Elimination of binary choice sequences

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Choice sequences

The theory of choice sequences **CS** was introduced by Troelstra (1968) and extensively studied by Kreisel and Troelstra (1970).

Formal systems for some branches of intuitionistic analysis. *Annals of Mathematical Logic*, 1(3):229–387, 1970.

- A sequence $f : \mathbf{N} \to \mathbf{N}$ is **lawlike** if we know a law (finite information) to generate it, e.g. recursive functions.
- **Choice sequences** are sequences of natural numbers which are more general than lawlike sequences.
- Operations on choice sequences are continuous in a strong sense: the continuous choice and bar induction are theorems of **CS**.
- **CS** can be considered as a formal system for Brouwer's intuitionism.

Elimination choice sequences

- Kreisel and Troelstra (1970) showed that **CS** is conservative extension of its lawlike part **IDB** using the **elimination translation**.
- Fourman (1982) observed that forcing over the site whose underlying category is a monoid of continuous functions $\mathbf{CONT}(\mathbf{N}^{\mathbf{N}}, \mathbf{N}^{\mathbf{N}})$ on Baire space with open cover topology corresponds to the elimination translation by Kreisel and Troelstra.
  - The correspondence between forcing and elimination translation was shown explicitly by van der Hoeven and Moerdijk (1982) by formalizing a fragment of sheaf semantics in **IDB**.

Outline

1. Theory of binary choice sequences BCS

2. Sheaf semantics of BCS

3. Formalization of sheaf semantics in EL

4. Elimination of choice sequences

Uniformly continuous functions on $2^{\mathbb{N}}$

$f : \mathbf{2}^{\mathbb{N}} \to \mathbb{N}$ is uniformly continuous

$$\iff \exists n \in \mathbb{N} \forall a, b \in \mathbf{2}^{\mathbb{N}} \left[\overline{a}n = \overline{b}n \to f(a) = f(b) \right]$$
$$\iff \exists n \in \mathbb{N} \forall a \in \mathbf{2}^{\mathbb{N}} \left[f(a) = f(\overline{a}n * 0^{\omega}) \right]$$

where $\overline{a}n * 0^{\omega} \equiv \overline{a}n * \langle 0, 0, 0, \cdots$.

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- Such a tree can be coded as a natural numbers.
- A uniformly continuous function $f : \mathbf{2}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ can be coded as a sequence of natural numbers.
- All these notions as well as composition of uniformly continuous function on $\mathbf{2}^{\mathbb{N}}$ and applications of uniformly continuous functions to binary sequences can be definable in **EL**.

EL: Elementary analysis

Elementary analysis **EL** is an (conservative) extension of **HA** based on two sorted intuitionistic predicate logic:

Language

- $\mathbf{N}, \mathbf{N}^{\mathbf{N}}$: sorts for natural numbers and lawlike sequences;
- $x, y, z, \cdots$: numerical variables;
- $a, b, c, \cdots$: lawlike variables;
- Symbols for all primitive recursive functions including 0 and S;
- App, λx, Rec, $=_{\mathbf{N}}$.

Terms

$$(\mathbf{N}\text{-Term}) \quad t, s ::= x \mid 0 \mid St \mid f(t_0, \ldots, t_{n-1}) \mid \mathrm{App}(\varphi, t) \mid \mathrm{Rec}(t, \varphi, s)$$

$$(\mathbf{N}^{\mathbf{N}}\text{-Term}) \quad \varphi ::= a \mid \lambda x.t$$

Formulas

$$A, B ::= t =_{\mathbf{N}} s \mid A \wedge B \mid A \to B \mid \forall x A \mid \exists x A \mid \forall a A \mid \exists a A$$

Axioms

EL has the axioms and rules of intuitionistic predicate logic with equality (on $\mathbf{N}$) and the following axioms:

(CON) $(\lambda x.t)(x) = t$

(REC) $\mathrm{Rec}(x, a, 0) = x, \quad \mathrm{Rec}(x, a, Sy) = a(\mathrm{Rec}(x, a, y), y)$

(PRIM) Defining equations for all primitive recursive functions.

