Degrees of unsolvability in topological spaces with countable cs-networks

Takayuki Kihara

Department of Mathematics, University of California, Berkeley, USA

Joint Work with

Arno Pauly

Université Libre de Bruxelles, Belgium

Workshop on Mathematical Logic and its Application, Kyoto, Sep 17, 2016

Goal

Develop the theory of degrees of unsolvability in topological spaces
(including spaces which are non-metrizable, not second-countable, etc.)

Goal

Develop the theory of degrees of unsolvability in topological spaces (including spaces which are non-metrizable, not second-countable, etc.)

— What is the motivation?

- In previous works [1,2], we utilized a generalization of the theory of degrees of unsolvability to give (partial/complete) solutions to *preexisting open problems in other areas of mathematics*.
- We are looking for more applications — but currently, the theory itself is still far from complete. So many things are yet to be done, even in the very basic part.

[1] V. Gregoriades, T. Kihara, and K. M. Ng, *Turing degrees in Polish spaces and decomposability of Borel functions*, submitted.

[2] T. Kihara, and A. Pauly, *Point degree spectra of represented spaces*, submitted.

Definition

1. An (ω^ω-)representation of a set $\mathcal{X}$ is a partial surjection $\delta :\subseteq \omega^\omega \to \mathcal{X}$.
2. A topological space $\mathcal{X}$ is admissibly represented if it has a universal continuous representation δ, that is,

   ($\forall$ continuous $\rho :\subseteq \omega^\omega \to \mathcal{X}$)($\exists$ continuous $\nu :\subseteq \omega^\omega \to \omega^\omega$) such that $\rho = \delta \circ \nu$.

Definition

1. An (ω^ω-)representation of a set $\mathcal{X}$ is a partial surjection $\delta :\subseteq \omega^\omega \to \mathcal{X}$.
2. A topological space $\mathcal{X}$ is admissibly represented if it has a universal continuous representation δ, that is,

   ($\forall$ continuous $\rho :\subseteq \omega^\omega \to \mathcal{X}$)($\exists$ continuous $\nu :\subseteq \omega^\omega \to \omega^\omega$) such that $\rho = \delta \circ \nu$.

Suppose that $\mathcal{X}$ is represented by δ.

- If $\delta(p) = x$, then we think of p as a name of x.
- The complexity of x is identified with that of $\delta^{-1}\{x\}$.
- The degree of x is the *degree of difficulty of calling a name of x*.

Degrees of difficulty of calling a name

$(\mathcal{X}, \delta_{\mathcal{X}})$, $(\mathcal{Y}, \delta_{\mathcal{Y}})$: represented spaces.

1. A point $x \in \mathcal{X}$ is (Turing) reducible to $y \in \mathcal{Y}$ ($x \leq_T y$) if there is a partial computable function $\Phi :\subseteq \omega^\omega \to \omega^\omega$ s.t.
$$(\forall p)\ [p \text{ is a name of } y \implies \Phi(p) \text{ is a name of } x].$$
2. $\mathbf{deg}(x) = \{z : z \equiv_T x\}$ is called the (Turing) degree of x.

Degrees of difficulty of calling a name

$(\mathcal{X}, \delta_{\mathcal{X}})$, $(\mathcal{Y}, \delta_{\mathcal{Y}})$: represented spaces.

1. A point $x \in \mathcal{X}$ is (Turing) reducible to $y \in \mathcal{Y}$ ($x \leq_T y$) if there is a partial computable function $\Phi :\subseteq \omega^\omega \to \omega^\omega$ s.t.

$$(\forall p)\ [p \text{ is a name of } y \implies \Phi(p) \text{ is a name of } x].$$

2. $\mathbf{deg}(x) = \{z : z \equiv_T x\}$ is called the (Turing) degree of x.

