Analytic Functions and Small Complexity Classes

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Workshop on Mathematical Logic and its Applications
Kyoto University, Kyoto, Japan

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TTE: Computable Real Numbers

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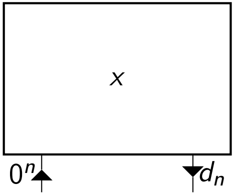

d_n rational approximations

$$|d_n - x| \leq 2^{-n}.$$

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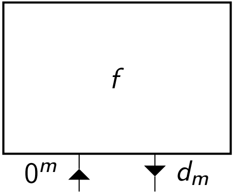

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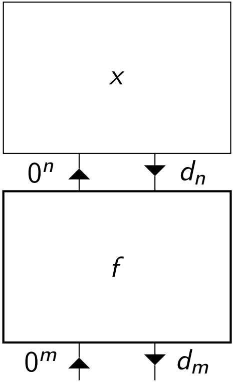

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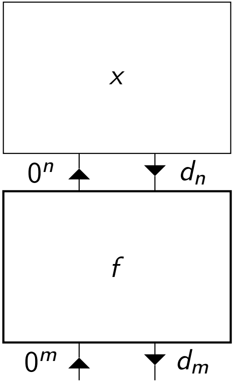

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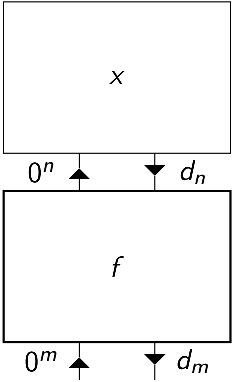

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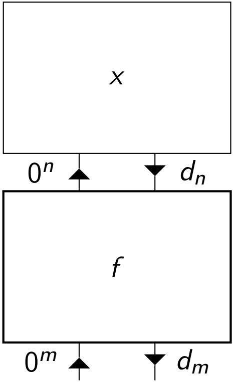

Notes

- closed under composition
- multidimensional functions: several input oracles and outputs.
- Computable functions are continuous.

Generalization

Representation

A representation for a set X is a partial surjective function $\rho : \mathcal{B} \to X$, that is, objects are encoded by string functions.

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$$\begin{tikzcd}
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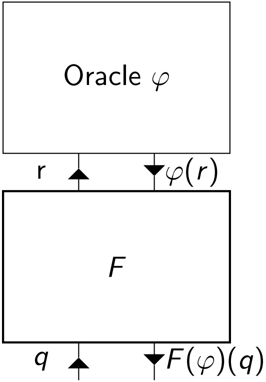
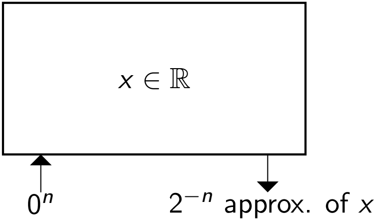


Representation for Real Numbers

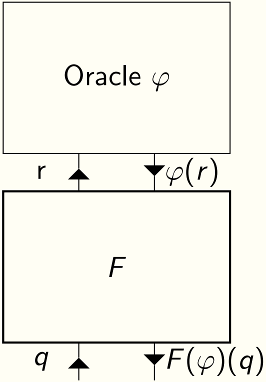
Example

Define a $\rho_{\mathbb{R}}$-name of $x \in \mathbb{R}$ by letting $\rho_{\mathbb{R}}(0^n)$ encode some $d \in \mathbb{D}$ such that $|d - x| \leq 2^{-n}$.

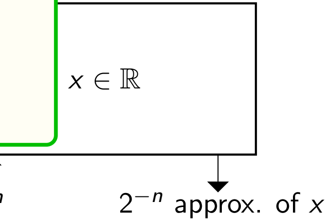


Numbers

$(\rho_{\mathbb{R}}^{|[0,1]}, \rho_{\mathbb{R}})$-computable



$\mathbb{R}$ by letting $\rho_{\mathbb{R}}(0^n)$ encode some $d \in \mathbb{D}$



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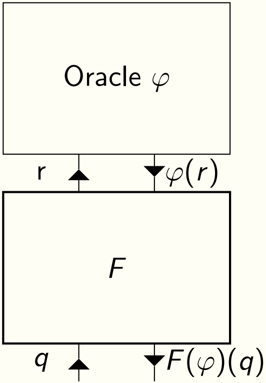
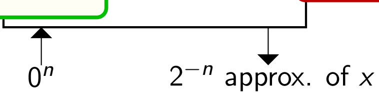
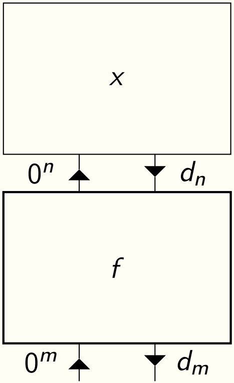


$\mathbb{R}$ by letting $\rho_{\mathbb{R}}$

$x \in \mathbb{R}$



Computable Function



Representation for Functions

Example

Define a $\delta_{\Box}$-name of a function $f \in C[0,1]$ as a pair $< \mu, \phi >$ where