(S) $0 \neq S0, \quad Sx = Sy \to x = y$

(IND) $A(0) \wedge \forall x\, [A(x) \to A(Sx)] \to \forall x A(x)$

(AC$_{00}$!) $\forall x \exists ! y A(x, y) \to \exists a \forall x A(x, a(x))$

BCS: Theory of binary choice sequences

BCS is an extension of **EL** with an additional sort $\mathbf{Ch}$:

Language

- The sort $\mathbf{Ch}$ for choice sequences;
- $\alpha, \beta, \gamma, \ldots$: choice sequence variables;
- Constants App^C, Rec^C, $\lambda^C x$.

Terms

$$(\mathbf{N})\quad t, s ::= x \mid 0 \mid St \mid f(t_0, \ldots, t_{n-1}) \mid \mathrm{App}(\varphi, t) \mid \mathrm{Rec}(t, \varphi, s) \mid \mathrm{App}^C(\sigma, t) \mid \mathrm{Rec}^C(t, \sigma, s)$$

$$(\mathbf{N}^{\mathbf{N}})\quad \varphi ::= a \mid \varphi[x/t] \mid \lambda x.t \quad (t \text{ does not contain choice variables})$$

$$(\mathbf{Ch})\quad \sigma ::= \alpha \mid \lambda^C x.t$$

Formulas

Formulas of **BCS** are built up as in **EL** but extended with quantifiers $\forall\alpha$ and $\exists\alpha$.

Axioms

- Logical axioms are those of **EL** and axioms of quantifiers for choice sequences.
- Non-logical axioms include those of **EL** with respect to the language of **BCS** except $\mathrm{AC}_{00}!$, which is restricted to formulas without free choice sequence variables, and the following:

(CON$^{\mathrm{C}}$) $(\lambda x.t)(x) = t$

(REC$^{\mathrm{C}}$) $\mathrm{Rec}^{C}(x, \alpha, 0) = x, \quad \mathrm{Rec}^{C}(x, \alpha, Sy) = \alpha(\mathrm{Rec}^{C}(x, \alpha, y), y)$

Axioms

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(REC^C) $\mathrm{Rec}^C(x, \alpha, 0) = x, \quad \mathrm{Rec}^C(x, \alpha, Sy) = \alpha(\mathrm{Rec}^C(x, \alpha, y), y)$

(ANL) $A(\alpha) \to \exists a \left[\exists \beta \in \mathbf{2}^{\mathbf{N}} \alpha = a|\beta \wedge \left(\forall \beta \in \mathbf{2}^{\mathbf{N}}\right) A(a|\beta)\right]$

where $\alpha \in \mathbf{2}^{\mathbf{N}} \equiv \forall x \left[\alpha x = 0 \vee \alpha x = 1\right]$.

(FC-C) $\forall \alpha \in \mathbf{2}^{\mathbf{N}} \exists \beta \, A(\alpha, \beta) \to \exists a \forall \alpha \in \mathbf{2}^{\mathbf{N}} A(\alpha, a|\alpha)$

(FC-F) $\forall \alpha \in \mathbf{2}^{\mathbf{N}} \exists b \, A(\alpha, b) \to \exists n \forall i < 2^n \exists b \forall \alpha \in \mathbf{2}^{\mathbf{N}} A(\mathrm{cons}_{(n,i)} \, |\alpha, b)$.

Consequences of axioms of BCS

Proposition

Quantifications over choice sequences can be reduced to quantifications over binary choice sequences.

$$\textbf{BCS} \vdash \forall\alpha A(\alpha) \leftrightarrow \forall a \forall\alpha \in \mathbf{2}^{\mathbf{N}} A(a|\alpha).$$

Proposition

Fan continuity is derivable from FC-F*.*

$$\textbf{BCS} \vdash \forall\alpha \in \mathbf{2}^{\mathbf{N}} \exists x\, A(\alpha, x) \rightarrow \exists n \forall\alpha \in \mathbf{2}^{\mathbf{N}} \exists y \forall\beta \in \mathbf{2}^{\mathbf{N}} \beta \in \overline{\alpha}n \rightarrow A(\beta, y).$$

Proposition

$$\textbf{BCS} \vdash \neg \left[\forall\alpha \in \mathbf{2}^{\mathbf{N}} \exists a\, \alpha = a\right] \;\&\; \forall\alpha \in \mathbf{2}^{\mathbf{N}} \neg\neg\exists a\, \alpha = a.$$

where $(\alpha = a) \equiv \forall x\, [\alpha x = ax]$*.*

Outline

Open cover topology over the monoid $\mathbf{UCONT}(2^{\mathbb{N}}, 2^{\mathbb{N}})$