Example of representation

- Let $(B_n)_{n \in \omega}$ be an open basis of a space $\mathcal{X}$. Then, each point $x \in \mathcal{X}$ is named by an enumeration p of its nbhd basis, that is,

$$\delta(p) = x \iff \mathrm{range}(p) = \{n \in \omega : x \in B_n\}.$$

- The degree of x is the enumeration degree of its nbhd basis.

A network for a space $\mathcal{X}$ is a collection $\mathcal{N}$ of subsets of $\mathcal{X}$ such that

$$(\forall x \in \mathcal{X})(\forall U \text{ open nbhd of } x)(\exists N \in \mathcal{N})\ x \in N \subseteq U.$$

Example of representation (II)

Let $(N_n)_{n\in\omega}$ be a network for a space $\mathcal{X}$. Then, each point $x \in \mathcal{X}$ is named by an enumeration p of a local subnetwork at x, that is,

- $x \in N_{p(n)}$ for any $n \in \omega$,
- $(\forall U$ open nbhd of $x)(\exists n)\ x \in N_{p(n)} \subseteq U$.

Fact (Schröder)

For a topological space $\mathcal{X}$, the following are equivalent:

1. $\mathcal{X}$ is admissibly represented.
2. $\mathcal{X}$ is a qcb_0 space.
3. $\mathcal{X}$ has a countable cs-network.

- A space is *qcb_0* if it is T_0, and is a quotient of a countably based space.
- (Michael 1966) A *cs-network* is a network $\mathcal{N}$ such that every convergent sequence converging to a point $x \in U$ with U open, is eventually in $N \subseteq U$ for some $N \in \mathcal{N}$.

T_0	enumeration degrees
T_1	?
Hausdorff	?
$T_{2\frac{1}{2}}$	?
metrizable	continuous degrees
transfinite dimensional	Turing degrees

Table: Degrees of second-countable spaces

Basic idea of "generalized" degree theory

- *Turing degrees* are degrees of calling names of points of *separable metrizable spaces having transfinite inductive dimension*.
- *Continuous degrees* are degrees of calling names of points of *separable metrizable spaces*.
- *Enumeration degrees* are degrees of calling names of points of *second-countable T_0 spaces*.

To develop our theory, we first deal with the following toy problem:

Toy Problem

Given $m < n$, does there exist a "*degree*" of a point of a T_m-space, which CANNOT be a degree of a point of a T_n-space?

T_3-degrees vs. $T_{2\frac{1}{2}}$-degrees.

- A space is T_3 if it is regular Hausdorff, that is, given any point and closed set are separated by nbhds.
- A space is $T_{2\frac{1}{2}}$ if any two distinct points are separated by closed nbhds.

T_3-degrees vs. $T_{2\frac{1}{2}}$-degrees.

- A space is T_3 if it is regular Hausdorff, that is, given any point and closed set are separated by nbhds.
- A space is $T_{2\frac{1}{2}}$ if any two distinct points are separated by closed nbhds.

Example

The Gandy-Harrington topology τ_{GH} is the topology on ω^ω generated by all computably analytic (i.e., lightface Σ^1_1) sets.

- $(\omega^\omega, \tau_{GH})$ is second-countable, $T_{2\frac{1}{2}}$, but not T_3.

Theorem (3 vs. $2\frac{1}{2}$)

Let x be a sufficiently complicated point in ω^ω.
deg(x): the degree of x w.r.t. the Gandy-Harrington topology.

1. **deg**(x) is realized as the degree of a point in a $T_{2\frac{1}{2}}$ space.
2. **deg**(x) cannot be realized as the degree of a point in a T_3 space.
3. Indeed, **deg**(x) cannot be a degree of a point of a Hausdorff space having a countable closed cs-network.

Remark

Regular $\Longrightarrow$ Having a countable closed cs-network.

The converse is not true, e.g., the sequential topology on the Kleene-Kreisel space $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$ has a countable closed cs-network, but not regular (Schröder).

$T_{2\frac{1}{2}}$-degrees vs. T_2-degrees.