- ϕ is such that $|\phi(d, 0^n) - f(d)| \leq 2^{-n}$ for all $d \in \mathbb{D} \cap [0,1]$
- μ encodes the modulus of continuity (in unary)

Representation for Fun

Modulus of Continuity

$$|x - y| \leq 2^{-\mu(n)} \Rightarrow |f(x) - f(y)| \leq 2^{-n}$$

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Define a $\delta_{\square}$-name of a function $f \in C[0,1]$ as a pair $< \mu, \phi >$ where

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$$\boxed{f \in C[0,1]}$$

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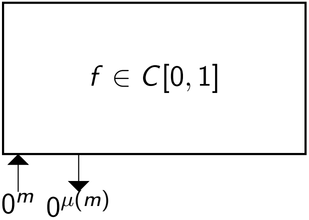

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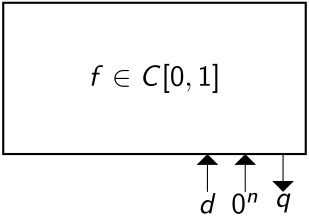

Type-2 Complexity Theory

- Let Σ^{**} be the set of length-monotone string-functions

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- Bound running time by second order polynomials $P(|\varphi|)(|x|)$

Type-2

Second Order Polynomials

Defined inductively by

1. $P(L,n) := m$, for $m \in \mathbb{N}$
2. $P(L,n) := n$
3. P, Q second order polynomials $\Rightarrow P + Q,\ P \cdot Q$
4. P a second order polynomial $\Rightarrow L(P)$

Example: $P(L,n) = 2L(L(L(n) \cdot L(n) + 2L(n)) + 2) + n + 10$

- Let
- Defi
- Bou

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- Can define complexity classes FP^2, $\#\mathrm{P}^2$ and $\mathrm{FPSPACE}^2$

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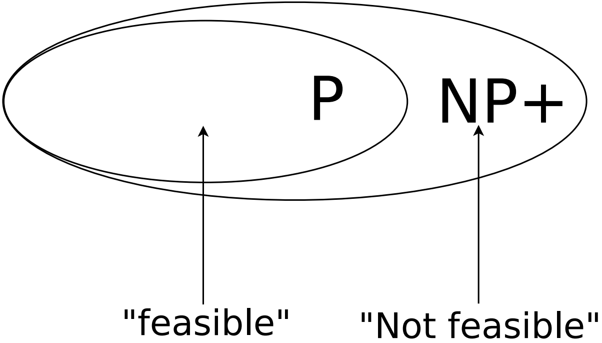
- Bound running time by second order polynomials $P(|\varphi|)(|x|)$
- Can define complexity classes FP^2, $\#\mathrm{P}^2$ and $\mathrm{FPSPACE}^2$
- Can also define complexity classes P^2, NP^2 and PSPACE^2 by considering functions $\varphi : \Sigma^{**} \to (\Sigma^* \to \{0, 1\})$

Small Complexity Classes



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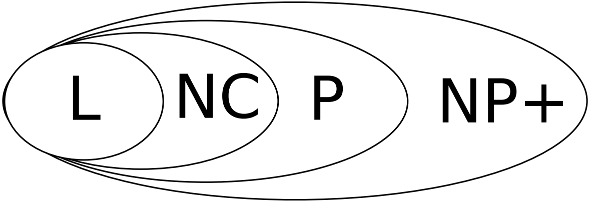

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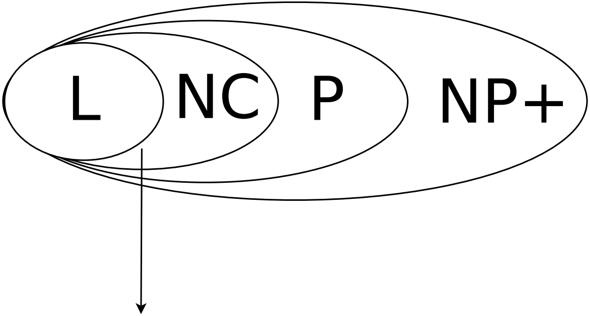
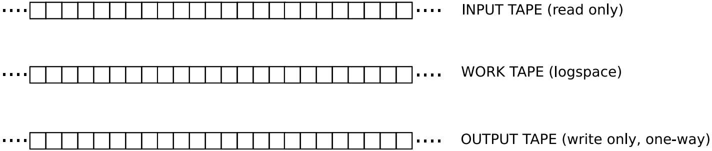


"efficiently solvable by a parallel computer"