The class $\mathbf{UCONT}(\mathbf{2}^{\mathbb{N}}, \mathbf{2}^{\mathbb{N}})$ of uniformly continuous functions on Cantor space $\mathbf{2}^{\mathbb{N}}$ is a monoid with unit $\mathbf{1} \stackrel{\text{def}}{=} \mathrm{id}_{\mathbf{2}^{\mathbb{N}}}$ and composition $\circ$ as operation. We regard $\mathbf{M} \stackrel{\text{def}}{=} \mathbf{UCONT}(\mathbf{2}^{\mathbb{N}}, \mathbf{2}^{\mathbb{N}})$ as a single object category $\{*\}$.

Definition

Open cover topology on $\mathbf{M}$ is generated by a coverage base $\mathcal{J}$ defined by

$$\mathcal{J}(*) \stackrel{\text{def}}{=} \left\{ S_n \subseteq \mathbf{UCONT}(\mathbf{2}^{\mathbb{N}}, \mathbf{2}^{\mathbb{N}}) \mid n \in \mathbf{N} \right\},$$

$$S_n \stackrel{\text{def}}{=} \{ \mathrm{cons}_u \mid u \in \mathbf{2}^* \ \& \ |u| = n \},$$

$$\mathrm{cons}_u : a \mapsto u * a.$$

N.B. We work in the coverage base $\mathcal{J}$ instead of the Grothendieck topology it generates.

Sheaves over the site $(\mathbf{M}, \mathcal{J})$ (where $\mathbf{M} = \mathbf{UCONT}(\mathbf{2}^{\mathbb{N}}, \mathbf{2}^{\mathbb{N}})$)

- A presheaf on $\mathbf{M}$ is an $\mathbf{M}$-set, i.e. a pair $(X, \upharpoonright)$ of set X and action $\upharpoonright : X \times \mathbf{M} \to X$ so that

$$x \upharpoonright \mathbf{1} = x,$$
$$(x \upharpoonright f) \upharpoonright g = x \upharpoonright (f \circ g).$$

  A morphism of $\mathbf{M}$-sets $(X, \upharpoonright)$ and $(Y, \upharpoonright')$ is function $\alpha : X \to Y$ which preserves action: $\alpha(x \upharpoonright f) = \alpha(x) \upharpoonright' f$.
- Given an $\mathbf{M}$-set $(X, \upharpoonright)$, a compatible family is just a family $(x_a)_{a \in S}$ of elements of X indexed by some $S \in \mathcal{J}$.
- Given a compatible family $(x_a)_{a \in S}$ ($S \in \mathcal{J}$), an amalgamation of the family is an element $x \in X$ such that $x \upharpoonright a = x_a$ for all $a \in S$.
- An $\mathbf{M}$-set is separated if every compatible family has at most one amalgamation; it is a sheaf if every compatible family has a unique amalgamation.

Sheaves over the site $(\mathbf{M}, \mathcal{J})$ (where $\mathbf{M} = \mathbf{UCONT}(\mathbf{2}^{\mathbb{N}}, \mathbf{2}^{\mathbb{N}})$)

Given a separated $\mathbf{M}$-set $(X, \upharpoonright)$, we can associate a sheaf $L(X, \upharpoonright)$, the sheafification of $(X, \upharpoonright)$. The elements of $L(X, \upharpoonright)$ are equivalence classes of compatible families $(x_a)_{a \in S}$ ($S \in \mathcal{J}$), where the equivalence is defined by

$$(x_a)_{a \in S} \sim (y_b)_{b \in T} \overset{\text{def}}{\iff} \exists U \in \mathcal{J} \forall c \in U \exists a \in S \exists b \in T \exists f, g \in \mathbf{M}$$
$$c = a \circ f = b \circ g \;\&\; x_a \upharpoonright f = y_b \upharpoonright g.$$

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Proposition

Let X be a set, and let $(X, \upharpoonright_C)$ be a constant $\mathbf{M}$-set with trivial action $x \upharpoonright_C f = x$. Then, $(X, \upharpoonright_C)$ is separated. Moreover

