On the duality of topological Boolean algebras

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Introduction

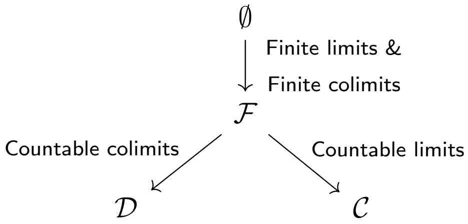
- Stone's representation of Boolean algebras (in $\mathbf{Set}$) as the set of clopen subsets of a compact zero-dimensional Hausdorff space is well known.
- It is slightly less well known that every compact zero-dimensional Hausdorff Boolean algebra is the powerset of a discrete space.
- Both dualities are based on character theories (in the same way as Pontraygin duality), where the two point discrete Boolean algebra 2 plays a pivotal role.
- The role of 2 can be highlighted by showing how the Boolean algebra structure arises naturally from a monad induced by 2.

Introduction

The material in this talk comes from the following sources:

- "The Pontryagin Duality of Compact 0-Dimensional Semilattices and its Applications" by K. Hofmann, M. Mislove, and A. Stralka
- "Topological Lattices" by D. Papert Strauss
- "Continuous lattices and domains" by G. Gierz, K. Hofmann, K. Keimel, J. Lawson, M. Mislove, and D. Scott
- "Stone Spaces" by P. Johnstone (particularly Chapter VI)
- "Sober spaces and continuations" by P. Taylor

We construct a few subcategories of **ZLCP** by starting with the empty subcategory and closing under certain limits/colimits



- $\emptyset$: Empty subcategory
- $\mathcal{F}$: Finite Hausdorff spaces ($=\mathcal{D} \cap \mathcal{C}$)
  - Ex: 0 (empty space), 1 (singleton space), $2 := 1 + 1$
- $\mathcal{D}$: Countable discrete spaces
  - Ex: $\mathbb{N} := \mu X.X + 1$ (inductive types)
- $\mathcal{C}$: 0-dim compact Polish spaces
  - Ex: $\mathbb{N}_\infty := \nu X.X + 1$ and $2^{\mathbb{N}} := \nu X.X \times 2$ (coinductive types)

The contravariant functor(s) $2^{(-)}$

$$\mathcal{D} \underset{2^{(-)}}{\overset{2^{(-)}}{\rightleftarrows}} \mathcal{C}$$

- For X in $\mathcal{D}$ or $\mathcal{C}$, the space 2^X is the space of all continuous functions from X to 2 (i.e., the clopen subsets of X) endowed with the compact-open topology.
  - If X is in $\mathcal{D}$ then 2^X is in $\mathcal{C}$
  - If X is in $\mathcal{C}$ then 2^X is in $\mathcal{D}$
  - Caution: $2^{(-)}$ is not defined on all of $\mathbf{ZLCP}$. The space $\mathbb{N} \times 2^{\mathbb{N}}$ is in $\mathbf{ZLCP}$, but $2^{(\mathbb{N} \times 2^{\mathbb{N}})} \cong \mathbb{N}^{\mathbb{N}}$ is not in $\mathbf{ZLCP}$.

The contravariant functor(s) $2^{(-)}$

$$\mathcal{D} \underset{2^{(-)}}{\overset{2^{(-)}}{\rightleftarrows}} \mathcal{C}$$

- A continuous function $f\colon X \to Y$ is mapped (contravariantly) to $2^f\colon 2^Y \to 2^X$ defined as $2^f := \lambda\phi.\lambda x.\phi(f(x))$.
    - Intuitively, 2^f maps a clopen $\phi \subseteq Y$ to the clopen $f^{-1}(\phi) \subseteq X$.