Small Complexity Classes

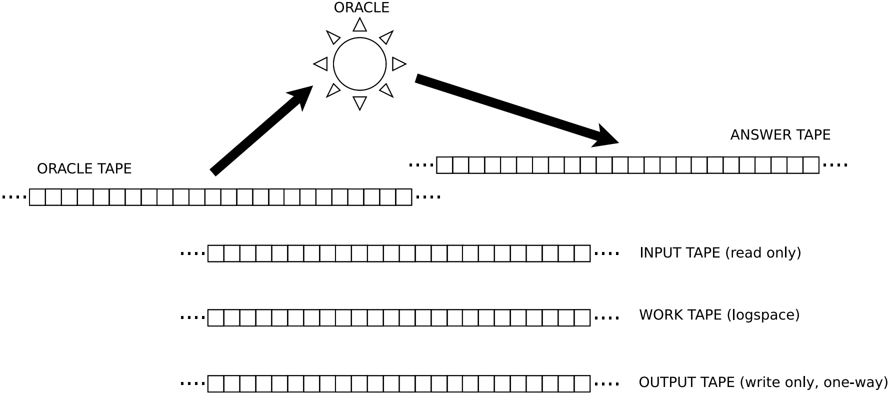


INPUT TAPE (read only)

WORK TAPE (logspace)

OUTPUT TAPE (write only, one-way)

Small Complexity Classes



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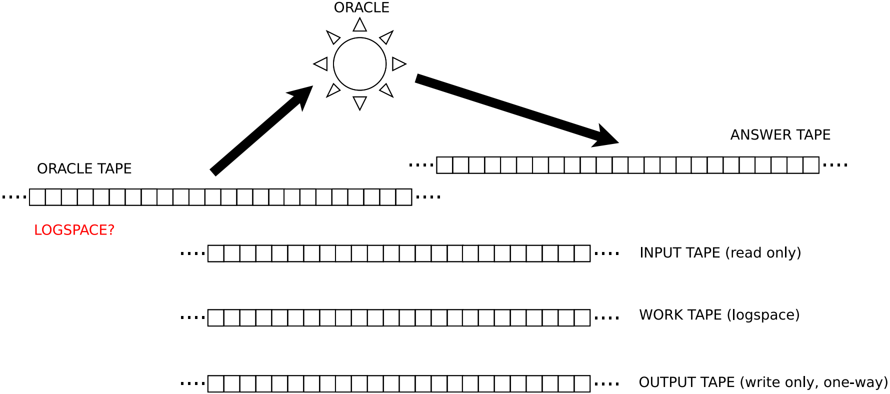

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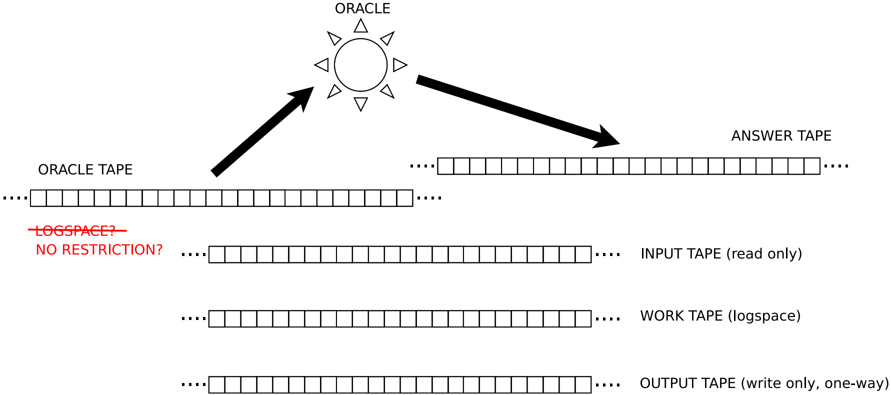

ORACLE

ANSWER TAPE

ORACLE TAPE

~~LOGSPACE?~~

NO RESTRICTION?

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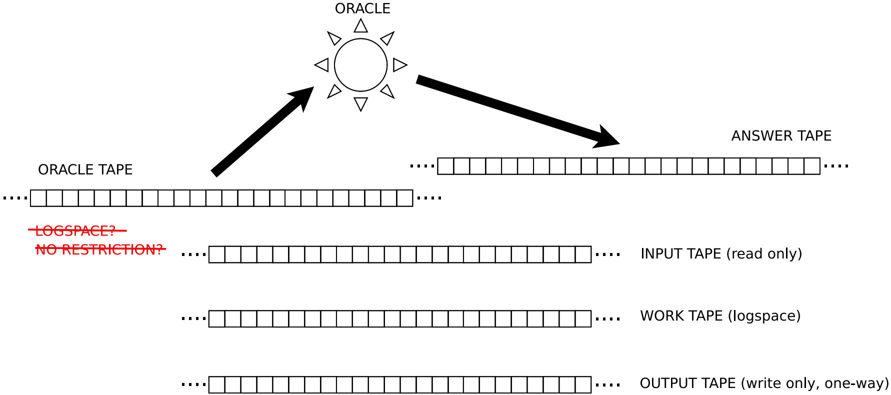

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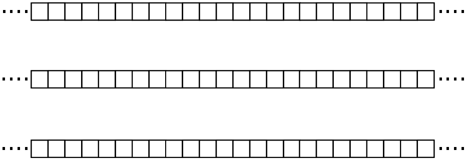
The stack model (Kawamura and Ota)