1. *The sheafification $L(X, \upharpoonright_C)$ is (isomorphic to) the set $\mathbf{UCONT}(\mathbf{2}^{\mathbb{N}}, X_{\text{disc}})$ of uniformly continuous functions with respect to the discrete topology on X with function composition as action.*
2. *For any two sets X, Y, there is a bijective correspondence between functions $f : X \to Y$ and morphisms $\alpha : L(X, \upharpoonright_C) \to L(Y, \upharpoonright_C)$.*

Interpretation of BCS in $\mathbf{Sh}(\mathbf{UCONT}(\mathbf{2}^{\mathbb{N}}, \mathbf{2}^{\mathbb{N}}), \mathcal{J})$

Let $\mathbf{N}, \mathbf{N}^{\mathbf{N}}, \mathbf{Ch}$ denote the sorts for natural numbers, lawlike sequences and choice sequences resp. Those sorts are interpreted as following sheaves:

- $[\![\mathbf{N}]\!]$: sheafification of the constant M-set $(\mathbb{N}, \upharpoonright_C)$.
- $[\![\mathbf{N}^{\mathbf{N}}]\!]$: sheafification of the constant M-set $(\mathbb{N}^{\mathbb{N}}, \upharpoonright_C)$.
- $[\![\mathbf{Ch}]\!]$: the exponential $[\![\mathbf{N}]\!]^{[\![\mathbf{N}]\!]}$ in $\mathbf{Sh}(\mathbf{M}, \mathcal{J})$.

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- $[\![\mathbf{N}^{\mathbf{N}}]\!]$: sheafification of the constant M-set $(\mathbb{N}^{\mathbb{N}}, 1_C)$.
- $[\![\mathbf{Ch}]\!]$: the exponential $[\![\mathbf{N}]\!]^{[\![\mathbf{N}]\!]}$ in $\mathbf{Sh}(\mathbf{M}, \mathcal{J})$.

Lemma

1. $[\![\mathbf{N}]\!]$ *is the set* $\mathbf{UCONT}(\mathbf{2}^{\mathbb{N}}, \mathbb{N}_{\mathrm{disc}})$ *of uniformly continuous functions with composition as action.*
2. $[\![\mathbf{N}^{\mathbf{N}}]\!]$ *is the set* $\mathbf{UCONT}(\mathbf{2}^{\mathbb{N}}, {\mathbb{N}^{\mathbb{N}}}_{\mathrm{disc}})$ *of uniformly continuous functions with composition as action.*
3. $[\![\mathbf{Ch}]\!]$ *is the set* $\mathbf{UCONT}(\mathbf{2}^{\mathbb{N}}, \mathbb{N}^{\mathbb{N}})$ *of uniformly continuous functions with composition as action.*

Interpretation of BCS in $\mathbf{Sh}(\mathbf{UCONT}(\mathbf{2}^{\mathbb{N}}, \mathbf{2}^{\mathbb{N}}), \mathcal{J})$

A term in context $\Gamma \vdash t : S$ (where $\Gamma \equiv x_1 : S_1, \cdots, x_n : S_n$ and $S, S_1, \cdots, S_n$ are sorts of **BCS**) is interpreted as a morphism $[\![\Gamma \vdash t : S]\!] : [\![\Gamma]\!] \to [\![S]\!]$, where $[\![\Gamma]\!] \equiv [\![S_1]\!] \times [\![S_n]\!]$:

$$
\begin{aligned}
[\![\Gamma \vdash x_i : S_i]\!] &\overset{\text{def}}{=} \pi_i : [\![\Gamma]\!] \to [\![S_i]\!], \\
[\![\Gamma \vdash f(t_0, \cdots, t_{n-1})]\!] &\overset{\text{def}}{=} f \circ \langle [\![t_0]\!], \cdots, [\![t_{n-1}]\!] \rangle, \\
[\![\Gamma \vdash \mathrm{App}(\varphi, t)]\!] &\overset{\text{def}}{=} \mathrm{ev}^{\mathrm{Sets}} \circ \langle [\![\varphi]\!], [\![t]\!] \rangle, \\
[\![\Gamma \vdash \mathrm{App}^C(\varphi, t)]\!] &\overset{\text{def}}{=} \mathrm{ev} \circ \langle [\![\varphi]\!], [\![t]\!] \rangle, \\
[\![\Gamma \vdash \mathrm{Rec}(t, \varphi, s)]\!] &\overset{\text{def}}{=} \mathrm{I}^{\mathrm{Sets}} \circ \langle [\![t]\!], [\![\varphi]\!], [\![s]\!] \rangle, \\
[\![\Gamma \vdash \mathrm{Rec}^C(t, \varphi, s)]\!] &\overset{\text{def}}{=} \mathrm{I} \circ \langle [\![t]\!], [\![\varphi]\!], [\![s]\!] \rangle, \\
[\![\Gamma \vdash \lambda x.t]\!] &\overset{\text{def}}{=} \Lambda^{\mathrm{Sets}}([\![t]\!]), \\
[\![\Gamma \vdash \lambda^C x.t]\!] &\overset{\text{def}}{=} \Lambda([\![t]\!]).
\end{aligned}
$$

