Harmony, the Curry-Howard(-Lambek) Correspondence, and Higher Proof Theory

Yoshihiro Maruyama

The Hakubi Centre for Advanced Research, Kyoto University

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Synopsis

- Starting with (what Dummett called) harmony, we elucidate
  - the universal architecture of the Curry-Howard correspondence;
  - the potential logical meaning of higher categories (or higher CH).
- Harmony allows us to generate the corresponding Curry-Howard isomorphism for a given class of logical constants.
  - The scope of CH, nonetheless, is broader than that of harmony, as witnessed, e.g., by Prior's tonk, which allows for a CH isomorphism.
- The meaning-theoretical content of harmony consists in the Curry-Howard-style verificationist semantics, rather than the Dummettian proof-theoretic validity semantics, which turned out to be incomplete after all (Schroeder-Heister 2012).

Remarks

Honestly, I do not know so much about proof/type theory.

- I thus take the CH iso. to be an iso. b/w logic and cats.
  - Type theory is basically a syntax-dependent presentation of cats. And everything I discuss should have type-theoretical meaning.
- This work, I believe, makes sense of categorical harmony better than my work in *Advances in Proof-Theoretic Semantics* (2016).
  - M., Categorical Harmony and Paradoxes in Proof-Theoretic Semantics, chapter of the book, 2016. Discussed Lawverian adjunction harmony.

The entire book is available for free at Springer.

Yet Another Remark

I do not discuss classical logic, the proof identity of which trivialises (Joyal Lemma), even in the absence of full contraction (Monoidal JL).

- CCC with dualising $\perp$ (i.e. DNE) are Boolean algebras.
  - No proof-relevant categorification of Boolean algebras.
  - Tripos-style proof-irrelevant categorical semantics is available for any substructural logic over FL (M. 2013).
- MCC with contr. dualising $\perp$ and terminal I are ordered monoids.
  - Some believe both contraction and weakening are crucial in JL; yet contraction is only needed for $\perp$ alone (M. 2012).

The latter is actually a version of No-Deleting Theorem in CQM.

General-Elimination Harmony

I do not philosophise harmony here, and directly get into technicalities.

- General-elimination harmony is a fundamental principle to generate elim. from intro. rules.
- To that end, however, you need higher-order rules, in which rules may be discharged as well as propositions. Fix n-ary S.
- I-rules $\Gamma \vdash \Phi_i(A_1, ..., A_n) \Rightarrow \Gamma \vdash S(A_1, ..., A_n)$ for $i = 1, ..., m$ yield E-rule $\Phi_1 \vdash A, ..., \Phi_m \vdash A \Rightarrow S(A_1, ..., A_n) \vdash A$ (contexts omitted).
- When you have no I-rule (resp. E-rule) you have got $\bot$ (resp. $\top$).

Schroeder-Heister, A Natural Extension of Natural Deduction, 1984 (the inferential completeness of intuitionistic connectives over the format of higher-order rules; Prawitz (1979) presents a similar idea). Incorporated into LF in Edinburgh.

Examples

Let us have a look at a couple of examples.

- GE harmoninous rules for $\vee$. I-rules: $\Gamma \vdash A_1 \Rightarrow \Gamma \vdash A_1 \vee A_2$; $\Gamma \vdash A_2 \Rightarrow \Gamma \vdash A_1 \vee A_2$. E-rule: $A_1 \vdash A, A_2 \vdash A \Rightarrow A_1 \vee A_2 \vdash A$
- Conjunction. I-rule: $\Gamma \vdash A_1, A_2 \Rightarrow \Gamma \vdash A_1 \wedge A_2$. (Monoidal) E-rule: $A_1, A_2 \vdash A \Rightarrow A_1 \wedge A_2 \vdash A$.
  - Cartesian E-rule: $A_1 \vdash A \Rightarrow A_1 \wedge A_2 \vdash A$ and $A_2 \vdash A \Rightarrow A_1 \wedge A_2 \vdash A$.
- Rules for implication are higher-order. $[A, ..., B]$ denotes a derivation from A to B. I-rule: $\Gamma \vdash [A, ..., B] \Rightarrow \Gamma \vdash A \to B$. E-rule: $[A, ..., B] \vdash C \Rightarrow A \to B \vdash C$.
- Liar logic. I-rule: $[R, ..., \bot] \vdash R$. E-rule: $[R, ..., \bot] \vdash A \Rightarrow R \vdash A$.

Cf. Sambin's reflection principle and Dosen's logical constants as punctuation marks.

I compared them with categorical harmony in *Advances in Proof-Theoretic Semantics*.

Three Layers of Structural Proof Theory

Fundamental proof theory, presumably, must be three-layered.

- Background-theory: the stuff you have to presuppose to be able to express rules.
  - Part of this is format-theory: e.g., the theory of ‘,’ and ‘[...]’ (MCC); we control structural rules at this level rather than the object-level.
- Object-theory: a formal system expressed on the basis of the background theory. E.g., intuitionisitc logic (bCCC).
  - In this case the format-theory (CCC) embeds into the object-theory (bCCC); yet this is coincidence.
- Meta-theory: the stuff you use to reason about properties of the formal system over the background theory. E.g., finitism or set th.