Topological Boolean algebras

- The discrete space $2 = \{\bot, \top\}$ is a Boolean algebra:
  - Disjunction (join) $\vee \colon 2 \times 2 \to 2$
  - Conjunction (meet) $\wedge \colon 2 \times 2 \to 2$
  - Negation $\neg \colon 2 \to 2$
- 2^X is a topological Boolean algebra:
  - $\top := \lambda x.\top$
  - $\bot := \lambda x.\bot$
  - $\vee \colon 2^X \times 2^X \to 2^X$ is the union of clopen sets

$$\phi \vee \psi := \lambda x.(\phi(x) \vee \psi(x))$$

  - $\wedge \colon 2^X \times 2^X \to 2^X$ is the intersection of clopen sets

$$\phi \wedge \psi := \lambda x.(\phi(x) \wedge \psi(x))$$

  - $\neg \colon 2^X \to 2^X$ is the complement of clopen sets

$$\neg\phi := \lambda x.\neg\phi(x)$$

- Let $(A, \top, \bot, \vee, \wedge, \neg)$ be a Boolean algebra in $\mathcal{D}$
  - (A has the discrete topology, so the operations are continuous)
- Then 2^A is a space in $\mathcal{C}$.
- Consider the subspace X of 2^A consisting of all Boolean algebra homomorphisms from A to 2:

$$X \xhookrightarrow{e} 2^A \underset{r}{\overset{\ell}{\rightrightarrows}} 2^{A \times A} \times 2^A \times 2$$

  X is the equalizer of the (continuous) maps ℓ and r:
  - $\ell := \lambda f. \langle \lambda \langle a, b \rangle . f(a \wedge b),\ \lambda c. f(\neg c),\ f(\top) \rangle$
  - $r := \lambda f. \langle \lambda \langle a, b \rangle . f(a) \wedge f(b),\ \lambda c. \neg f(c),\ \top \rangle$

  (ℓ and r also imply that every $f \in X$ preserves finite joins)
- Therefore, X is a space in $\mathcal{C}$ because it is the equalizer of a pair of maps between spaces in $\mathcal{C}$.

Stone Duality

- There is a bijection between ultrafilters of a Boolean algebra A and Boolean algebra homomorphisms from A to 2.
- So X can equivalently be viewed as the set of ultrafilters of A.
- X inherits the subspace topology from 2^A, which is generated by the clopen sets

$$U_a := \{f \in X \mid f(a) = \top\}$$

  for $a \in A$.
- $X \in \mathcal{C}$ is the Stone space associated to $A \in \mathcal{D}$, and Stone's representation theorem shows that 2^X and A are isomorphic Boolean algebras.
  - The isomorphism $h \colon A \to 2^X$ is defined as $h(a) = \lambda f. f(a)$, but the proof that it is an isomorphism is non-constructive.

Topological (?) Stone Duality

- Next consider a Boolean algebra $(A, \top, \bot, \vee, \wedge, \neg)$ in $\mathcal{C}$
  - (A has a non-trivial topology, and we will assume that the operations are continuous)
- Applying Stone duality directly to A will yield a Stone space C which is compact and Hausdorff.
- However, in general C is "too big" to be in $\mathbf{ZLCP}$.
  - The Stone dual of $2^{\mathbb{N}}$ is $\beta\mathbb{N}$, the Stone-Cech compactification of the natural numbers.
- Instead, we can just repeat the equalizer construction to get a more reasonably sized dual space.

Topological (?) Stone Duality

- Let $(A, \top, \bot, \vee, \wedge, \neg)$ be a (topological) Boolean algebra in $\mathcal{C}$
- Then 2^A is a (discrete) space in $\mathcal{D}$.
- Consider the subspace X of 2^A consisting of all continuous Boolean algebra homomorphisms from A to 2:

$$X \xhookrightarrow{e} 2^A \underset{r}{\overset{\ell}{\rightrightarrows}} 2^{A \times A} \times 2^A \times 2$$

  X is the equalizer of the (continuous) maps ℓ and r:
  - $\ell := \lambda f.\big\langle \lambda\langle a,b\rangle.f(a \wedge b),\ \lambda c.f(\neg c),\ f(\top)\big\rangle$
  - $r := \lambda f.\big\langle \lambda\langle a,b\rangle.f(a) \wedge f(b),\ \lambda c.\neg f(c),\ \top\big\rangle$

  (ℓ and r also imply that every $f \in X$ preserves finite joins)
- Therefore, X is in $\mathcal{D}$ because $\mathcal{D}$ is closed under subspaces.

