Strong normalization for the parameter-free polymorphic lambda calculus based on the Ω-rule

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Motivation

- Girard's proof of the strong normalization of his system $\boldsymbol{F}$ requires the third-order arithmetic on the meta-level.
- Natural question: can we have a more predicative proof of the normalization for fragments of $\boldsymbol{F}$?
    - predicative proof = proof without circular reasoning.

Aim of This Talk

- In this talk, we present a predicative proof of the strong normalization for F_n^p by studying Buchholz' Ω-rule.
  - F_n^p: a parameter-free polymorphic lambda calculus allowing n-times nested second-order quantifier.
  - We transfer an important method in proof theory called the Ω-rule into computer science.
  - Moreover, we give a proof-theoretic bound of the strong normalization for it.

Akiyoshi and Terui, "Strong normalization for the parameter-free polymorphic lambda calculus based on the Omega-rule", *First International Conference on Formal Structures for Computation and Deduction (FSCD)*, 2016.

Definition of Syntax

Definition (Cf. Aehlig08)

For each $n \in \mathbb{N} \cup \{-1\}$, we define $\mathbf{Tp}_n$ as

$$A_n, B_n \quad ::= \quad \alpha \mid A_n \Rightarrow B_n \mid \forall \alpha. A_{n-1}.$$

where $FV(A_{n-1}) \subseteq \{\alpha\}$ in the last clause.

We write $\mathbf{Tp_{simp}} = \mathbf{Tp}_{-1}$.
Types in this set are "parameter-free".

$$\begin{array}{lcll} N & := & \forall \alpha.(\alpha \Rightarrow \alpha) \Rightarrow (\alpha \Rightarrow \alpha) & \in \mathbf{Tp}_0 \\ T & := & \forall \alpha.(\alpha \Rightarrow \alpha \Rightarrow \alpha) \Rightarrow (\alpha \Rightarrow \alpha) & \in \mathbf{Tp}_0 \\ O & := & \forall \alpha.((N \Rightarrow \alpha) \Rightarrow \alpha) \Rightarrow (\alpha \Rightarrow \alpha) \Rightarrow (\alpha \Rightarrow \alpha) & \in \mathbf{Tp}_1 \end{array}$$

Remark

An important property: $A, B \in \mathbf{Tp}_n$ *implies* $A[B/\alpha] \in \mathbf{Tp}_n$.

Term Rules and Conversions

Definition

Terms (**Tm**) and Conversions of $\mathbf{F}^p$ are defined in the standard way:

$$\frac{}{x^A \in \mathsf{Tm}}\ (\mathsf{var}) \qquad \frac{}{c^A \in \mathsf{Tm}}\ (\mathsf{con}) \qquad \frac{M^B \in \mathsf{Tm}}{(\lambda x^A.M)^{A\Rightarrow B} \in \mathsf{Tm}}\ (\mathsf{abs})$$

$$\frac{M^{A\Rightarrow B} \in X \quad N^A \in \mathsf{Tm}}{(MN)^B \in \mathsf{Tm}}\ (\mathsf{app}) \qquad \frac{M^A \in \mathsf{Tm} \cap \mathsf{Ec}(\alpha)}{(\Lambda\alpha.M)^{\forall\alpha.A} \in \mathsf{Tm}}\ (\mathsf{Abs})$$

$$\frac{M^{\forall\alpha.A} \in \mathsf{Tm}}{(MB)^{A[B/\alpha]} \in \mathsf{Tm}}\ (\mathsf{App})$$

$$(\lambda x^A.M)N \to M[N/x^A], \quad (\Lambda\alpha.M)B \to M[B/\alpha].$$

Definition

$\mathbf{F}^p_n$ is obtained by restricting types to Tp_n.

Previous Results by Alternkirch, Coquand, and Aehlig

- Girard's proof of $\boldsymbol{SN}(\boldsymbol{F})$ requires the third-order arithmetic on the meta-level.
- Question: can we have a more predicative proof of the normalization for fragments of system F?

