Proof-theoretic strength and indicator arguments

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Indicators

What is indicator?

- It is introduced by Kirby and Paris in 1970's, and the general frame work is given by Kaye.
- It is used to prove the independence of the Paris-Harrington principle from PA.
- A tool to study cuts of nonstandard models of arithmetic.

Indicators are useful to analyze the proof-theoretic strength of combinatorial statements in arithmetic.

Note that most theorems in this talk are more or less folklores (in the field of nonstandard models of arithmetic).

Nonstandard models of arithmetic

In this talk we will mainly use the base system EFA $= \mathrm{I}\Delta_0 + \exp$ or RCA_0^*, which consists of $\mathrm{I}\Delta_0^0 + \exp$ plus Δ_1^0-comprehension, and models we will consider will be countable nonstandard.
Let $M \models \mathrm{EFA}$.

- $I \subseteq M$ is said to be a cut (abbr. $I \subseteq_e M$) if $a < b \in I \to a \in I$ and I is closed under addition $+$ and multiplication $\cdot$.
- $\mathrm{Cod}(M) = \{X \subseteq M \mid X \text{ is } M\text{-finite}\}$, where M-finite set is a set coded by an element in M (by means of the usual binary coding).
- for $Z \in \mathrm{Cod}(M)$, $|Z|$ denotes the internal cardinality of Z in M.
- for $I \subseteq_e M$, $\mathrm{Cod}(M/I) := \{X \cap I \mid X \in \mathrm{Cod}(M)\}$.

Proposition

If $I \subseteq_e M$, then I is a Σ_0-elementary substructure of M.

Cuts

There are several important types of cuts.

Theorem (exponentially closed cut, Simpson/Smith)

Let $M \models \mathrm{EFA}$, and let $I \subsetneq_e M$. Then the following are equivalent.

1. $(I, \mathrm{Cod}(M/I)) \models \mathrm{WKL}_0^*$.
2. *I is closed under* exp.

Theorem (semi-regular cut)

Let $M \models \mathrm{EFA}$, and let $I \subsetneq_e M$. Then the following are equivalent.

1. $(I, \mathrm{Cod}(M/I)) \models \mathrm{WKL}_0$.
2. *I is semi-regular, i.e., if $X \in \mathrm{Cod}(M)$ and $|X| \in I$, then $X \cap I$ is bounded in I.*

These combinatorial characterization of cuts play key roles in the definition of indicators.

Indicators

Let T be a theory of second-order arithmetic.
A Σ_0-definable function $Y : [M]^2 \to M$ is said to be an *indicator* for $T \supseteq \mathrm{WKL}_0^*$ if

- $Y(x, y) \leq y$,
- if $x' \leq x < y \leq y'$, then $Y(x, y) \leq Y(x', y')$,
- $Y(x, y) > \omega$ if and only if there exists a cut $I \subseteq_e M$ such that $x \in I < y$ and $(I, \mathrm{Cod}(M/I)) \models T$.

  (Here, $Y(x, y) > \omega$ means that $Y(x, y) > n$ for any standard natural number n.)

Example

- $Y(x, y) = \max\{n : \exp^n(x) \leq y\}$ is an indicator for WKL_0^*.
- $$Y(x, y) = \max\{n : \text{any } f[[x, y]]^n \to 2 \text{ has a homogeneous set } Z \subseteq [x, y] \text{ such that } |Z| > \min Z\}$$

  is an indicator for ACA_0.

Basic properties of indicators

Theorem

If Y is an indicator for a theory T, then for any $n \in \omega$,

$$T \vdash \forall x \exists y Y(x, y) \geq n.$$

Theorem

If Y is an indicator for a theory T, then, T is a Π^0_2-conservative extension of $\mathrm{EFA} + \{\forall x \exists y Y(x, y) \geq n \mid n \in \omega\}$.

Let $F^Y_n(x) = \min\{y \mid Y(x, y) \geq n\}$.

