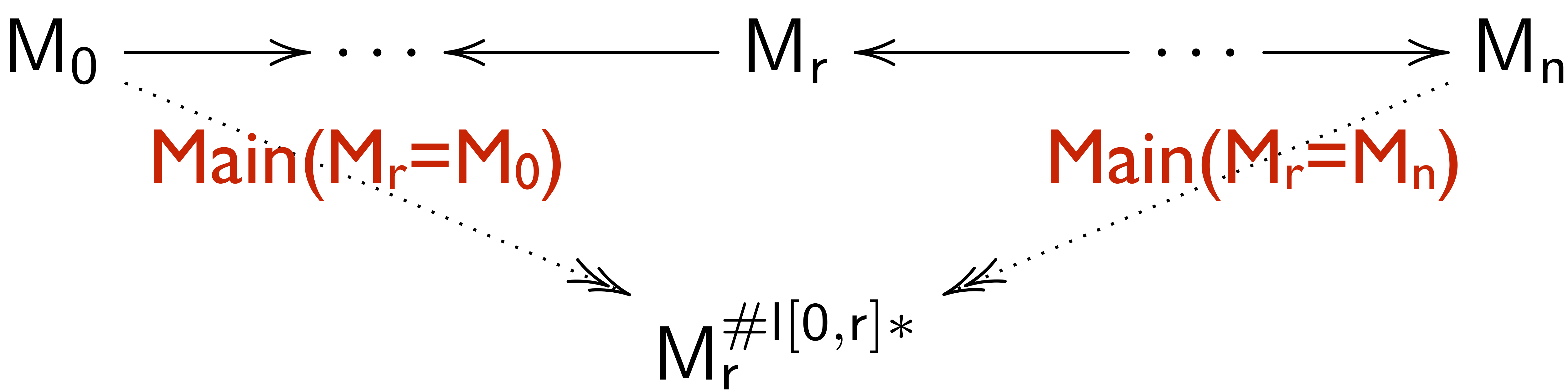

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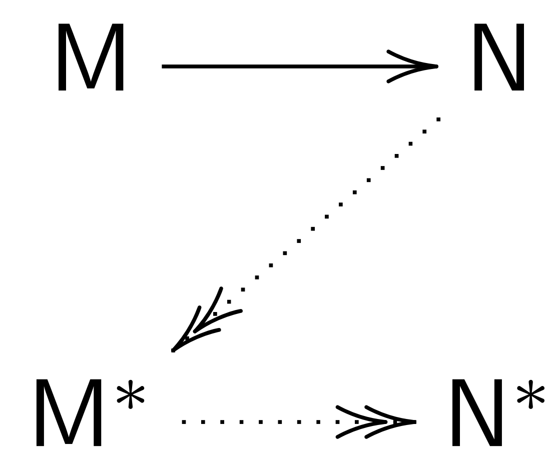
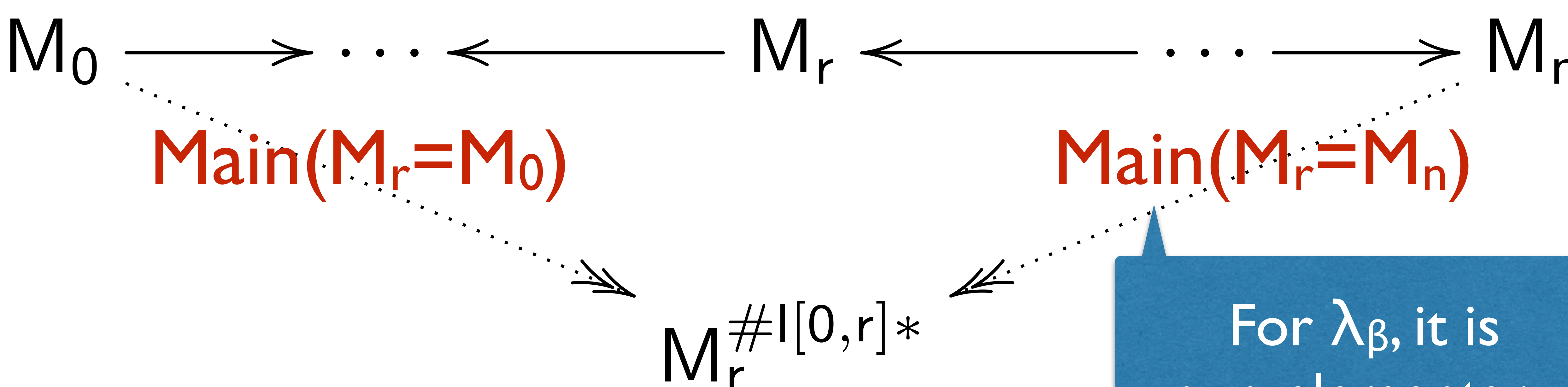