Topological (?) Stone Duality

- X can be viewed as the set of clopen ultrafilters of A.
- Proving that A and 2^X are isomorphic requires a little topological algebra.
- The crucial observation (D. Papert Strauss, 1968, see also G. Bezhanishvili & J. Harding, 2015) is that every compact Hausdorff Boolean algebra is complete and atomic.
  - a is an atom if $a \neq \bot$ and for all $b \leq a$ either $b = \bot$ or $b = a$.
  - A is atomic if every element is the join of the atoms below it.
  - Complete atomic Boolean algebras are isomorphic to the powerset of its atoms with the usual set-theoretical join and meet operations.
- The main work remaining is to show that every $f \in X$ is of the form $\uparrow a := \{b \in A \mid a \leq b\}$ for some atom $a \in A$.

Topological (?) Stone Duality

- For every atom $a \in A$, the set $\uparrow a$ is a clopen ultrafilter:
  - Ultrafilter: $a \leq b \vee \neg b$ hence $a = (a \wedge b) \vee (a \wedge \neg b)$ which implies $a \leq b$ or $a \leq \neg b$.
  - Closed: $\uparrow a$ is the preimage of the closed singleton $\{a\}$ under the continuous map $\lambda b.(b \wedge a)$.
  - Open: $\downarrow(\neg a)$ is closed and equals the complement of $\uparrow a$ because if $a \not\leq b$ then $a \leq \neg b$ hence $b = \neg\neg b \leq \neg a$.

  Therefore, $\uparrow a$ is in X.
- For the converse, fix $f \in X$. Note that f is a clopen subset of A, hence compact.
  - Since f is a filter, the family of closed sets $\{\downarrow b \mid b \in f\}$ has the finite intersection property.
  - Using compactness of f, this implies there is a unique minimal element $a \in f$.
  - Clearly $a \neq \bot$ because $\bot \notin f$, and if $b < a$ then $a \leq \neg b$ (f is an ultrafilter) hence $b = b \wedge a \leq b \wedge \neg b = \bot$.

  Therefore, $f = \uparrow a$ for some atom $a \in A$.

Topological (?) Stone Duality

- Wrapping up, we again define an isomorphism $h\colon A \to 2^X$ as $h(b) = \lambda f. f(b)$.
  - Each $f \in X$ is of the form $\uparrow a$ for some atom in A, and $f(b) = \top$ iff $a \leq b$. Therefore, we can interpret $h(b)$ as the set of atoms below b.
  - The result of D. Papert Strauss guarantees that h is an isomorphism of Boolean algebras
  - h is continuous by definition, and every continuous bijection between compact Hausdorff spaces is a homeomorphism.

Therefore, 2^X and A are isomorphic topological Boolean algebras in $\mathcal{C}$.

Summary so far

$$\mathcal{D} \underset{2^{(-)}}{\overset{2^{(-)}}{\rightleftarrows}} \mathcal{C}$$

- For every topological Boolean algebra A in $\mathcal{D}$ there is a space $\mathbf{pt}(A)$ in $\mathcal{C}$ such that $A \cong 2^{\mathbf{pt}(A)}$.
- For every topological Boolean algebra A in $\mathcal{C}$ there is a space $\mathbf{pt}(A)$ in $\mathcal{D}$ such that $A \cong 2^{\mathbf{pt}(A)}$.

$$\mathbf{pt}(A) \hookrightarrow 2^A \underset{r}{\overset{\ell}{\rightrightarrows}} 2^{A\times A} \times 2^A \times 2$$

$$\begin{aligned} \ell &:= \lambda f.\langle \lambda\langle a,b\rangle.f(a\wedge b),\ \lambda c.f(\neg c),\ f(\top)\rangle \\ r &:= \lambda f.\langle \lambda\langle a,b\rangle.f(a)\wedge f(b),\ \lambda c.\neg f(c),\ \top\rangle \end{aligned}$$

Morphisms

- Clearly, the functor $2^{(-)}$ sends a continuous map $f \colon X \to Y$ (in either $\mathcal{D}$ or $\mathcal{C}$) to a Boolean algebra homomorphism $2^f \colon 2^Y \to 2^X$ (in the other category).
- Furthermore, a (continuous) Boolean algebra homomorphism $h \colon A \to B$ uniquely determines a map $u \colon \mathbf{pt}(B) \to \mathbf{pt}(A)$
  - For $f \in \mathbf{pt}(B)$ we have that $2^h(f) = \lambda a.f(h(a)) = f \circ h$ is a Boolean algebra homormorphism from A to 2, hence in $\mathbf{pt}(A)$.