- A space is $T_{2\frac{1}{2}}$ if any two distinct points are separated by closed nbhds.
- A space is T_2 if any two distinct points are separated by open nbhds.

$T_{2\frac{1}{2}}$-degrees vs. T_2-degrees.

- A space is $T_{2\frac{1}{2}}$ if any two distinct points are separated by closed nbhds.
- A space is T_2 if any two distinct points are separated by open nbhds.

Example

The relatively prime integer topology is the topology on the positive integers generated by $\{U_b(a) : a \text{ and } b \text{ are relatively prime}\}$ where $U_b(a) = \{a + bn : n \in \mathbb{Z}\}$.

- This is second-countable, Hausdorff, but not $T_{2\frac{1}{2}}$.

Consider the countable product of the relatively prime integer topology:

Theorem ($\mathbf{2\frac{1}{2}}$ vs. $\mathbf{2}$)

Let $x \in \mathbb{Z}^{\omega}_{>0}$ be sufficiently generic w.r.t. Baire topology.
$\mathbf{deg}(x)$: the degree of x w.r.t. the product relatively prime topology

1. $\mathbf{deg}(x)$ is realized as the degree of a point in a T_2 space.
2. $\mathbf{deg}(x)$ cannot be realized as the degree of a point in a $T_{2\frac{1}{2}}$ space.

Consider the countable product of the relatively prime integer topology:

Theorem ($2\frac{1}{2}$ vs. 2)

Let $x \in \mathbb{Z}^\omega_{>0}$ be sufficiently generic w.r.t. Baire topology.
$\mathbf{deg}(x)$: the degree of x w.r.t. the product relatively prime topology

1. $\mathbf{deg}(x)$ is realized as the degree of a point in a T_2 space.
2. $\mathbf{deg}(x)$ cannot be realized as the degree of a point in a $T_{2\frac{1}{2}}$ space.

Moreover, even if we know a name of such an x, we cannot get any new information on names of points in a T_3 space...

1. (Medvedev 1955) A point x is *quasi-minimal* if
   - it has no computable name, but
   - it has no nontrivial information on names of points in $\mathbf{2}^\omega$

$$x \not\leq_T \emptyset \text{ and } (\forall y \in \mathbf{2}^\omega)[y \leq_T x \implies y \leq_T \emptyset].$$

2. A point x is *quasi-minimal w.r.t. $\mathcal{P}$* if
   - it has no computable name, but
   - it has no nontrivial information on names of points in $\mathcal{P}$-spaces

1. (Medvedev 1955) A point x is *quasi-minimal* if
   - it has no computable name, but
   - it has no nontrivial information on names of points in $\mathbf{2}^\omega$

$$x \not\leq_T \emptyset \text{ and } (\forall y \in \mathbf{2}^\omega)[y \leq_T x \implies y \leq_T \emptyset].$$

2. A point x is *quasi-minimal w.r.t. $\mathcal{P}$* if
   - it has no computable name, but
   - it has no nontrivial information on names of points in $\mathcal{P}$-spaces

Theorem (3 vs. 2 — the quasi-minimal version)

Let $x \in \mathbb{Z}^\omega_{>0}$ be Cohen $\mathbf{1}$-generic w.r.t. Baire topology.
$\mathbf{deg}(x)$: the degree of x w.r.t. the product relatively prime topology

1. $\mathbf{deg}(x)$ is realized as the degree of a point in a T_2 space.
2. $\mathbf{deg}(x)$ is quasi-minimal w.r.t. $T_{2\frac{1}{2}}$ spaces having countable closed cs-networks.

T_2-degrees vs. T_1-degrees.

- A space is T_2 if the diagonal is closed.
- A space is T_1 if every singleton is closed.

T_2-degrees vs. T_1-degrees.

- A space is T_2 if the diagonal is closed.
- A space is T_1 if every singleton is closed.