For λβ, it is non-elementary

Quantitative analysis via compositional Z [Fujita&N'16]

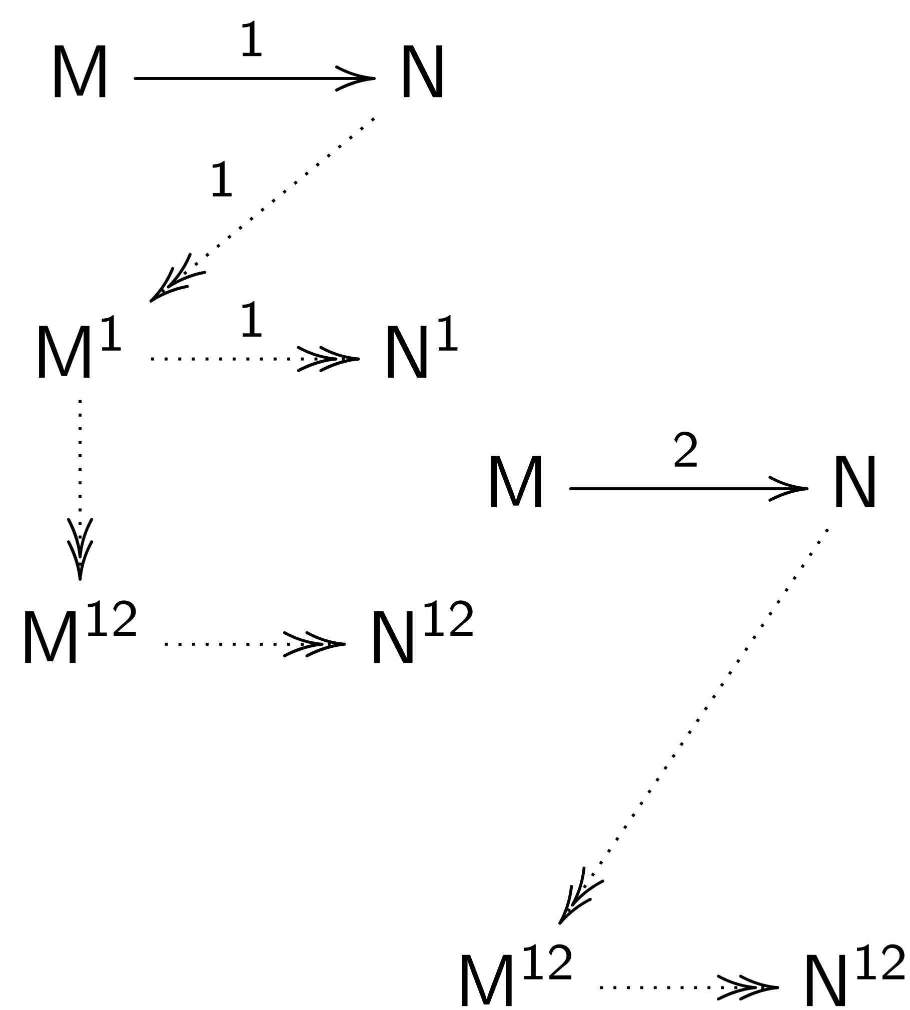
- From bounds of steps in compositional Z (given below), we can give a bound Main(M=N) in CR

$$M \to_1 N \quad \Rightarrow \quad N \to^{\mathrm{Rev1}(|M|)} M^1$$

$$M \to^{\mathrm{Eval2}(|M|)} M^2$$

$$M \to_2 N \quad \Rightarrow \quad N \to^{\mathrm{Rev2}(|M|)} M^{12}$$

$$M \to^n N \quad \Rightarrow \quad M^{12} \to^{\mathrm{Mon}(|M|,n)} N^{12}$$



Quantitative analysis via compositional Z

Main(M=N) is defined as

$\mathrm{Main}(M \leftarrow N) = 1$
$\mathrm{Main}(M \rightarrow_1 N) = \mathrm{Rev1}(|M|) + \mathrm{Eval2}(|M'|)$
$\mathrm{Main}(M \rightarrow_2 N) = \mathrm{Rev2}(|M|)$
$\mathrm{Main}(M = P \leftarrow Q) = \mathrm{Main}(M=P) + 1$
$\mathrm{Main}(M = P \rightarrow_1 Q) = \mathrm{Mon}(n, \mathrm{Main}(M=P)) + \mathrm{Eval2}(n) + \mathrm{Rev1}(n)$
$\mathrm{Main}(M = P \rightarrow_2 Q) = \mathrm{Mon}(n, \mathrm{Main}(M=P)) + \mathrm{Rev2}(n)$

where n = maximum term size in M=N

Summary

- Confluence of λ with permutative conversions becomes much simpler with compositional Z
- Compositional Z suggests (quasi-)modular proofs of confluence
- Quantitative analysis for CR via Z can be extended to compositional Z