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      Provably total in $\boldsymbol{HA}$ = representable in $\boldsymbol{F}_0^p$.

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    - Aehlig: an indirect predicative proof of WN for F_n^p for a specific terms;
      Provably total in ID_n = representable in F_n^p.
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    Provably total in ID_n = representable in F_n^p.
    (The problem of SN was left open in his Ph.D thesis)
- Our aim is to improve the situation by giving a direct predicative proof of the strong normalization of such fragments for all terms.

Altenkirch and Coquand, "A Finitary Subsystem of the Polymorphic λ-calculus", *TLCA 2001*.

Aehlig, "Parameter-free polymorphic types", *APAL*, 2008.

Our Results

- Systems of inductive definitions:
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- Systems of inductive definitions:
  1. $\boldsymbol{ID_1} = \boldsymbol{PA} +$ the least fixed points for $\boldsymbol{PA}$-definable monotone operators.
  2. $\boldsymbol{ID_{n+1}} = \boldsymbol{ID_n} +$ the least fixed points for $\boldsymbol{ID_n}$-definable monotone operators with $1 \leq n$.
  3. $\boldsymbol{ID_{<\omega}} := \bigcup_{n\in\omega} \boldsymbol{ID_n}$.
  4. $\boldsymbol{ID_\omega}$: a proper extension of $\boldsymbol{ID_{<\omega}}$.

Theorem
$\boldsymbol{ID_{n+1}} \vdash \boldsymbol{SN}(\boldsymbol{F^p_n})$ *for all* $n < \omega$.

Theorem
$\boldsymbol{ID_\omega} \vdash \boldsymbol{SN}(\boldsymbol{F^p})$ *with* $\boldsymbol{F^p} := \bigcup_{n\in\omega} \boldsymbol{F^p_n}$.

Theorem (Aehlig 08)
Every representable function in $\boldsymbol{F^p_n}$ *is provably total in* $\boldsymbol{ID_n}$.

What is the Ω-Rule?

- The Ω-rule: infinitary rule introduced by Buchholz (1977) for ordinal analysis of iterated inductive definitions.
  - Schütte's ω-rule: branching over natural numbers.
  - The Ω-rule: branching over arithmetical cut-free proofs.
- Main theorems by Buchholz:

  Embedding: BI (parameter free Π^1_1-CA) is embedded to BI^Ω.
  Collapsing: weak normalization for arithmetical formulas for BI^Ω.

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- Recent developments:
  1. For a stronger system (μ-calculus): H.Towsner (2008).
  2. modal μ-calculus like ID_1: G. Jäger and T. Studer (2010).
  3. Complete cut-elimination theorem: R.Akiyoshi and G.Mints (2016, AML).

Buchholz' Ω-Rule

- Idea of the Ω-rule: BHK-reading of $\forall XA \rightarrow B$.
    - Meaning of $\forall XA \rightarrow B$: some transformation f (function) from any (cut-free) proof of $\forall XA$ to a proof of B (BHK-reading).

Buchholz' Ω-Rule

- Idea of the Ω-rule: BHK-reading of $\forall XA \to B$.
  - Meaning of $\forall XA \to B$: some transformation f (function) from any (cut-free) proof of $\forall XA$ to a proof of B (BHK-reading).
- So, if we have a proof $f(d)$ of B for any (cut-free) proof d of $\forall XA$, then we have a proof of $\forall XA \to B$.

$$\dfrac{\begin{matrix}\{d : \forall XA(X)\} \\ \vdots \\ \ldots B \ldots\end{matrix}}{\forall XA(X) \to B}\ \Omega$$

Remark

The Ω-rule works well not only for a formal system based on intuitionistic logic, but for one based on classical logic as well.