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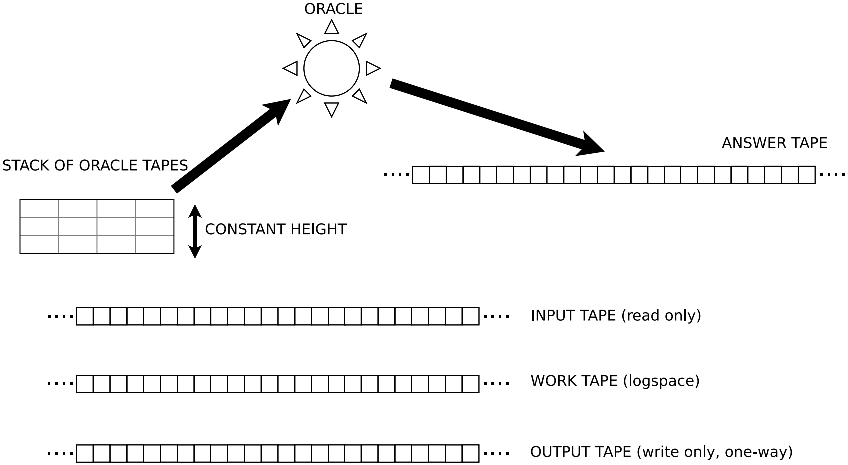

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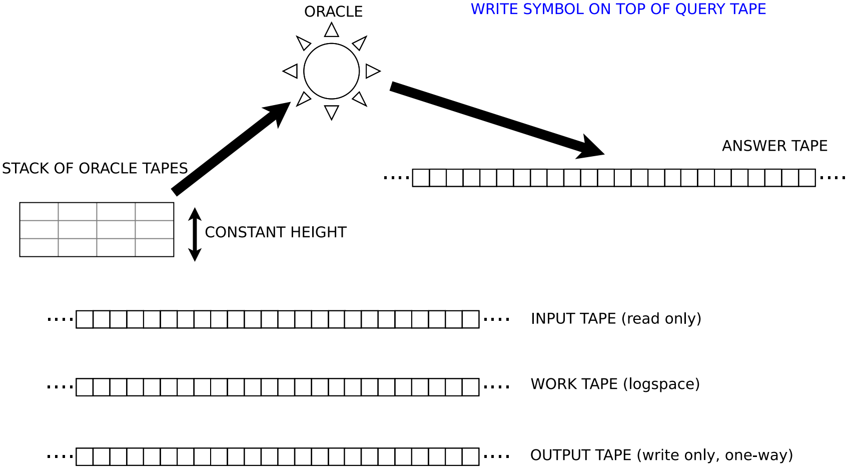

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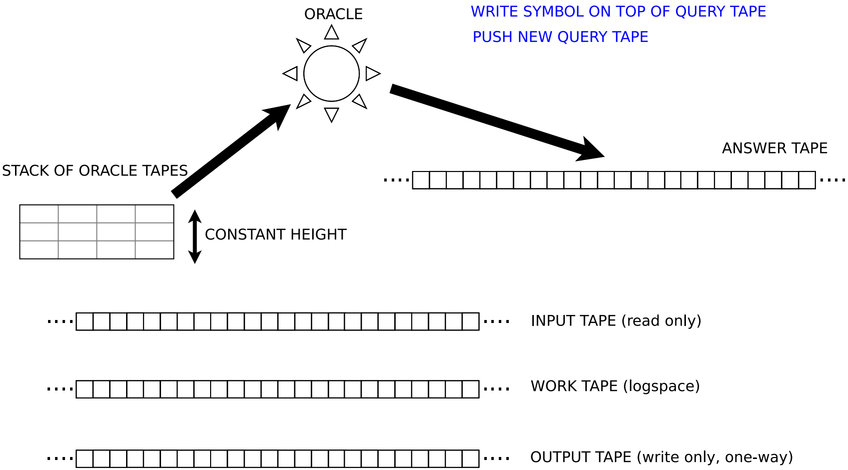

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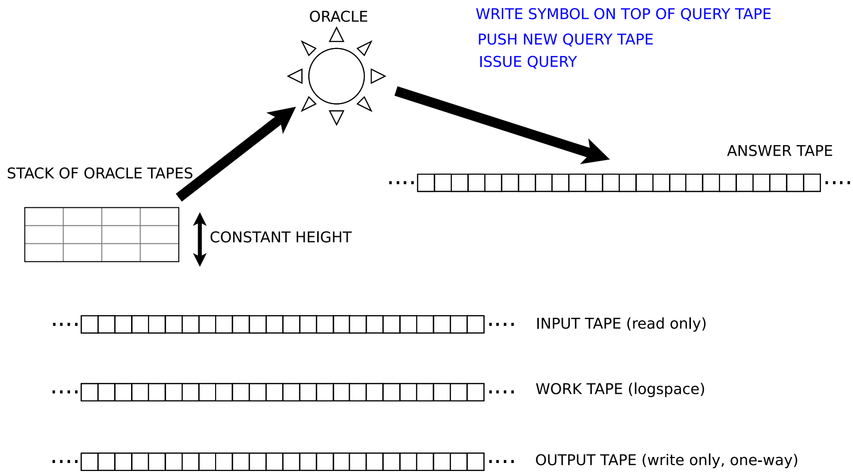