Example

The cocylinder topology is the topology on ω^ω generated by $\{\omega^\omega \setminus [\sigma] : \sigma \in \omega^{<\omega}\}$, where $[\sigma] = \{x \in \omega^\omega : \sigma \prec x\}$.

- This is second-countable, T_1, but not Hausdorff.

Theorem (2 vs. 1)

Let $x \in \omega^\omega$ be sufficiently fast-growing as a function on ω.
deg(x): the degree of x w.r.t. the cocylinder topology.

1. **deg(x)** is realized as the degree of a point in a T_1 space.
2. **deg(x)** cannot be realized as the degree of a point in a T_2-space.
3. **deg(x)** is quasi-minimal w.r.t. T_2 spaces having countable closed cs-networks.

T_1-degrees vs. T_0-degrees.

T_1-degrees vs. T_0-degrees.

Example

The lower topology is the topology on $\mathbb{R}$ generated by $\{(q, \infty) : q \in \mathbb{Q}\}$.

- This is second-countable, T_0, but not T_1.

Theorem (1 vs. 0)

Let $x \in \mathbb{R}$ be neither left- nor right-c.e.
deg(x): the degree of x w.r.t. the lower topology.

1. **deg(x)** is realized as the degree of a point in a T_0 space.
2. **deg(x)** is quasi-minimal w.r.t. T_1 spaces.

[second-countable]-degrees vs. [non-second-countable]-degrees.

Remark

The category of admissibly represented sps. is cartesian closed. Thus, if $\mathcal{X}$ is admissibly represented, then so is the following space:

$$\mathcal{A}_1(\mathcal{X}) = \{f \in C(\mathcal{X}, \mathbb{S}) : f^{-1}\{\bot\} \text{ is singleton}\},$$

where $\mathbb{S} = \{\top, \bot\}$ is the Sierpiński space, whose open sets are $\emptyset$, $\{\top\}$, and $\{\top, \bot\}$.

Roughly speaking, $\mathcal{A}_1(\mathcal{X})$ is the space of closed singletons in $\mathcal{X}$.

Recursion-theoretic view

The degree of difficulty of calling a name of a point $\{x\}$ in $\mathcal{A}_1(\mathcal{X})$ $\approx$ that of finding an oracle z making x be a $\Pi^0_1(z)$ singleton.

One may think of $\mathcal{A}_1(\omega^\omega)$ as one of the easiest non-second-countable spaces.

We say that $x \in \omega^\omega$ is a *lost melody* if there is $z \in \omega^\omega$ such that $\{x\}$ is a $\Pi^0_1(z)$ singleton (i.e., $\{x\} \leq_T z$), but $x \not\leq_T z'$.

Theorem ([second-countable] vs. [non-second-countable])

Let $x \in \omega^\omega$ be a lost melody s.t. $\{x\}$ is not computable.
$\mathbf{deg}(\{x\})$: the degree of $\{x\}$ as a point in $\mathcal{A}_1(\omega^\omega)$.
Then, $\mathbf{deg}(\{x\})$ is quasi-minimal w.r.t. second-countable spaces.

More remarks on $\mathcal{A}_1(\mathcal{X})$

Proposition

1. If $\mathcal{X}$ is Hausdorff, $\{\{x\}\} \mapsto x : \mathcal{A}_1\mathcal{A}_1(\mathcal{X}) \to \mathcal{X}$ is continuous.
2. There is a T_1 space $\mathcal{X}$ such that $\{\{x\}\} \mapsto x : \mathcal{A}_1\mathcal{A}_1(\mathcal{X}) \to \mathcal{X}$ is not continuous (indeed, not Borel).

Proof of Theorem (**3** vs. $\mathbf{2\frac{1}{2}}$)

The degree of a complicated point in the Gandy-Harrington space cannot be a degree of a point of a Hausdorff space having a countable closed cs-network.

Recall: a point x in a space $\mathcal{X}$ with a countable cs-network $\mathcal{N}$ is named by an enumeration p of a local subnetwork at x, that is,

- $x \in N_{p(n)}$ for any $n \in \omega$,
- $(\forall U$ open nbhd of $x)(\exists n)\ x \in N_{p(n)} \subseteq U$.