Mints' Question

- Around 2008, Mints asked the following question:
  - There should be the connection between the computability predicate and the Ω-rule.
- We can prove the strong normalization by the following argument:
  1. Every reducible terms is S.N.
  2. All terms are reducible (Reducibility Theorem).
- The difficulty in $\boldsymbol{F}$ comes from the impredicativity of $\forall \boldsymbol{X}$:
  - $\boldsymbol{t} : \forall \boldsymbol{X}\boldsymbol{A}$ is reducible iff for any type B, $\boldsymbol{tB}$ is reducible of type $\boldsymbol{A}[\boldsymbol{X}/\boldsymbol{B}]$.
- The definition by induction on type breaks down.

  (Girard's solution: "Reducibility Candidate")
- Indeed, the Ω-rule uses the substitution in the embedding. It avoids "induction on type" as well.

Analogy between Embedding and Reducibility Theorems

Buchholz' embedding of $\forall^2 E$ via the Ω-rule:

$$\dfrac{\overset{\vdots}{A(T) \to B}}{\forall X A(X) \to B} \quad \Rightarrow \quad \dfrac{\dfrac{\dfrac{[d : A(X)]}{A(T)}\, S^X_T \qquad \overset{\vdots}{A(T) \to B}}{\ldots B \ldots}}{\forall X A(X) \to B}\, \Omega \quad \to E$$

- Idea: Embedding corresponds to Reducibility:
  - $T \ni d \vdash \Gamma \Rightarrow T^\infty \ni d^\infty \vdash \Gamma$.
  - All terms are reducible.
- We extend the JM method using the Ω-rule.

Joachimski and Matthes, "Short Proofs of Normalization for the simply-typed lambda-calculus, permutative conversions and Gödel's T", AML, 2003.

Towards Strong Normalization Theorem

Our strategy is to find a suitable set X such that

1. Prove $\mathsf{Tm} \subseteq X$ by showing that X is closed under the term rules (*Embedding*).
2. Prove $X \subseteq \mathsf{SN}$ (*Collapsing*).

Remark

In proof-theory, X is a suitable infinitary proof system, say $\boldsymbol{PA}(\omega)$.

- To consider the strong normalization, *explicit* bound variables are replaced by constant.
  - These variables are unchanged in the process of the normalization.
- If M is a term, then $M^\circ := M\bar{t}$ is a term of a suitable atomic type.

Cf. Akiyoshi and Mints, "An extension of the Omega-rule", AML, 2016.

JM Rules

First, we define a suitable set of terms $\mathbf{JM}_{\mathsf{simp}} \subseteq \mathsf{SN}$. In this case, we essentially follow Joachimski and Matthes' way.

Definition

$\mathbf{JM}_{\mathsf{simp}}$ is defined to be the least set X ($\subseteq \mathsf{Dom}_{\mathsf{simp}}$) closed under the following rules:

$$\frac{\overline{M} \in X}{x\overline{M} \in X}\ (\mathsf{vap}^-) \qquad \frac{\overline{T}^\circ \in X}{c\overline{T} \in X}\ (\mathsf{cap}^\circ) \qquad \frac{M \in X}{\lambda x^A.M \in X}\ (\mathsf{abs})$$

$$\frac{M \in X \cap \mathsf{Ec}(\alpha)}{\Lambda\alpha.M \in X}\ (\mathsf{Abs}) \qquad \frac{M[N/x^A]\overline{T} \in X \quad N^\circ \in X}{(\lambda x^A.M)N\overline{T} \in X}\ (\beta^\circ)$$

$$\frac{M[B/\alpha]\overline{T} \in X}{(\Lambda\alpha.M)B\overline{T} \in X}\ (\mathsf{B})$$

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with

$$\mathbf{Dom_{simp}} := \{M \in \mathbf{Tm} : \mathbf{type}(\mathbf{fv}(M)) \subseteq \mathbf{Tp_{simp}},\ \mathbf{type}(M) \in \forall\mathbf{Tp_{simp}}\},$$

where $\forall\mathbf{Tp_{simp}} := \mathbf{Tp_{simp}} \cup \{\forall\alpha.A : A \in \mathbf{Tp_{simp}}\}$.