Theorem

If Y is an indicator for a theory T and $T \vdash \forall x \exists y \theta(x, y)$ for some Σ_1-formula θ, then, there exists $n \in \omega$ such that

$$T \vdash \forall x \exists y < F^Y_n(x) \theta(x, y).$$

Let F_k be the k-th fast-growing function.

Example

- $Y(x, y) = \max\{k : F_k(x) < y\}$ is an indicator for WKL_0.

Thus, we have

- $\mathrm{WKL}_0 \vdash \forall x \exists y F_k(x) < y$ for any $k \in \omega$,
- if $\mathrm{WKL}_0 \vdash \forall x \exists y \theta(x, y)$ then there exists some $k \in \omega$ such that $\mathrm{WKL}_0 \vdash \forall x \exists y < F_k(x) \theta(x, y)$,
- the proof-theoretic strength of WKL_0 is the same as the totality of all primitive recursive functions.

Once you find an indicator for a theory T, one can characterize its Π^0_2-part.

Theorem

Any consistent recursive theory $T \supseteq \mathrm{WKL}^*_0$ (or first-order theory extending EFA*) has an indicator.*

Set indicators

One can generalize indicators to capture wider class of formulas.

Let T be a theory of second-order arithmetic.
A Σ_0-definable function $Y : \mathrm{Cod}(M) \to M$ is said to be a *set indicator* for $T \supseteq \mathrm{WKL}_0^*$ if

- $Y(F) \leq \max F$,
- if $F \subseteq F'$, then $Y(F) \leq Y(F')$,
- $Y(F) > \omega$ if and only if there exists a cut $I \subseteq_e M$ such that $\min F \in I < \max F$ and $(I, \mathrm{Cod}(M/I)) \models T$, and $F \cap I$ is unbounded in I.

Note that if Y is a set indicator, then $Y'(x, y) = Y([x, y])$ is an indicator function.

Example

- $Y(F) = \max\{m : F \text{ is } \omega^m\text{-large}\}$ is an indicator for WKL_0.

Basic properties of indicators (review)

Theorem

If Y is an indicator for a theory T, then for any $n \in \omega$,

$$T \vdash \forall x \exists y Y(x, y) \geq n.$$

Theorem

If Y is an indicator for a theory T, then, T is a Π^0_2-conservative extension of $\mathrm{EFA} + \{\forall x \exists y Y(x, y) \geq n \mid n \in \omega\}$.

Let $F^Y_n(x) = \min\{y \mid Y(x, y) \geq n\}$.

Theorem

If Y is an indicator for a theory T and $T \vdash \forall x \exists y \theta(x, y)$ for some Σ_1-formula θ, then, there exists $n \in \omega$ such that

$$T \vdash \forall x \exists y < F^Y_n(x) \theta(x, y).$$

Basic properties of set indicators

Theorem

If Y is a set indicator for a theory T, then for any $n \in \omega$,

$$T \vdash \forall X \subseteq_{\mathrm{inf}} \mathbb{N}\, \exists F \subseteq_{\mathrm{fin}} X (Y(F) \geq n).$$

Theorem

If Y is a set indicator for a theory T, then, T is a $\tilde{\Pi}^0_3$-conservative extension of $\mathrm{RCA}^*_0 + \{\forall X \subseteq_{\mathrm{inf}} \mathbb{N}\, \exists F \subseteq_{\mathrm{fin}} X(Y(F) \geq n) \mid n \in \omega\}$.

Here, a $\tilde{\Pi}^0_3$-formula is of the form $\forall X \psi(X)$ where ψ is Π^0_3.