$$\begin{array}{ccc}
\mathbf{pt}(B) & \hookrightarrow & 2^B \\
{\scriptstyle u}\downarrow & & \downarrow{\scriptstyle 2^h} \\
\mathbf{pt}(A) & \hookrightarrow & 2^A
\end{array}$$

Duality

- Let $\mathbf{Bool}(\mathcal{D})$ and $\mathbf{Bool}(\mathcal{C})$ denote the subcategories of (topological) Boolean algebras and (continuous) Boolean algebra homomorphisms in $\mathcal{D}$ and $\mathcal{C}$, respectively.
- The contravariant functors $2^{(-)}$ and $\mathbf{pt}$ define a dual equivalence between $\mathcal{D}$ and $\mathbf{Bool}(\mathcal{C})$ (also $\mathcal{C}$ and $\mathbf{Bool}(\mathcal{D})$)

$$
\begin{array}{ccc}
\mathcal{D} & \underset{\mathbf{pt}}{\overset{2^{(-)}}{\rightleftarrows}} & \mathbf{Bool}(\mathcal{C})^{op} \\
\uparrow & & \downarrow \\
\mathbf{Bool}(\mathcal{D}) & \underset{2^{(-)}}{\overset{\mathbf{pt}}{\rightleftarrows}} & \mathcal{C}^{op}
\end{array}
$$

Boolean algebras

In either $\mathcal{D}$ or $\mathcal{C}$ we have:

- The trivial Boolean algebra 1 is the terminal object (in both categories)
- 2 is the initial object
- Products $\otimes$ of Boolean algebras are given as $2^X \otimes 2^Y = 2^X \times 2^Y = 2^{X+Y}$
- Coproducts $\oplus$ of Boolean algebras are given as $2^X \oplus 2^Y = 2^{X \times Y}$
- 2^{2^X} is the free topological Boolean algebra on X
- $\mathbf{Bool}(\mathcal{D})$ is closed under countable colimits
- $\mathbf{Bool}(\mathcal{C})$ is closed under countable limits

The monad $2^{2^{(-)}}$

$$2^{2^{(-)}} \circlearrowright \mathcal{D} \qquad\qquad \mathcal{C} \circlearrowleft 2^{2^{(-)}}$$

- Applying $2^{(-)}$ twice yields a monad (for both $\mathcal{D}$ and $\mathcal{C}$).
  - $f\colon X \to Y$ maps to $2^{2^f} := \lambda F.\lambda\phi.F(\lambda x.\phi(f(x)))$.
- The unit $\eta_X : X \to 2^{2^X}$ is defined as $\eta_X := \lambda x.\lambda\phi.\phi(x)$.
  - $\eta_X(x)$ can be thought of as the set $\{\phi \in 2^X \mid x \in \phi\}$
- The multiplication $\mu_X : 2^{2^{2^{2^X}}} \to 2^{2^X}$ is defined as

$$\mu_X := 2^{\eta_{2^X}} = \lambda\mathfrak{F}.\lambda\phi.\mathfrak{F}(\lambda F.F(\phi))$$

$$\begin{array}{ccc} T^3 & \xrightarrow{T\mu} & T^2 \\ {\scriptstyle \mu T}\downarrow & & \downarrow{\scriptstyle \mu} \\ T^2 & \xrightarrow[\mu]{} & T \end{array} \qquad\qquad \begin{array}{ccc} T & \xrightarrow{\eta T} & T^2 \\ {\scriptstyle T\eta}\downarrow & \searrow^{1} & \downarrow{\scriptstyle \mu} \\ T^2 & \xrightarrow[\mu]{} & T \end{array}$$