Proof of Theorem ($\mathbf{3}$ vs. $\mathbf{2\frac{1}{2}}$)

The degree of a complicated point in the Gandy-Harrington space cannot be a degree of a point of a Hausdorff space having a countable closed cs-network.

Recall: a point x in a space $\mathcal{X}$ with a countable cs-network $\mathcal{N}$ is named by an enumeration p of a local subnetwork at x, that is,

- $x \in N_{p(n)}$ for any $n \in \omega$,
- $(\forall U$ open nbhd of $x)(\exists n)\ x \in N_{p(n)} \subseteq U$.

Consider another representation $\bar{\delta}_{\mathcal{N}}$ of $\mathcal{X}$ defined by $\bar{\delta}_{\mathcal{N}}(p) = x$ iff

- $x \in \overline{N_{p(n)}}$ for any $n \in \omega$,
- $(\forall U$ open nbhd of $x)(\exists n)\ x \in N_{p(n)} \subseteq U$.

Proof of Theorem (**3** vs. $\mathbf{2\frac{1}{2}}$)

The degree of a complicated point in the Gandy-Harrington space cannot be a degree of a point of a Hausdorff space having a countable closed cs-network.

Regular $\Longrightarrow$ Having a countable closed cs-network.

Proposition

If $\mathcal{X}$ is a Hausdorff space having a countable closed cs-network $\mathcal{N}$ then $\mathbf{id} : (\mathcal{X}, \bar{\delta}_{\mathcal{N}}) \to (\mathcal{X}, \delta_{\mathcal{N}})$ is continuous.

Proof of Theorem (**3** vs. **$2\frac{1}{2}$**)

The degree of a complicated point in the Gandy-Harrington space cannot be a degree of a point of a Hausdorff space having a countable closed cs-network.

Regular $\Longrightarrow$ Having a countable closed cs-network.

Proposition

If $\mathcal{X}$ is a Hausdorff space having a countable closed cs-network $\mathcal{N}$ then $\mathbf{id} : (\mathcal{X}, \bar{\delta}_{\mathcal{N}}) \to (\mathcal{X}, \delta_{\mathcal{N}})$ is continuous.

Lemma

$(\mathcal{X}, \delta_{\mathcal{N}})$: Hausdorff space having a countable cs-network.
Let $z \in (\omega^\omega, \tau_{\mathbf{GH}})$ and $x \in \mathcal{X}$.
If a $\delta_{\mathcal{N}}$-name of x is computable relative to a $\mathbf{GH}$-name of z, then no $\mathbf{GH}$-name of z is computable relative to a $\bar{\delta}_{\mathcal{N}}$-name of x.

Lemma

If a $\delta_{\mathcal{N}}$-name of x is computable relative to a $\mathbf{GH}$-name of z, then no $\mathbf{GH}$-name of z is computable relative to a $\bar{\delta}_{\mathcal{N}}$-name of x.

Lemma

If a $\delta_{\mathcal{N}}$-name of x is computable relative to a $\mathbf{GH}$-name of z, then no $\mathbf{GH}$-name of z is computable relative to a $\bar{\delta}_{\mathcal{N}}$-name of x.

- S_e: the e-th lightface Σ^1_1 set.
- A $\mathbf{GH}$-name of x is an enumeration of $G_x = \{e : x \in S_e\}$.

Lemma

If a $\delta_{\mathcal{N}}$-name of x is computable relative to a $\mathbf{GH}$-name of z, then no $\mathbf{GH}$-name of z is computable relative to a $\bar{\delta}_{\mathcal{N}}$-name of x.