WRITE SYMBOL ON TOP OF QUERY TAPE

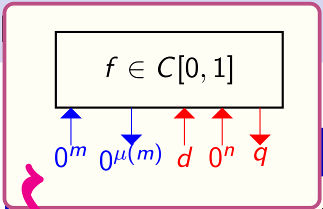
PUSH NEW QUERY TAPE

ISSUE QUERY

Example

Function Application

The function Apply : $C[0,1] \times [0,1] \to \mathbb{R}, (f,x) \mapsto f(x)$ is $([\delta_{\square}, \rho_{\mathbb{R}}^{|[0,1]}], \rho_{\mathbb{R}})-$FL computable.



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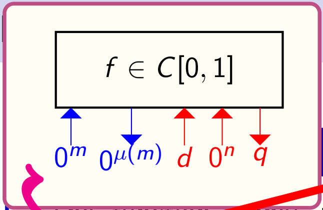
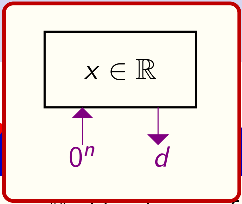


$f \in C[0,1]$

0^m $0^{\mu(m)}$ d 0^n q



$x \in \mathbb{R}$

0^n d

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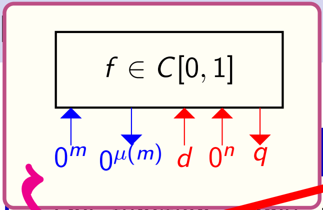

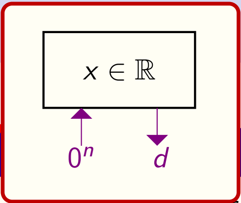

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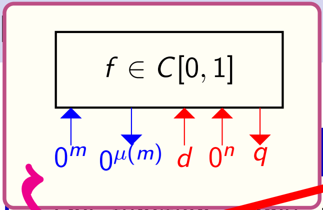
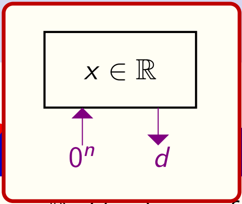
Input:	0^n
Stack:	-
Oracle Output:	ε
Work:	ε
Output:	ε





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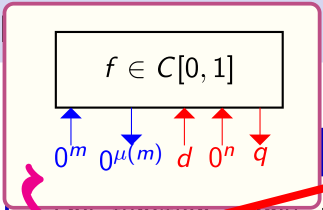
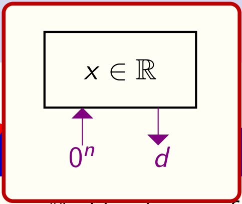
```
Input:          0^n
Stack:          ε;ε;ε
Oracle Output:  ε
Work:           ε
Output:         ε
```





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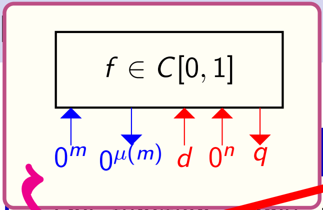
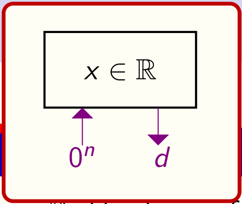
Input:	0^n
Stack:	$0^{n+1};\varepsilon;\varepsilon$
Oracle Output:	ε
Work:	ε
Output:	ε





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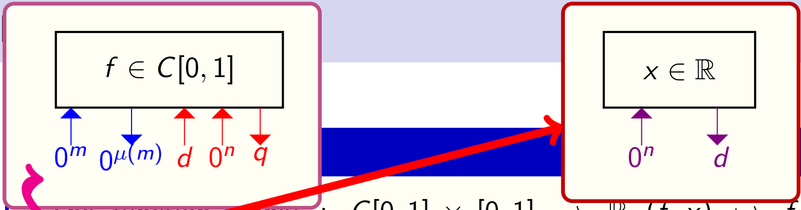
`Input:`	0^n
`Stack:`	ε;ε
`Oracle Output:`	0^m
`Work:`	ε
`Output:`	ε

$$|x-y| \le 2^{-m} \Rightarrow |f(x)-f(y)| \le 2^{-(n+1)}$$



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Stack:	0^m;ε
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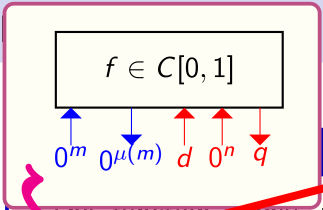