Inductive Case: the Ω-Rule

As to JM Rules, we can show Embedding ($\mathbf{Tm_{simp}} \subseteq \mathbf{JM_{simp}}$).
Next, we extend Buchholz' Ω-rule for the strong normalization proof. In this talk, we focus on the simplest case $\mathbf{JM_0}$.

Definition

$\mathbf{JM_0}$ is defined to be the least set $X (\subseteq \mathbf{Dom_0})$ closed under the JM rules and Ω_0 :.

$$\frac{M^{\forall\alpha.A} \in X \quad \{\ K[B/\alpha]\overline{T} \in X\ \}_{K^A \in \mathbf{JM_{simp}} \cap \mathbf{Ec}(\alpha)}}{MB\overline{T} \in X}\ \Omega_0$$

This rule is a "hidden-redex". In a proof-figure notation, this is visualized as:

$$\dfrac{\dfrac{A[\gamma/\alpha]}{\forall\alpha.A}\ \forall^2 I \qquad \dfrac{\begin{array}{c}\{K : A\}\\ \vdots \\ \ldots A[B/\alpha]\ldots\end{array}}{\forall\alpha.A \to A[B/\alpha]}\ {\to^{\Omega} I}}{A[B/\alpha]}\ \to E$$

Inductive Case: the Ω-Rule

- To eliminate Ω_0 is to eliminate the second-order redex (collapsing).

Remark

- *In Buchholz' original Ω-rule, the domain (to which* $\boldsymbol{K}$ *belongs) is the set of normal arithmetical terms.*
- *In fact,* $\mathbf{JM}_{\mathsf{simp}} \subseteq \mathsf{SN}$. *So, we quantify over the set of strongly normalizable terms. To define the domain in a suitable way is the key for defining the Ω-rule.*
- *Iterating this definition, we can define* $\mathbf{JM}_{\boldsymbol{n}}$ *with* $\Omega_{\boldsymbol{n}}$ *for* $\boldsymbol{n} \geq \mathbf{1}$.

Key Lemma for Embedding

Lemma

$\mathbf{JM_0}$ *is closed under* ($\mathsf{App_0}$):

$$\frac{M^{\forall\alpha.A} \in X \quad B \in \mathsf{Tp_0}}{MB \in X}\ (\mathsf{App_0})$$

Proof. Suppose that $M^{\forall\alpha.A} \in \mathbf{JM_0}$ and $B \in \mathsf{Tp_0}$. We use

$$\frac{M^{\forall\alpha.A} \in X \quad \{K[B/\alpha] \in X\}_{K^A \in \mathbf{JM}_{\mathsf{simp}} \cap \mathsf{Ec}(\alpha)}}{MB \in X}\ \Omega_0$$

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Take any $K^A \in \mathbf{JM}_{\mathsf{simp}} \cap \mathsf{Ec}(\alpha)$, then we have $K^A \in \mathbf{JM_0}$.

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Moreover, we can show $K[B/\alpha] \in \mathbf{JM_0}$.
Hence, we obtain $MB \in \mathbf{JM_0}$ by Ω_0. □

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Hence, we obtain $MB \in \mathbf{JM_0}$ by Ω_0. □

Remark

This lemma is the crucial case of Embedding in proof-theory, that is, Π^1_1-CA is interpreted into inifinitary system using the Ω-rule.