where I, ev and Λ are the iterator, evaluation morphism and exponential transpose respectively.

Interpretation of BCS in $\mathbf{Sh}(\mathbf{UCONT}(\mathbf{2}^{\mathbb{N}}, \mathbf{2}^{\mathbb{N}}), \mathcal{J})$

The truth of formula $\Gamma \vdash A$ in context $\Gamma \equiv x_1 : S_1, \ldots, x_n : S_n$ can be defined by forcing relation $\vec{\zeta} \Vdash \Gamma \vdash A$ between finite list $\vec{\zeta} \equiv \zeta_1, \ldots, \zeta_n$ of elements $(\zeta_i \in [\![S_i]\!])$ and formula $\Gamma \vdash A$ in context:

1. $\vec{\zeta} \Vdash \Gamma \vdash t = s \overset{\text{def}}{\Longleftrightarrow} [\![t]\!](\vec{\zeta}) = [\![s]\!](\vec{\zeta})$;
2. $\vec{\zeta} \Vdash \Gamma \vdash A \wedge B \overset{\text{def}}{\Longleftrightarrow} \left(\vec{\zeta} \Vdash \Gamma \vdash A\right) \wedge \left(\vec{\zeta} \Vdash \Gamma \vdash B\right)$;
3. $\vec{\zeta} \Vdash \Gamma \vdash A \rightarrow B \overset{\text{def}}{\Longleftrightarrow} \forall f \in \mathbf{M} \left(\vec{\zeta} \circ f \Vdash \Gamma \vdash A \rightarrow \vec{\zeta} \circ f \Vdash \Gamma \vdash B\right)$;
4. $\vec{\zeta} \Vdash \Gamma \vdash \forall x : S\, A \overset{\text{def}}{\Longleftrightarrow} \forall f \in \mathbf{M}\, \forall g \in [\![S]\!]\, \vec{\zeta} \circ f, g \Vdash \Gamma, x : S \vdash A$;
5. $\vec{\zeta} \Vdash \Gamma \vdash \exists x : S\, A \overset{\text{def}}{\Longleftrightarrow} \exists T \in \mathcal{J}\, \forall g \in T \exists f \in [\![S]\!]$
$\vec{\zeta} \circ g, f \Vdash \Gamma, x : S \vdash A$.

Some refinements

- For the truth of $\Gamma \vdash A$, it suffices to consider list $\vec{\zeta}$ such that if S_i is either $\mathbf{N}$ or $\mathbf{N}^{\mathbf{N}}$ then $\zeta_i \in [\![S_i]\!]$ is a constant function, i.e. it can be identified with element of $\mathbb{N}$ or $\mathbb{N}^{\mathbb{N}}$
- For the clauses for quantifiers, if the sort S of variable is either $\mathbf{N}$ or $\mathbf{N}^{\mathbf{N}}$, quantifications over $[\![S]\!]$ can be restricted to quantifications over $\mathbb{N}$ and $\mathbb{N}^{\mathbb{N}}$.
- The base case is equivalent to the following.

$$\begin{aligned} &\vec{a} \Vdash \Gamma \vdash t = s \\ &\overset{\text{def}}{\iff} [\![t]\!](\vec{a}) = [\![s]\!](\vec{a}) \\ &\iff \forall b \in \mathbf{2}^{\mathbb{N}} [\![t^N[\Gamma/\vec{a}(b)]]\!]^* = [\![s^N[\Gamma/\vec{a}(b)]]\!]^*. \end{aligned}$$

where $t^N[\Gamma/\vec{a}(b)]$ is obtained from t by replacing λ^C by λ, and x_i by $a_i(b)$ (regarded as formal symbols.). The resulting term is informally interpreted in the base set theory, which is denoted by $[\![t^N[\Gamma/\vec{a}(b)]]\!]^*$.