- S_e: the e-th lightface Σ^1_1 set.
- A $\mathbf{GH}$-name of x is an enumeration of $G_x = \{e : x \in S_e\}$.
- Assume that $x \leq_T z$ via Ψ, that is,
  - $(e, D) \in \Psi$ and $D \subseteq G_x \Longrightarrow z \in N_e$.
  - U open nbhd of $z \Longrightarrow \exists (e, D) \in \Psi\ [D \subseteq G_x \text{ and } z \in N_e \subseteq U]$

Lemma

If a $\delta_{\mathcal{N}}$-name of x is computable relative to a $\mathbf{GH}$-name of z, then no $\mathbf{GH}$-name of z is computable relative to a $\bar{\delta}_{\mathcal{N}}$-name of x.

- S_e: the e-th lightface Σ^1_1 set.
- A $\mathbf{GH}$-name of x is an enumeration of $G_x = \{e : x \in S_e\}$.
- Assume that $x \leq_T z$ via Ψ, that is,
    - $(e, D) \in \Psi$ and $D \subseteq G_x \Longrightarrow z \in N_e$.
    - U open nbhd of $z \Longrightarrow \exists (e, D) \in \Psi$ $[D \subseteq G_x$ and $z \in N_e \subseteq U]$
- $L = \{n : \forall (m, D) \in \Psi\ [D \subseteq G_x \implies N_m \cap N_n \neq \emptyset]\}$.
- If $n \in L$ then $z \in \overline{N_n}$.

Lemma

If a $\delta_{\mathcal{N}}$-name of x is computable relative to a $\mathbf{GH}$-name of z, then no $\mathbf{GH}$-name of z is computable relative to a $\bar{\delta}_{\mathcal{N}}$-name of x.

- S_e: the e-th lightface Σ^1_1 set.
- A $\mathbf{GH}$-name of x is an enumeration of $G_x = \{e : x \in S_e\}$.
- Assume that $x \leq_T z$ via Ψ, that is,
  - $(e, D) \in \Psi$ and $D \subseteq G_x \Longrightarrow z \in N_e$.
  - U open nbhd of $z \Longrightarrow \exists(e, D) \in \Psi\ [D \subseteq G_x$ and $z \in N_e \subseteq U]$
- $L = \{n : \forall(m, D) \in \Psi\ [D \subseteq G_x \implies N_m \cap N_n \neq \emptyset]\}$.
- If $n \in L$ then $z \in \overline{N_n}$.
- Suppose $z \leq_T (x, \bar{\delta}_{\mathcal{N}})$ via an enumeration Γ:

$$e \in G_x \iff (\exists D \text{ finite})[(e, D) \in \Gamma \text{ and } D \subseteq L].$$

Lemma

If a $\delta_{\mathcal{N}}$-name of x is computable relative to a $\mathbf{GH}$-name of z, then no $\mathbf{GH}$-name of z is computable relative to a $\bar{\delta}_{\mathcal{N}}$-name of x.

- S_e: the e-th lightface Σ^1_1 set.
- A $\mathbf{GH}$-name of x is an enumeration of $G_x = \{e : x \in S_e\}$.
- Assume that $x \leq_T z$ via Ψ, that is,
  - $(e, D) \in \Psi$ and $D \subseteq G_x \Longrightarrow z \in N_e$.
  - U open nbhd of $z \Longrightarrow \exists (e, D) \in \Psi\ [D \subseteq G_x$ and $z \in N_e \subseteq U]$
- $L = \{n : \forall (m, D) \in \Psi\ [D \subseteq G_x \implies N_m \cap N_n \neq \emptyset]\}$.
- If $n \in L$ then $z \in \overline{N_n}$.
- Suppose $z \leq_T (x, \bar{\delta}_{\mathcal{N}})$ via an enumeration Γ:

$$e \in G_x \iff (\exists D \text{ finite})[(e, D) \in \Gamma \text{ and } D \subseteq L].$$

- Since L is $\Pi^1_1(x)$, this gives a $\Pi^1_1(x)$ definition of G_x; however G_x is clearly $\Sigma^1_1(x)$ complete, a contradiction.