$$T(X) := 2^{2^X}$$

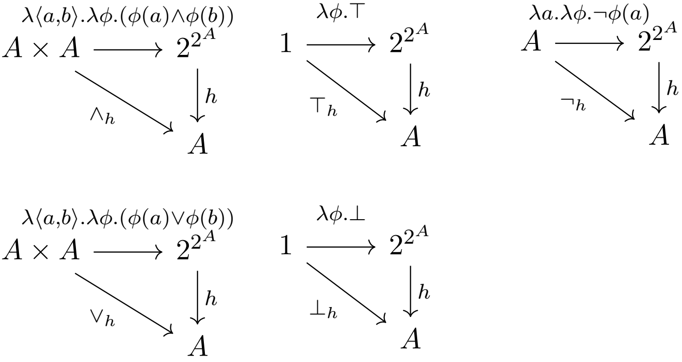
Every Boolean algebra $A \cong 2^{\mathbf{pt}(A)}$ is an algebra for the monad $2^{2^{(-)}}$ with structure map $h\colon 2^{2^A} (\cong 2^{2^{2^{\mathbf{pt}(A)}}}) \to A (\cong 2^{\mathbf{pt}(A)})$ defined as

$$h = 2^{\eta_{\mathbf{pt}(A)}} = \lambda \mathcal{F}.\lambda x.\mathcal{F}(\lambda \phi.\phi(x))$$

$$\begin{array}{ccc} T^2(A) & \xrightarrow{T(h)} & T(A) \\ {\scriptstyle \mu_A}\downarrow & & \downarrow{\scriptstyle h} \\ T(A) & \xrightarrow[h]{} & A \end{array} \qquad\qquad \begin{array}{ccc} A & \xrightarrow{\eta_A} & T(A) \\ & {\scriptstyle 1}\searrow & \downarrow{\scriptstyle h} \\ & & A \end{array}$$

Monad algebras are Boolean algebras

You can retrieve the Boolean algebra structure from a monad algebra (A, h) as follows:



Monad algebras are Boolean algebras

- We provide an example of how to prove this really makes (A, h) a Boolean algebra.
- The associative law $h \circ 2^{2^h} = h \circ \mu_A$ yields

$$h(\lambda\phi.\mathfrak{A}(\lambda F.\phi(h(F)))) = h(\lambda\phi.\mathfrak{A}(\lambda F.F(\phi)))$$

  for $\mathfrak{A} \colon 2^{2^{2^{2^A}}}$.
- The unit law $h \circ \eta_A = 1_A$ gives

$$h(\lambda\phi.\phi(b)) = b$$

  for $b \in A$.

Monad algebras are Boolean algebras

For $a, b, c \in A$, we show that

$$a \wedge_h (b \vee_h c) = (a \wedge_h b) \vee_h (a \wedge_h c).$$

- Plugging $\mathfrak{A}_1 := \lambda\mathcal{F}.\mathcal{F}(\lambda\psi.\psi(a)) \wedge \mathcal{F}(\lambda\psi.(\psi(b) \vee \psi(c)))$ into the associative law reduces to

$$\begin{aligned} & h\big(\lambda\phi.\phi(a) \wedge \phi(h(\lambda\psi.(\psi(b) \vee \psi(c))))\big) \\ = \quad & h\big(\lambda\phi.\phi(a) \wedge (\phi(b) \vee \phi(c))\big) \end{aligned}$$

  The left hand side is the definition of $a \wedge_h (b \vee_h c)$.

- Next plug in $\mathfrak{A}_2 := \lambda\mathcal{F}.\mathcal{F}(\lambda\psi.(\psi(a) \wedge \psi(b))) \vee \mathcal{F}(\lambda\psi.(\psi(a) \wedge \psi(c)))$ and get

$$\begin{aligned} & h\big(\lambda\phi.\phi(h(\lambda\psi.(\psi(a) \vee \psi(b)))) \vee \phi(h(\lambda\psi.(\psi(a) \vee \psi(c))))\big) \\ = \; & h\big(\lambda\phi.(\phi(a) \wedge \phi(b)) \vee (\phi(a) \wedge \phi(c))\big) \end{aligned}$$

  The left hand side is the definition of $(a \wedge_h b) \vee_h (a \wedge_h c)$. The right hand side equals $h\big(\lambda\phi.\phi(a) \wedge (\phi(b) \vee \phi(c))\big)$ because $\wedge$ distributes over $\vee$ in 2. The previous slide showed this is equal to $a \wedge_h (b \vee_h c)$.

- As another example, $\mathfrak{A} := \lambda\mathcal{F}.\mathcal{F}(\lambda\psi.\psi(b)) \vee \mathcal{F}(\lambda\psi.\neg\psi(b))$ can be used to show that $(b \vee_h \neg_h b) = \top_h$.