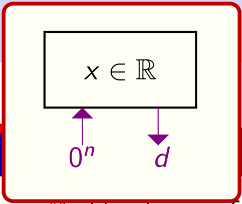

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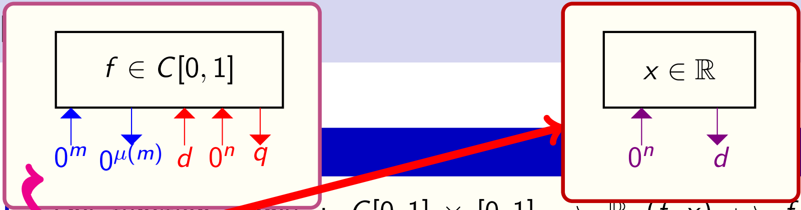
`Input:`	0^n
`Stack:`	ε
`Oracle Output:`	d
`Work:`	ε
`Output:`	ε

$|x - d| \leq 2^{-m}$



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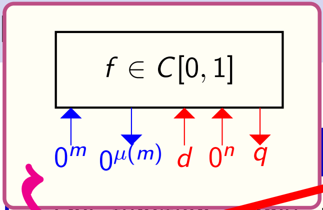
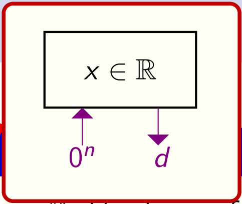
`Input:`	0^n
`Stack:`	$d, 0^{n+1}$
`Oracle Output:`	d
`Work:`	ε
`Output:`	ε





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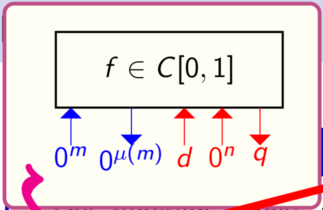
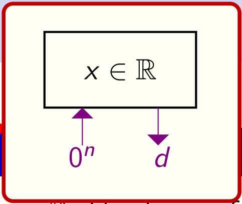
`Input:`	0^n
`Stack:`	-
`Oracle Output:`	q
`Work:`	ε
`Output:`	ε

$$|q - f(d)| \leq 2^{-(n+1)}$$





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`Input:`	0^n
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`Work:`	ε
`Output:`	q

$|q - f(x)| \leq 2^{-n}$

Example

Function Application

The function Apply $: C[0,1] \times [0,1] \to \mathbb{R}, (f,x) \mapsto f(x)$ is $([\delta_\square, \rho_\mathbb{R}^{|[0,1]}], \rho_\mathbb{R})-$FL computable.

Similarly, Apply^c $: C[0,1] \times [0,1] \to [0,1]\,(f,x) \mapsto f^c(x)$ is $([\delta_\square, \rho_\mathbb{R}^{|[0,1]}], \rho_\mathbb{R})-$FL computable for constant $c \in \mathbb{N}$ using a stack of size $2c+1$.

Some Results from Real Complexity Theory

Fact

For general polynomial time computable functions, many important operators have been shown to be computationally hard.

For example

- Polynomial time computable functions may have non computable derivatives. (Ko 1983)
- Parametric maximization is NP-hard. (Ko/Friedman (1982))
- Integration is #P-hard. (Friedman (1984))

Analytic Function

An analytic function is a function locally given by a complex power series.

Definition (Analytic Function)

$f : D \to \mathbb{C}$, $D \subseteq \mathbb{C}$ is analytic if for any $x_0 \in D$ the Taylor-series

$$T(x) := \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

converges to $f(x)$ for x in a neighborhood of x_0.

Analytic Function

An analytic f es.

Computable Sequence

$(a_m)_{m \in \mathbb{N}}$ is computable if there is a machine that on input $1^n, 1^m$ outputs a rational approximation $d_{n,m}$ with $|d_{n,m} - a_m| \leq 2^{-n}$

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converges to $f(x)$ for x in a neighborhood of x_0.

Theorem (Pour-El, Richards, Ko, Friedman, Müller (1987/1989))

f is (polytime) computable iff $(a_m)_{m\in\mathbb{N}}$ is.

From that polynomial time computability of the derivative and the anti-derivative of a function follows immediately.

Representation for Analytic Functions

How to represent analytic functions?

Representation for Analytic Functions

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Theorem (Müller (1995))

The evaluation operator $((a_m)_{m\in\mathbb{N}}, x) \to f(x)$ *is not computable.*

Representation for Analytic Functions

How to represent analytic functions?

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Lemma

Let $f : \overline{B}(0,1) \to \mathbb{R}$ be analytic and $(a_n)_{n\in\mathbb{N}}$ its power series around 0. Then there exist $k, A \in \mathbb{N}$ such that

1. $\sqrt[k]{2}$ *is a lower bound on the radius of convergence*
2. $|a_n| \leq A \cdot 2^{-\frac{n}{k}}$

How to represent analytic functions?