- K. Nakazawa and K. Fujita. Compositional Z: confluence proofs for permutative conversion. Studia Logica, to appear.
- K. Fujita and K. Nakazawa. Church-Rosser Theorem and Compositional Z-Property. In Proceedings of 33rd JSSST, 2016

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“I feel that the new proofs (…) are more beautiful than those we started with, and this is my actual motivation.”

— [Pollack 1995]

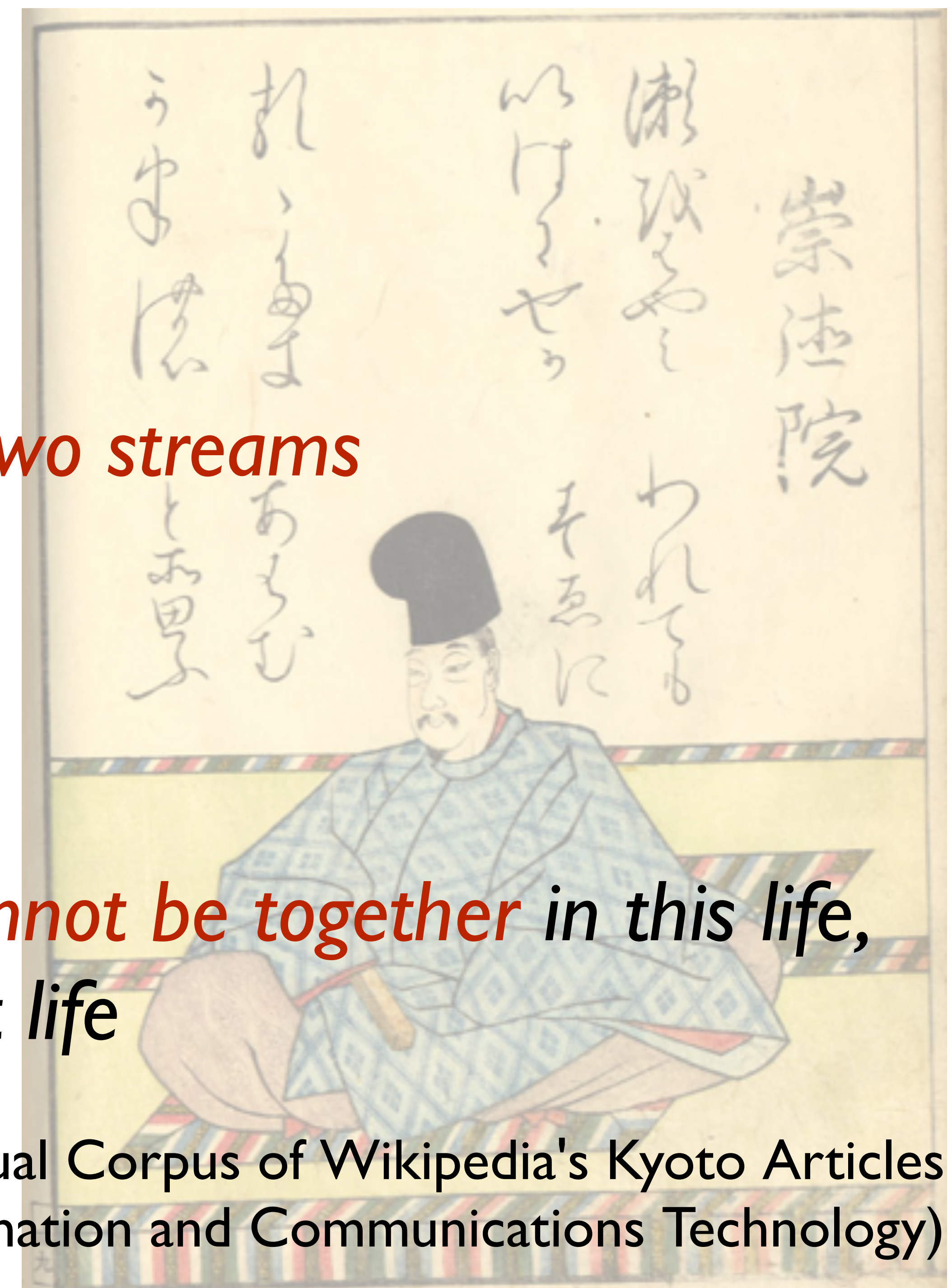
A Classical Japanese Poem

composed by Sutoku-In (崇徳院) in 12th cent.