Theorem

If Y is a set indicator for a theory T and $T \vdash \forall X \subseteq_{\mathrm{inf}} \mathbb{N}\, \exists F \subseteq_{\mathrm{fin}} X \theta(F)$ for some Σ_1-formula θ, then, there exists $n \in \omega$ such that

$$T \vdash \forall Z \subseteq_{\mathrm{fin}} \mathbb{N} (Y(Z) \geq n \rightarrow \exists F \subseteq Z\, \theta(F)).$$

Example

- $Y(F) = \max\{m : F \text{ is } \omega^m\text{-large}\}$ is an indicator for WKL_0.

Thus,

- all the $\tilde{\Pi}^0_3$-consequences of WKL_0 can be captured by ω^m-largeness notion.

Question

Is there a canonical way to find indicators?

Ramsey-like statements

Definition (Ramsey-like formulas)

A *Ramsey-like-Π^1_2-formula* is a Π^1_2-formula of the form

$$(\forall f : [\mathbb{N}]^n \to k)(\exists Y)(Y \text{ is infinite} \wedge \Psi(f, Y))$$

where $\Psi(f, Y)$ is of the form $(\forall G \subseteq_{\mathrm{fin}} Y)\Psi_0(f \upharpoonright [[0, \max G]_{\mathbb{N}}]^n, G)$ such that Ψ_0 is a Δ^0_0-formula.

- In particular, RT^n_k is a Ramsey-like-Π^1_2-statement where $\Psi(f, Y)$ is the formula "Y is homogeneous for f".

Theorem

Any restricted Π^1_2-formula of the form $\forall X \exists Y \Theta(X, Y)$ where Θ is a Σ^0_3-formula is equivalent to a Ramsey-like formula over WKL_0.

Note that this theorem can be proved by a canonical syntactical calculation.

Density

Definition (EFA, Density notion)

Given a Ramsey-like formula

$$\Gamma = (\forall f : [\mathbb{N}]^n \to k)(\exists Y)(Y \text{ is infinite} \land \Psi(f, Y)),$$

- $Z \subseteq_{\text{fin}} \mathbb{N}$ is said to be 0-*dense*(Γ) if $|Z|, \min Z > 2$,
- $Z \subseteq_{\text{fin}} \mathbb{N}$ is said to be $(m+1)$-*dense*(Γ) if
  - (for any $n, k < \min Z$ and) for any $f : [[0, \max Z]]^n \to k$, there is an m-dense(Γ) set $Y \subseteq Z$ such that $\Psi(f, Y)$ holds, and,
  - for any partition $Z_0 \sqcup \cdots \sqcup Z_{\ell-1} = Z$ such that $\ell \leq Z_0 < \cdots < Z_{\ell-1}$, one of Z_i's is m-dense(Γ).

Note that "Z is m-dense(Γ)" can be expressed by a Δ_0-formula.

Put $Y_\Gamma(F) := \max\{m \mid F \text{ is } m\text{-dense}(\Gamma)\}$.

Theorem

Y_Γ is a set indicator for $\mathrm{WKL}_0 + \Gamma$.

Characterizing proof-theoretic strength by indicators

One can characterize the proof-theoretic strength of a finite restricted Π^1_2-theory $T \supseteq \mathsf{WKL}_0$ as follows.

- Find a Ramsey-like formula Γ such that $T \leftrightarrow \mathsf{WKL}_0 + \Gamma$.
- Then, m-dense(Γ) sets capture $\tilde{\Pi}^0_3$-part of T.
- In particular, the provably recursive functions of T are $\{F_m \mid F_m(x) = \min\{y \mid [x, y] \text{ is } m\text{-dense}(\Gamma)\}\}$.

Actually, one can generalize the above argument for infinite theories.

One can also replace the base theory WKL_0 with other systems, e.g., WKL_0^* or ACA_0.