Key Lemma for Collapsing (Normalization)

Lemma (Collapsing)

JM$_{\mathsf{simp}}$ *satisfies* Ω_0:

$$\frac{M^{\forall\alpha.A} \in \mathbf{JM}_{\mathsf{simp}} \quad \{\, K[B/\alpha]\overline{T} \in \mathbf{JM}_{\mathsf{simp}} \,\}_{K^A \in \mathbf{JM}_{\mathsf{simp}} \cap \mathsf{Ec}(\alpha)}}{MB\overline{T} \in \mathbf{JM}_{\mathsf{simp}}}$$

Proof. By induction on the derivation of $M^{\forall\alpha.A} \in \mathbf{JM}_{\mathsf{simp}}$.

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Proof. By induction on the derivation of $M^{\forall\alpha.A} \in \mathbf{JM}_{\mathsf{simp}}$.
If $M \equiv \Lambda\alpha.N \in \mathbf{JM}_{\mathsf{simp}}$ is derived by $(\mathbf{Abs})$, then $N^A \in \mathbf{JM}_{\mathsf{simp}} \cap \mathbf{Ec}(\alpha)$.

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Let $K := N$ to obtain $N[B/\alpha]\overline{T} \in \mathbf{JM}_{\textsf{simp}}$.

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Thus $MB\overline{T} \in \mathbf{JM}_{\mathsf{simp}}$ by (B).

$$\frac{M[B/\alpha]\overline{T} \in \mathbf{JM}_{\mathsf{simp}}}{(\Lambda\alpha.M)B\overline{T} \in \mathbf{JM}_{\mathsf{simp}}}\ (\mathsf{B})$$

□

Main Result

By iterating the arguments, we have:

Theorem

For each $\boldsymbol{n} \in \mathbb{N} \cup \{\textsf{simp}\}$, $\mathbf{F}^p_n$ *admits strong normalization. Hence* $\mathbf{F}^p$ *admits strong normalization too.*

Proof. Consider a term t in $\mathbf{F}^p$. Then t belongs to $\mathbf{F}^p_n$ for some $n < \omega$.
So, by Embedding, t is in $\mathbf{JM}_n$.
By the previous lemma (Collapsing), we see that
$t \in \mathbf{JM}_n \subseteq \mathbf{JM}_{n-1}, \ldots, \subseteq \mathbf{JM}_{\textsf{simp}} \subseteq \textsf{SN}$. □

Global Formalization in ID_{n+1} and ID_ω

To formalize our argument, the only strong method needed is the Ω_n-rule:

$$\frac{M^{\forall\alpha.A} \in X \quad \{ K[B/\alpha]\overline{T} \in X \}_{K^A \in \mathbf{JM}_{n-1} \cap \mathsf{Ec}(\alpha)}}{MB\overline{T} \in X}\ \Omega_n$$

This definition is by iterated inductive definitions. So, our arguments using Ω_n are formalized in ID_{n+1}.

Theorem

$ID_{n+1} \vdash SN(F^p_n)$ for all n.

Remark

This gives a sharp bound since $ID_n \nvdash SN(F^p_n)$.

In ID_ω, we can "speak" about any ID_n at once, so we have

Theorem

$ID_\omega \vdash SN(F^p)$.

Local Formalization in ID_n

- In general, the computability argument is by non-monotonic inductive definition.

  Cf. Martin-Löf, "Hauptsatz for the intuitionistic theory of iterated inductive definitions", 1971.

- But, if we consider a specific term, then Gödel-Tait method (the computability argument) works well.

Theorem (Aehlig 08)

Every representable function in $\boldsymbol{F}_n^p$ is provably total in $\boldsymbol{ID}_n$.

Proof. We refer to Section 4.2 of our paper. □

Summary

- Girard's proof of $SN(F)$ requires the third-order arithmetic.
- If we consider a parameter-free subsystem F_n^p, we can give a predicative proof of $SN(F_n^p)$.
- Instead of "Reducibility candidate", we used the idea of the Ω-rule.

Theorem

$ID_{n+1} \vdash SN(F_n^p)$ *for all* n.

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Theorem

$ID_\omega \vdash SN(F^p)$.

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