Outline

Forcing in EL

1. $\vec{a} \Vdash \Gamma \vdash t = s \overset{\text{def}}{\iff} \forall b \in \mathbf{2}^{\mathbb{N}} [\![t^N[\Gamma/\vec{a}(b)]]\!]^* = [\![s^N[\Gamma/\vec{a}(a)]]\!]^*$;
2. $\vec{a} \Vdash \Gamma \vdash A \wedge B \overset{\text{def}}{\iff} (\vec{a} \Vdash \Gamma \vdash A) \wedge (\vec{a} \Vdash \Gamma \vdash B)$;
3. $\vec{a} \Vdash \Gamma \vdash A \to B \overset{\text{def}}{\iff} \forall f \in \mathbf{M}\ (\vec{a} \circ f \Vdash \Gamma \vdash A \to \vec{a} \circ f \Vdash \Gamma \vdash B)$;
4. $\vec{a} \Vdash \Gamma \vdash \forall x : S\, A \overset{\text{def}}{\iff} \forall f \in \mathbf{M}\, \forall g \in [\![S]\!]\ \vec{a} \circ f, g \Vdash \Gamma, x : S \vdash A$;
5. $\vec{a} \Vdash \Gamma \vdash \exists x : S\, A \overset{\text{def}}{\iff} \exists T \in \mathcal{J}\, \forall g \in T \exists f \in [\![S]\!]$
   $\vec{a} \circ g, f \Vdash \Gamma, x : S \vdash A$.

The sheaf semantics for **BCS** involves following notions:

- Uniformly continuous functions of the types $\mathbf{2}^{\mathbb{N}} \to \mathbb{N}$, $\mathbf{2}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$, and $\mathbf{2}^{\mathbb{N}} \to \mathbf{2}^{\mathbb{N}}$.
- Compositions between them.
- Applications of those functions to elements of $\mathbf{2}^{\mathbb{N}}$.

Forcing in EL

By a **context** Γ, we mean a finite list of choice sequence variables. Let A be a formula of **BCS** in a context Γ, where $\Gamma \equiv \alpha_0, \dots, \alpha_{n-1}$ Let $\vec{\varphi} \equiv \varphi_0, \dots, \varphi_{n-1}$ be a list of lawlike terms of **EL**. We define a formula $\vec{\varphi} \Vdash \Gamma \vdash A$ of **EL** by induction on A.

1. $\vec{\varphi} \Vdash \Gamma \vdash u = v \stackrel{\text{def}}{\equiv} \forall a \in \mathbf{2}^{\mathbf{N}}\ u^N[\Gamma/\vec{\varphi}|a] = v^N[\Gamma/\vec{\varphi}|a]$;
2. $\vec{\varphi} \Vdash \Gamma \vdash A \wedge B \stackrel{\text{def}}{\equiv} (\vec{\varphi} \Vdash \Gamma \vdash A) \wedge (\vec{\varphi} \Vdash \Gamma \vdash B)$;
3. $\vec{\varphi} \Vdash \Gamma \vdash A \to B \stackrel{\text{def}}{\equiv} \forall a \in K_C\ (\vec{\varphi} \cdot a \Vdash \Gamma \vdash A \to \vec{\varphi} \cdot a \Vdash \Gamma \vdash B)$;
4. $\vec{\varphi} \Vdash \Gamma \vdash \forall \mathsf{a} A \stackrel{\text{def}}{\equiv} \forall \mathsf{b}\, \vec{\varphi} \Vdash \Gamma \vdash A[\mathsf{a}/\mathsf{b}]$;
5. $\vec{\varphi} \Vdash \Gamma \vdash \forall \alpha A \stackrel{\text{def}}{\equiv} \forall a \in K_C\, \forall b\, \vec{\varphi} \cdot a, b \Vdash \Gamma, \beta \vdash A[\alpha/\beta]$;
6. $\vec{\varphi} \Vdash \Gamma \vdash \exists \mathsf{a} A \stackrel{\text{def}}{\equiv} \exists d\, \forall i < 2^d \exists \mathsf{b}\, \vec{\varphi} \cdot \mathrm{cons}_{(d,i)} \Vdash \Gamma \vdash A[\mathsf{a}/\mathsf{b}]$;
7. $\vec{\varphi} \Vdash \Gamma \vdash \exists \alpha A \stackrel{\text{def}}{\equiv} \exists d\, \forall i < 2^d \exists a\, \vec{\varphi} \cdot \mathrm{cons}_{(d,i)}, a \Vdash \Gamma, \beta \vdash A[\alpha/\beta]$.