Density with the base ACA_0

Definition (EFA, Density notion with the base ACA_0)

Given a Ramsey-like formula

$$\Gamma = (\forall f : [\mathbb{N}]^n \to k)(\exists Y)(Y \text{ is infinite} \wedge \Psi(f, Y)),$$

- $Z \subseteq_{\text{fin}} \mathbb{N}$ is said to be 0-*dense′*(Γ) if $|Z| > 4, \min Z > 2$,
- $Z \subseteq_{\text{fin}} \mathbb{N}$ is said to be $(m+1)$-*dense′*(Γ) if
  - (for any $n, k < \min Z$ and) for any $f : [[0, \max Z]]^n \to k$, there is an m-dense′(Γ) set $Y \subseteq Z$ such that $\Psi(f, Y)$ holds, and,
  - for any partition $f : [Z]^3 \to \ell$ such that $\ell < \min Z$ there is an m-dense′(Γ) set $Y \subseteq Z$ which is f-homogeneous.

Put $Y'_\Gamma(F) := \max\{m \mid F \text{ is } m\text{-dense'}(\Gamma)\}$.

Theorem

Y'_Γ is a set indicator for $\mathsf{ACA}_0 + \Gamma$.

With ACA_0, one can always characterize the Π^1_1-part of Γ.

Density with the base WKL_0^*

Definition (EFA, Density notion with the base WKL_0^*)

Given a Ramsey-like formula

$$\Gamma = (\forall f : [\mathbb{N}]^n \to k)(\exists Y)(Y \text{ is infinite} \wedge \Psi(f, Y)),$$

- $Z \subseteq_{\mathrm{fin}} \mathbb{N}$ is said to be 0-*dense*$^*(\Gamma)$ if $Z \neq \emptyset$,
- $Z \subseteq_{\mathrm{fin}} \mathbb{N}$ is said to be $(m+1)$-*dense*$^*(\Gamma)$ if
  - (for any $n, k < \min Z$ and) for any $f : [[0, \max Z]]^n \to k$, there is an m-dense$^*(\Gamma)$ set $Y \subseteq Z$ such that $\Psi(f, Y)$ holds, and,
  - $Z \setminus [0, \exp(\min Z)]$ is m-dense$^*(\Gamma)$.

Put $Y_\Gamma^*(F) := \max\{m \mid F \text{ is } m\text{-dense}^*(\Gamma)\}$.

Theorem

Y_Γ^* is a set indicator for $\mathrm{WKL}_0^* + \Gamma$.

Conservation results

- $\mathrm{WKL}_0^* + \mathrm{RT}^n_k$ is a $\tilde{\Pi}^0_3$-conservative extension of $\mathrm{RCA}_0^* + \{\forall X \subseteq_{\mathrm{inf}} \mathbb{N}\, \exists F \subseteq_{\mathrm{fin}} X(F \text{ is } n\text{-dense}^*(\mathrm{RT}^n_k)) \mid n \in \omega\}$.
  $= \mathrm{RCA}_0^*$
- $\mathrm{WKL}_0 + \mathrm{RT}^2_2$ is a $\tilde{\Pi}^0_3$-conservative extension of $\mathrm{RCA}_0^* + \{\forall X \subseteq_{\mathrm{inf}} \mathbb{N}\, \exists F \subseteq_{\mathrm{fin}} X(F \text{ is } n\text{-dense}(\mathrm{RT}^2_2)) \mid n \in \omega\}$.
  $= \mathrm{RCA}_0$
- $\mathrm{ACA}_0 + \mathrm{RT} = \mathrm{ACA}_0'$ is a Π^1_1-conservative extension of $\mathrm{RCA}_0^* + \{\forall X \subseteq_{\mathrm{inf}} \mathbb{N}\, \exists F \subseteq_{\mathrm{fin}} X(F \text{ is } n\text{-dense}(\mathrm{RT})) \mid n \in \omega\}$.
- ...

Question

Can indicator arguments be converted to proof-interpretation style conservation results?

Thank you!

- Richard Kaye, Models of Peano Arithmetic, Oxford University Press, 1991.
- Ludovic Patey and Y, The proof-theoretic strength of Ramsey's theorem for pairs and two colors, draft, available at http://arxiv.org/abs/1601.00050
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