Representation 1

A function $\varphi \in \Sigma^{**}$ is a name for a power series $(a_k)_{k\in\mathbb{N}}$ iff it is a concatenation of the following

1. *An integer A encoded in binary*
2. *An integer k encoded in unary*
3. *A name for a sequence $(a_k)_{k\in\mathbb{N}}$*

Such that $|a_n| \leq A \cdot 2^{-\frac{n}{k}}$ for all $n \in \mathbb{N}$.

How to represent analytic functions?

Representation 2

A (length-monotone) function $\varphi : \Sigma^* \to \Sigma^*$ *is a name for an analytic function* $f : \bar{B}(0,1) \to \mathbb{R}$ *iff it is a concatenation of the following*

1. *An integer* A *encoded in binary,*
2. *An integer* k *encoded in unary,*
3. *A name for the function* f

Such that f *extends analytically to* $B(0, \sqrt[k]{2})$ *and* $|f(z)| \leq A$ *for all* $z \in B(0, \sqrt[k]{2})$

Analytic Functions and Computational Complexity

Theorem (Kawamura, Rösnick, Müller, Ziegler (2013))

With the previous two representations the following operations can be performed in polynomial time

1. *evaluation*
2. *addition and multiplication*
3. *differentiation and anti-differentiation*
4. *parametric maximization*

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Further, when identifying an analytic function with its power series, the operators that compute one representation from the other are polynomial-time computable.

Complexity of Ordinary Differential Equations

Theorem (Kawamura, 2010)

Consider the IVP

$$y'(t) = f(t, y(t)) \quad ; \quad y(0) = 0.$$

There exists functions $f : [0,1] \times [-1,1] \to \mathbb{R}$ *and* $y : [0,1] \to [-1,1]$ *as above such that*

1. f *is Lipschitz-continuous and polynomial time computable*
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For analytic right-hand side y will again be analytic and polynomial time computable.

Analytic Functions and Small Complexity Classes

Consider functions complex analytic on the closed unit disc.

Representation 1

Integers A, k and the series sequence $(a_k)_{k \in \mathbb{N}}$.
$|a_n| \leq A \cdot 2^{-\frac{n}{k}}$ *for all* $n \in \mathbb{N}$.

Representation 2

Integers A, k, and name for function f.
f *extends analytically to* $B(0, \sqrt[k]{2})$ *and* $|f(z)| \leq A$ *for all* $z \in B(0, \sqrt[k]{2})$

Those two representations are logspace equivalent.

Representation 1 $\Rightarrow$ Representation 2

Task

Given A, k s.t. $|a_n| \leq A \cdot 2^{-\frac{n}{k}}$ for all $n \in \mathbb{N}$.
Find A', k' s.t. for all $z \in B(0, \sqrt[k']{2})$ $|f(z)| \leq A'$

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Solution

Let $k' = 2k$ and $A' = 4kA$

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Solution

Let $k' = 2k$ and $A' = 4kA$

Proof

For $z \in B(0, \sqrt[k']{2}) :\ f(z) = \sum_{n=0}^{\infty} a_n z^n \leq A \sum_{n=0}^{\infty} 2^{-\frac{n}{k}} \cdot 2^{\frac{n}{2k}} \leq 4Ak$

Representation 1 ⇒ Representation 2

Task

Compute a function name for f.

Representation 1 ⇒ Representation

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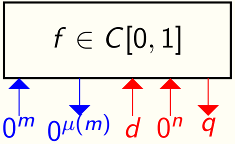
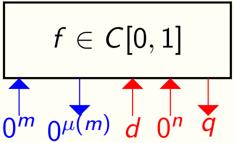


$f \in C[0,1]$

0^m $0^{\mu(m)}$ d 0^n q

Representation 1 ⇒ Representation



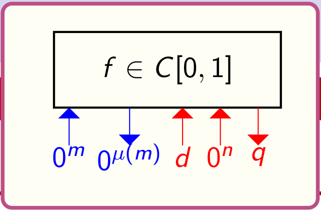
Task

Compute a function name for f.

Solution

$m \mapsto m + \log_2(A) + 2\log_2(k) + 5$ is a modulus of continuity for the function.

Representation 1 ⇒ Representation



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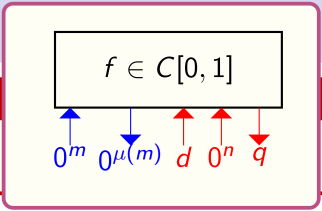

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We need to evaluate $N \approx n \cdot k + \log(A)$ terms of the sum.
$\sum_{j=0}^{poly(n)} a_j x^j$ is logspace computable.

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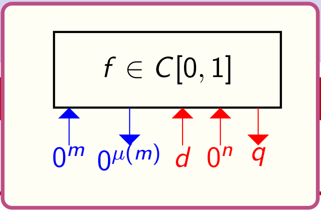

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Proof

The following is logspace computable

1. Addition and Multiplication of polynomially many n-bit integers
2. x^m with precision polynomial in n for polynomial length m
3. Composition of constantly many logspace computable functions

Representation 2 $\Rightarrow$ Representation 1

Task

Given A, k s.t. for all $z \in B(0, \sqrt[k']{2})$ $|f(z)| \leq A$
Compute A', k' such that $|a_m| \leq A \cdot 2^{-\frac{m}{k}}$ for all $m \in \mathbb{N}$.