Theorem (Soundness)

Let A *be a formula of* **BCS** *in the context* $\Gamma \equiv \alpha_0, \ldots, \alpha_{n-1}$. *Then*

$$\textbf{BCS} \vdash A \implies \textbf{EL} \vdash \forall a_0, \ldots, a_{n-1} \left[\vec{a} \Vdash \Gamma \vdash A \right],$$

where $\vec{a} \equiv a_0, \ldots, a_{n-1}$.

Outline

Definition

The class $\mathbf{Form}(\mathbb{B})$ of formulas is defined by the clauses defining the formulas of **BCS** together with the following clause:

- If $A \in \mathbf{Form}(\mathbb{B})$, then $(\forall \alpha \in \mathbb{B})A, (\exists \alpha \in \mathbb{B})A \in \mathbf{Form}(\mathbb{B})$.

N.B. $(\forall \alpha \in \mathbb{B})$ and $(\exists \alpha \in \mathbb{B})$ are added as primitive symbols, not as abbreviations of quantifiers for choice sequence followed by a predicate $\mathbf{2}^{\mathbf{N}}$.

Elimination Translation

A mapping $A \mapsto \ulcorner A \urcorner$ of formulas A in $\mathrm{Form}(\mathbb{B})$ *without free choice sequence variables* to formulas $\ulcorner A \urcorner$ of **EL** is defined as follows:

$$
\begin{aligned}
\ulcorner u = v \urcorner &\equiv u^N = v^N, \\
\ulcorner A \wedge B \urcorner &\equiv \ulcorner A \urcorner \wedge \ulcorner B \urcorner, \\
\ulcorner A \rightarrow B \urcorner &\equiv \ulcorner A \urcorner \rightarrow \ulcorner B \urcorner, \\
\ulcorner \forall \mathsf{a} A \urcorner &\equiv \forall \mathsf{a} \ulcorner A \urcorner, \\
\ulcorner \forall \alpha A \urcorner &\equiv \forall a \ulcorner \forall \gamma \in \mathbb{B} A[\alpha/a|\gamma] \urcorner, \\
\ulcorner \exists \mathsf{a} A \urcorner &\equiv \exists \mathsf{a} \ulcorner A \urcorner, \\
\ulcorner \exists \alpha A \urcorner &\equiv \exists a \ulcorner \forall \gamma \in \mathbb{B} A[\alpha/a|\gamma] \urcorner,
\end{aligned}
$$