Putnam on Quine and Wittgenstein (ultimately, Carroll)

It would be impossible to fully explicate our background-theory in view of the fundamental circularity lurking behind the scene:

> *The 'exciting' thesis that logic is true by convention reduces to the unexciting claim that logic is true by convention plus logic. (Putnam 1979)*

We can still explicate the underlying format-theory, which is indeed crucial for categorical semantics because categories must model the structure of the format-theory (',', '[...]', type contexts, etc.) as well.

The Universal Curry-Howard Correspondence

S can be regarded as an n-ary categorical construction such that

- Given $\{f_i : \Gamma \to \Phi_i\}_{i=1,...,m}$ and $\{g_i : \Phi_i \to A\}_{i=1,...,m}$, there are arrows $S_r(f_i) : \Gamma \to S$ and $S_l(\{g_i\}_{i=1,...,m}) : S \to A$ such that
  - β equation: $S_l(\{g_i\}_{i=1,...,m}) \circ S_r(f_i) = g_i \circ f_i$.
  - η equation: $S_l(\{S_r(id_{\Phi_i})\}_{i=1,..,m}) = id_S$.
- NB: ',' and '[...]' are replaced by monoidal product $\otimes$ and internal hom [-, -] resp.
- You can recover the equational characterisation of CCC from the universal CH.
  - In general, the above (equationally characterised) S-construction cannot be characterised via universality, as monoidal products and modalities cannot.
- This encompasses any of positive, negative, and mixed constructions, giving a logically symmetrised account of different categorical constructions.
  - You can also define categories with $\to$ and without $\times$.

The Liar Logic and its Categorical Models

The universal CH gives you a CH iso. for logic with the Liar sentence.

- I-rule: $[R, ..., \bot] \vdash R$. E-rule: $[R, ..., \bot] \vdash A \Rightarrow R \vdash A$.
    - It is just another way to say $R \leftrightarrow \neg R$.
    - This Liar equation $R \leftrightarrow \neg R$ can be solved by self-duality $A \simeq A^*$.
- The corresponding “Liar categories" include any compact closed category with self-duality (i.e., $A \simeq A^*$ coherently for any A).
    - A compact closed category is a category with dual objects (namely, A^*, $ev : A \otimes A^* \to I$, and $cv : I \to A \otimes A^*$ for any object A).
    - E.g., $A = A^*$ in **Rel**. Almost the same as structures in CQM.

If “linear logic" means the logic of vector spaces, it is inconsistent (and yet you can make sense of the inconsistency as in comp. closed cats).

Prior's tonk

The scope of CH is wider than that of GE as witnessed by Prior's tonk.

- I-rules: $\Gamma \vdash A_i \Rightarrow \Gamma \vdash A_1 \mathrm{tonk} A_2$. E-rule: $A_i \vdash A \Rightarrow A_1 \mathrm{tonk} A_2 \vdash A$.
  - Causes explosion: $A \vdash B$ for any A, B.
- This pair of rules is not harmonious, and yet admits the CH iso. in the presence of a zero object (terminal and initial at once).
  - This, therefore, is not an instance of the universal CH above.
- Tonk can be modelled by biproduct, which presupposes a zero object 0, which, in turn, yields a zero proof $0 : A \to B$ for any A, B.
  - In **Vect**, any V indeed follows from any W.
  - cf. Dosen-Petric (2007). Zero proofs for classical proof identity.

Inconsistency is not a limitation for the Curry-Howard correspondence.

This particular case of Prior's tonk is discussed in my *Synthese* paper (2016).

Consistency

- A rule is pure iff it includes no logical constant in the assumptions; it is impure otherwise.
- Purity is the key to ensuring consistency; otherwise there is room for inconsistency, which is, however, not that bad.
- If you keep the logic linear, you could keep consistency even in the presence of impure rules. No proof in a general setting yet.

Levels of Inconsistency

You have to be careful of different levels of inconsistency (provability-level, proof-level, object-level, and their combinations).

- The Joyal lemma is about the proof-level inconsistency.
- The object-level inconsistency includes the collapse of cartesian comp. closed cats. and the collapse of toposes with fixpoints.
  - Collapsing phenomena are interesting on their own.

NB: all this is proof/type theory in categorical disguise. Where exactly does category theory deviate from proof/type theory? Nowhere at all?

Not Much Time Left

NB: I omit the higher part due to the time limitation. The idea is simple enough to say in two words: define *n*-reduction. (No logical meaning of higher categories explicated so far; HoTT is just about identity types.)

Concluding Remarks

Harmony does not imply anything (conservativity, normalisability, etc.).

- Yet GE harmony induces a generic family of CH iso's, including, e.g., the one b/w Liar logic and cats. with self-duality.
- The converse does not hold: the GE-harmonious CH iso's are not all of the CH iso's, as witnessed by Prior's tonk.
    - The CH correspondence is consistent with inconsistency. Inconsistency has structures on its own right as in CQM.
- The machinery tells us categorical constructions share a common logical structure, hinting at something like logical category theory.

Where is the border b/w CH isomorphism and CH non-isomorphism?