Solution

We can just set $A' = A$ and $k' = k$.

Representation 2 $\Rightarrow$ Representation 1

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Given A, k s.t. for all $z \in B(0, \sqrt[k']{2})$ $|f(z)| \leq A$
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Solution

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Proof

By Cauchy's integral formula $|a_m| = \frac{f^{(m)}(0)}{m!} \leq A \cdot 2^{-\frac{n}{k}}$ for all $n \in \mathbb{N}$.

Representation 2 $\Rightarrow$ Representation 1

Task

Compute the coefficients for the power series around 0.

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Solution

- Approximate function by a polynomial
- Differentiate this polynomial and evaluate at 0
- Divide by factorial

Representation 2 ⇒ Representation 1

Task

Compute the coefficients for the power series around 0.

Solution

- Approximate function by a polynomial
- Differentiate this polynomial and evaluate at 0
- Divide by factorial

Proof

- Computing factorials and binomial coefficients of polynomial size is logspace computable.
- To get the coefficient a_m the function has to be evaluated at $2m+1$ equidistant points with polynomial precision (see e.g.Müller).

Further Operations

Logspace computable operations

Similarly, the following operations on analytic functions are computable in logartihmic-space

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Similarly, the following operations on analytic functions are computable in logartihmic-space

1. Addition, Subtraction, Multiplication of two analytic functions
2. Computing the d-fold derivative
3. Computing the d-fold anti-derivative

Multidimensional Analytic Functions

Multidimensional Power Series

$$\begin{aligned}\sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}} a_{i,j}x_1^i x_2^j &= \sum_{i\in\mathbb{N}} b_i x_1^i \\ \text{with } b_i &:= \sum_{j\in\mathbb{N}} a_{i,j}x_2^j\end{aligned}$$

Computing $b_i \rightarrow$ evaluating an analytic function.

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Computing b_i → evaluating an analytic function.

$|a_{i,j}| \leq A \cdot 2^{-\frac{i+j}{k}}$ for all $n \in \mathbb{N}$.

$|b_i| \leq \sum_{j\in\mathbb{N}} |a_{i,j}|\,|x_2|^j \leq A2^{-\frac{i}{k}} \sum_{j\in\mathbb{N}} 2^{-\frac{j}{k}} = A2^{-\frac{j}{k}} \frac{\sqrt[k]{2}}{\sqrt[k]{2}-1} \leq \mathbf{(2Ak)}2^{-\frac{i}{k}}$

Multidimensional analytic functions

Representation

A function $\varphi \in \Sigma^{**}$ is a name for a d dimensional power series $(a_{n_1,\dots,n_d})_{n_1,\dots,n_d \in \mathbb{N}}$ iff it is a concatenation of the following

1. An integer A encoded in binary
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Such that $|a_{n_1,\dots,n_d}| \leq A \cdot 2^{-\frac{n_1+\dots+n_d}{k}}$ for all $n \in \mathbb{N}$.

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Note

Complexity of evaluation (and most other operations) is **exponential** in the dimension!

Open Problem: P-completeness

Kawamura and Ota also define the notions of reductions and completeness.

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Theorem

For the set M of bijective functions in $C[0,1]$, define the $\delta_{\square INV}$ representation by adding a modulus of continuity for the inverse function to the $\delta_{\square}$ representation.
The function $Inv : M \to C[0,1]$, $Inv(f) \mapsto f^{-1}$ is $(\delta_{\square INV}, \delta_{\square})$-FP-complete.

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Which operations on analytic functions are P-hard?

- Initial Value Problem?
- Maximization?

Open Problem: P-completeness

Initial Value Problem

$$\dot{y}(t) = f(y(t)) \quad ; \quad y(0) = 0.$$

Open Problem: P-completeness

Initial Value Problem

$$\dot{y}(t) = f(y(t)) \quad ; \quad y(0) = 0.$$

Power series

$$\begin{aligned} f^{[0]}(y) &= y \\ f^{[k+1]}(y) &= \frac{1}{k+1} f(y) \frac{d}{dy} f^{[k]}(y) \end{aligned}$$

Then

$$a_k = \frac{1}{k!} y^{(k)}(0) = f^{[k]}(0)$$

This can be used to get a recurrence relation that makes it possible to compute the coefficients in polynomial time.

Can they also be computed without such a recurrence?

Conclusion

- Presented Kawamura and Ota's model for logspace computability in analysis.
- In this model many operations on analytic functions are logspace computable, when considering representations that have previously been considered for polynomial time computability.
- Open Problems: Parametrized Maximization, Ordinary Differential Equations
- Connection to parallelization in exact real arithmetic
- Implementations