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\begin{aligned}
\ulcorner\forall\alpha\in\mathbb{B}u=v\urcorner &\equiv \forall a\in\mathbf{2}^{\mathbf{N}}u[\alpha/a]^{N}=v[\alpha/a]^{N},\\
\ulcorner\forall\alpha\in\mathbb{B}A\wedge B\urcorner &\equiv \ulcorner\forall\alpha\in\mathbb{B}A\urcorner\wedge\ulcorner\forall\alpha\in\mathbb{B}B\urcorner,\\
\ulcorner\forall\alpha\in\mathbb{B}A\to B\urcorner &\equiv \forall a\in K_C\left(\ulcorner\forall\gamma\in\mathbb{B}A[\alpha/a|\gamma]\urcorner\to\ulcorner\forall\gamma\in\mathbb{B}B[\alpha/a|\gamma]\urcorner\right),\\
\ulcorner\forall\alpha\in\mathbb{B}\forall\mathsf{a}A\urcorner &\equiv \forall\mathsf{b}\ulcorner\forall\alpha\in\mathbb{B}A[\mathsf{a}/\mathsf{b}]\urcorner,\\
\ulcorner\forall\alpha\in\mathbb{B}\forall\beta A\urcorner &\equiv \forall a\forall b\in K_C\ulcorner\forall\gamma\in\mathbb{B}A[\alpha/b|\gamma,\beta/a|\gamma]\urcorner,\\
\ulcorner\forall\alpha\in\mathbb{B}\forall\beta\in\mathbb{B}A\urcorner &\equiv \forall a,b\in K_C\ulcorner\forall\gamma\in\mathbb{B}A[\alpha/b|\gamma,\beta/a|\gamma]\urcorner,\\
\ulcorner\forall\alpha\in\mathbb{B}\exists\mathsf{a}A\urcorner &\equiv \exists d\forall i<\mathbf{2}^{d}\exists\mathsf{b}\ulcorner\forall\gamma\in\mathbb{B}A[\alpha/\operatorname{cons}_{(d,i)}|\gamma,\mathsf{a}/\mathsf{b}]\urcorner,\\
\ulcorner\forall\alpha\in\mathbb{B}\exists\beta A\urcorner &\equiv \exists a\ulcorner\forall\gamma\in\mathbb{B}A[\alpha/\gamma,\beta/a|\gamma]\urcorner,\\
\ulcorner\forall\alpha\in\mathbb{B}\exists\beta\in\mathbb{B}A\urcorner &\equiv \exists a\in K_C\ulcorner\forall\gamma\in\mathbb{B}A[\alpha/\gamma,\beta/a|\gamma]\urcorner,\\
\ulcorner\exists\alpha\in\mathbb{B}A\urcorner &\equiv \exists a\in K_C\ulcorner\forall\gamma\in\mathbb{B}A[\alpha/a|\gamma]\urcorner.
\end{aligned}
$$

Theorem

Let A *be a formula of* **BCS** *in a context* $\Gamma \equiv \alpha_0, \ldots, \alpha_{n-1}$*. Then*

$$\textbf{EL} \vdash \forall a_0, \ldots, a_{n-1} \, (\vec{a} \Vdash \Gamma \vdash A \leftrightarrow \ulcorner \forall \beta \in \mathbb{B} A[\Gamma/\vec{a}|\beta] \urcorner) \, .$$

where $A[\Gamma/\vec{a}|\beta] \equiv A[\alpha_0/a_0|\beta, \ldots, \alpha_{n-1}/a_{n-1}|\beta]$*.*

Corollary

Let A *be a formula of* **BCS** *which does not contain free choice sequence variables. Then*

$$\textbf{EL} \vdash (\Vdash A) \leftrightarrow \ulcorner A \urcorner,$$

where $(\Vdash A) \equiv (\langle\rangle \Vdash \langle\rangle \vdash A)$*.*

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Theorem

If A is a formula of **EL**, *then $\ulcorner A \urcorner \equiv A$. Thus* **BCS** $\vdash A \Rightarrow$ **EL** $\vdash A$.

The main results

Theorem

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Let A be a formula of **BCS** *which does not contain free choice sequence variables. Then*

$$\mathbf{BCS} \vdash A \iff \mathbf{EL} \vdash (\Vdash A)\,.$$

Future work

Clarify the connection between elimination translation and internal language.

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1. **EL** $\vdash \forall a_0, \ldots, a_{n-1} \left(\vec{a} \Vdash \Gamma \vdash A \leftrightarrow \ulcorner \forall \beta \in \mathbb{B} A[\Gamma/\vec{a}|\beta] \urcorner\right)$, where $A[\Gamma/\vec{a}|\beta] \equiv A[\alpha_0/a_0|\beta, \ldots, \alpha_{n-1}/a_{n-1}|\beta]$.

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2. On the other hand, we have a correspondence between forcing and derivability in the internal language of $\mathbf{Sh}(\mathbf{M}, \mathcal{J})$.

$$\vec{a} \Vdash \Gamma \vdash A \iff \vdash_{\mathbf{Sh}(\mathbf{M},\mathcal{J})} \forall \alpha \in \mathbf{2}^{\mathbf{N}} A[\Gamma/\vec{a}(\alpha)].$$

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3. The elimination translation seems to be a translation of forcing expressed in the internal language of $\mathbf{Sh}(\mathbf{M}, \mathcal{J})$ into the forcing expressed in the language of **EL**.

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4. Can we understand other elimination translations (choice sequences, lawlike sequences, binary lawlike sequences, etc) in the siminlar way by considering suitable sheaf category and theory of arithmetics?

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