The fan theorem and convexity

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- $\{0,1\}^*$ the set of finite binary sequences
- $u, v, w \in \{0,1\}^*$
- $|u|$ the length of u
- $\overline{u}n$ the restriction of u to the first n elements
- $u * v$ the concatenation of u and v
- $i \in \{0,1\}$
- α, β infinite binary sequences

$\mathrm{B} \subseteq \{0,1\}^*$ is

- *detachable* if $\forall u \, (u \in \mathrm{B} \vee u \notin \mathrm{B})$
- a *bar* if $\forall \alpha \, \exists n \, (\overline{\alpha} n \in \mathrm{B})$
- a *uniform bar* if $\exists N \, \forall \alpha \, \exists n \leq N \, (\overline{\alpha} n \in \mathrm{B})$

FAN every detachable bar is a uniform bar

neither provable nor falsifiable in Bishop's constructive mathematics

Lemma (Julian, Richman 1984)

The following are equivalent:

- FAN
- $f : [0,1] \to \mathbb{R}^+$ u/c $\;\Rightarrow\;$ $\inf f > 0$

Lemma (B., Svindland 2016)

$f : [0,1] \to \mathbb{R}^+$ u/c + **convex** $\;\Rightarrow\;$ $\inf f > 0$

Is there a corresponding extra condition on bars such that FAN becomes constructively valid?

$$u < v \;:\Leftrightarrow\; |u| = |v| \wedge \exists i < |u| \left(\overline{u}i = \overline{v}i \wedge u_i = 0 \wedge v_i = 1\right)$$

$$u \le v \;:\Leftrightarrow\; u = v \vee u < v.$$

$\mathrm{A} \subseteq \{0,1\}^*$ is

- *convex* if

$$u \le v \le w \;\wedge\; u \in \mathrm{A} \;\wedge\; w \in \mathrm{A} \;\Rightarrow\; v \in \mathrm{A}$$

- *co-convex* if $\{0,1\}^* \setminus \mathrm{A}$ is convex

Proposition. *Every detachable co-convex bar is a uniform bar.*

Fix a detachable co-convex bar B. We can assume that B is *closed under extension*:

$$u \in \mathrm{B} \Rightarrow u * 0 \in \mathrm{B} \wedge u * 1 \in \mathrm{B}$$

u is *secure* if

$$\exists n \forall w \in \{0,1\}^n (u * w \in \mathrm{B})$$

Claim 1. *For every u, either $u * 0$ is secure or $u * 1$ is secure.*

There exists a function

$$F : \{0, 1\}^* \to \{0, 1\}$$

such that

$$\forall u \ (u * F(u) \text{ is secure}) \,.$$

Define α by

$$\alpha_n = 1 - F(\overline{\alpha}n).$$

Claim 2. $\forall n \forall u \in \{0,1\}^n \, (u \neq \overline{\alpha}n \Rightarrow u \textit{ is secure})$

There exists n such that $\overline{\alpha}n$ is secure. Therefore, every u of length n is secure. Therefore, B is a uniform bar.

■

Proof of Claim 1. For

$$\beta := 1 * 0 * 0 * 0 * \ldots$$

there exists a positive l with $\overline{\beta} l \in \mathrm{B}$. Set $m = l - 1$. By co-convexity of B, we either have

$$\left\{ v \mid v \leq \overline{\beta} l \right\} \subseteq \mathrm{B} \quad \text{or} \quad \left\{ v \mid \overline{\beta} l \leq v \right\} \subseteq \mathrm{B}.$$

In the first case,

$$0 * w \in \mathrm{B}$$

for every w of length m, which implies that 0 is secure. In the second case,

$$1 * w \in \mathrm{B}$$

for every w of length m, which implies that 1 is secure.

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