瀬を早み 岩にせかるる滝川の
われても末に 逢はむとぞ思ふ

(direct translation)
A stream of the river separates into two streams
after hitting the rock,
but it will become one stream again

(that is,)
although if I love someone but we cannot be together in this life,
I can be together with her in the next life



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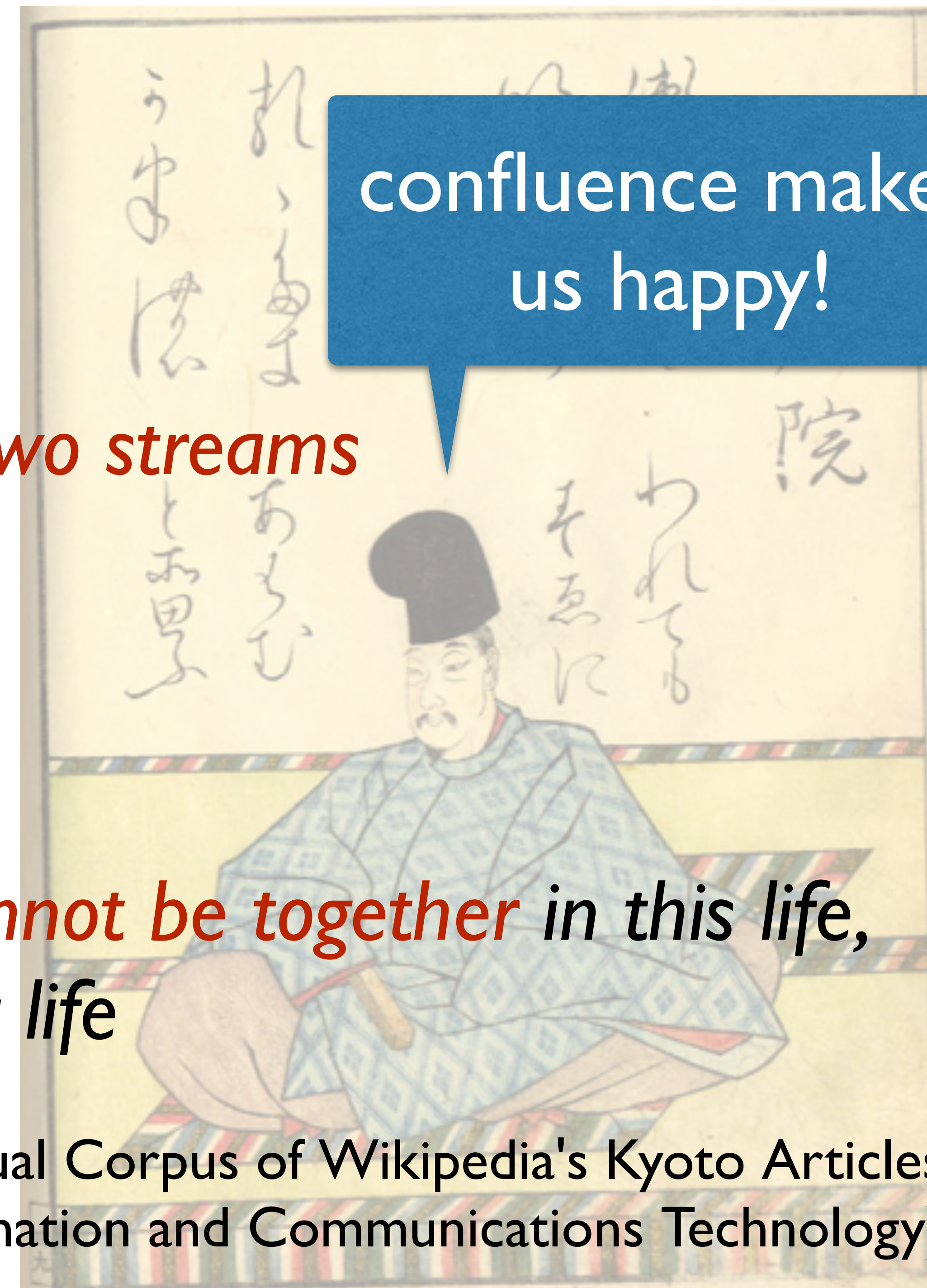

confluence makes us happy!

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