Monad algebra morphisms

Similarly, you can show that monad algebra morphisms correspond to Boolean algebra morphisms.

$$
\begin{array}{ccc}
2^{2^A} & \xrightarrow{2^{2^f}} & 2^{2^B} \\
{\scriptstyle h}\big\downarrow & & \big\downarrow{\scriptstyle h'} \\
A & \xrightarrow[f]{} & B
\end{array}
$$

We obtain that the subcategory of $2^{2^{(-)}}$ algebras (in $\mathcal{D}$ or $\mathcal{C}$) is precisely the subcategory of Boolean algebras (in $\mathcal{D}$ or $\mathcal{C}$).

Vietoris space and modal logic

- When X is in $\mathcal{D}$ or $\mathcal{C}$, we have that $X \hookrightarrow 2^{2^X}$ embeds as the subspace of Boolean algebra homomorphisms ($X = \mathbf{pt}(2^X)$).
- If instead we take the subspace of 2^{2^X} of meet semilattice morphisms (maps preserving $\wedge$ and $\top$, but not necessarily $\neg$) then we get the Vietoris space $\mathcal{V}(X)$.
  - $\mathcal{V}(X)$ is defined as the space of compact subsets of X with topology generated by the clopen sets:

$$\begin{aligned} \Box\phi &:= \{\kappa \in \mathcal{V}(X) \mid \kappa \subseteq \phi\}, \text{ and} \\ \Diamond\phi &:= \{\kappa \in \mathcal{V}(X) \mid \kappa \cap \phi \neq \emptyset\} \end{aligned}$$

    for $\phi \in 2^X$. Note that $\Box\phi = \neg\Diamond\neg\phi$ and $\Diamond\phi = \neg\Box\neg\phi$.
  - Using the homeomorphism $\lambda F.\lambda\phi.\neg F(\neg\phi) \colon 2^{2^X} \to 2^{2^X}$ we can see that taking join semilattice morphisms instead would yield a space homeomorphic to $\mathcal{V}(X)$.
- $\mathcal{V}(X)$ is the free topological semilattice on X (in $\mathcal{D}$ or $\mathcal{C}$)

Vietoris space and modal logic

- There is a bijection between continuous maps $f\colon X \to \mathcal{V}(X)$ in $\mathcal{D}$ (resp., $\mathcal{C}$) and continuous meet semilattice morphisms $\widehat{f}\colon 2^X \to 2^X$ in $\mathcal{C}$ (resp., $\mathcal{D}$)
  - $\widehat{f}$ is the double transpose of f.
- A map $f\colon X \to \mathcal{V}(X)$ can be viewed as a non-deterministic transition system, or Kripke frame
- A meet semilattice morphism $\widehat{f}\colon 2^X \to 2^X$ can be viewed as a modal operator $\Box$ on the Boolean algebra.

Conclusion

- We have looked at the following dualities:

$$\begin{array}{ccc} \mathcal{D} & \underset{\mathbf{pt}}{\overset{2^{(-)}}{\rightleftarrows}} & \mathbf{Bool}(\mathcal{C})^{op} \\ \uparrow & & \downarrow \\ \mathbf{Bool}(\mathcal{D}) & \underset{2^{(-)}}{\overset{\mathbf{pt}}{\rightleftarrows}} & \mathcal{C}^{op} \end{array}$$

- The objects of $\mathbf{Bool}(\mathcal{C})$ and $\mathbf{Bool}(\mathcal{D})$ are topological Boolean algebras, and are the algebras of the monad $2^{2^{(-)}}$
- Can the correspondence between A and $\mathbf{pt}(A)$ be made more constructive if we have inductive/coinductive definitions of the spaces?
    - Replace the coproduct and terminal object (from $\mathcal{D}$) in $\mathbb{N} = \mu X.X + 1$ with the product and initial object (from $\mathbf{Bool}(\mathcal{C})$) to get $2^{\mathbb{N}} = \nu X.X \times 2$
    - In general, can we convert a coinductive definition interpreted in $\mathbf{Bool}(\mathcal{C})$ into a coinductive definition for the same space in $\mathcal{C}$ (or similarly convert inductive definitions in $\mathbf{Bool}(\mathcal{D})$ to $\mathcal{D}$)?