# Imaginary Hypercubes

Hideki Tsuiki and Yasuyuki Tsukamoto

Graduate School of Human and Environmental Studies, Kyoto University
{tsuiki,tsukamoto}@i.h.kyoto-u.ac.jp

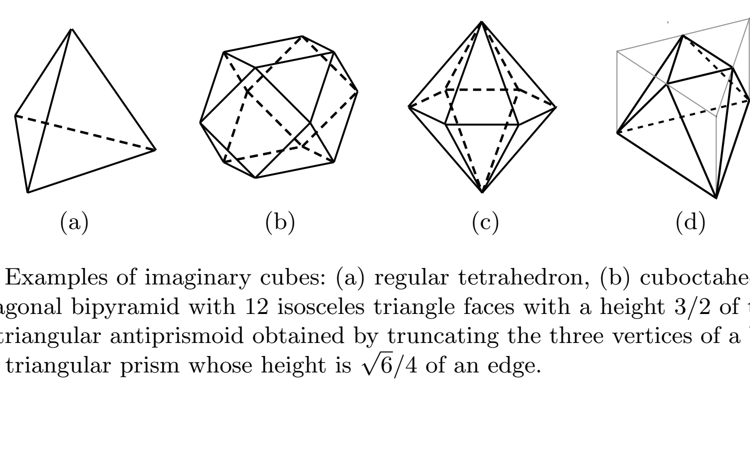
**Abstract.** Imaginary cubes are three-dimensional objects that have square projections in three orthogonal ways, just like a cube has. In this paper, we introduce higher-dimensional extensions of imaginary cubes and study their properties.

## 1 Introduction

Imaginary cubes are three-dimensional objects that have square projections in three orthogonal ways, just like a cube has [1]. A regular tetrahedron and a cuboctahedron are examples of imaginary cubes (Fig. 1(a,b)). There are two imaginary cubes with remarkable geometric properties: a hexagonal bipyramid imaginary cube (Fig. 1(c); we simply call it an H) and a triangular antiprismoid imaginary cube (Fig. 1(d); we call it a T). Figure 2 shows how they can be considered as imaginary cubes. The first author of this paper has studied imaginary cubes, in particular minimal convex imaginary cubes and fractal imaginary cubes. He has also designed sculptures and puzzles based on them [1–4].

In this paper, we study higher-dimensional extensions of imaginary cubes. In particular, we study $n$-dimensional counterparts of regular tetrahedron, H, and T for each $n \geq 2$, which we call $\mathrm{S}^n, \mathrm{H}^n$, and $\mathrm{T}^n$, respectively. We also study fractal imaginary cubes that correspond to these three series of polytopes.

In Section 2, we review properties of imaginary cubes based on [1]. Then, we study higher-dimensional extensions of them in Section 3, and fractal imaginary hypercubes in Section 4.



(a) (b) (c) (d)

**Fig. 1.** Examples of imaginary cubes: (a) regular tetrahedron, (b) cuboctahedron, (c) H: hexagonal bipyramid with 12 isosceles triangle faces with a height 3/2 of the base, (d) T: triangular antiprismoid obtained by truncating the three vertices of a base of a regular triangular prism whose height is $\sqrt{6}/4$ of an edge.

### Objects and Polytopes

Here, we only study imaginary cubes that are compact subsets of $\mathbb{R}^n$. Therefore, an *object* means a non-empty compact subset of $\mathbb{R}^n$ in this paper. We say that two objects are *similar* if one can be transformed to the other by scaling and isometry. We call this equivalence class a *shape*. Each shape S is also regarded as a name of an object, and we say that an object $A$ is an S if $A$ belongs to the class S. We use roman font to denote a shape, but italic font is used for objects.

A *polytope* is a convex hull of a finite set of points in $\mathbb{R}^n$. We denote by $\mathrm{conv}(A)$ the convex hull of an object $A$, and by $\mathrm{vert}(P)$ the set of vertices of a polytope $P$. A *facet* of an $n$-dimensional polytope $P$ is an $(n-1)$-dimensional face of $P$. We simply call an $n$-dimensional hypercube an *$n$-cube*. We refer the reader to [5] for background material on polytopes.

For any two objects $A$ and $B$, and for any scalar $c \in \mathbb{R}$, we set their Minkowski sum $A + B = \{\boldsymbol{a} + \boldsymbol{b} \in \mathbb{R}^n \mid \boldsymbol{a} \in A, \boldsymbol{b} \in B\}$, and scaling $cA = \{c\boldsymbol{a} \mid \boldsymbol{a} \in A\}$. In this paper, $\mathbf{1}$ is the vector $(1, \ldots, 1) \in \mathbb{R}^n$, and "$\cdot$" is the dot product on $\mathbb{R}^n$.

## 2 Imaginary Cubes

Imaginary cubes are three-dimensional objects with square projections in three orthogonal ways. Note that a regular octahedron also has square projections in three orthogonal ways, but its square projections are arranged differently. We exclude such a case by defining an imaginary cube more precisely as follows.

**Definition 1.** Let $C$ be a 3-cube, and $A$ be an object.

1. $A$ is an *imaginary cube of* $C$ if $A$ has the same three square projections as $C$ has.
2. $A$ is an *imaginary cube* if it is an imaginary cube of a cube.
3. $A$ is a *minimal convex imaginary cube (MCI for short) of* $C$ if $A$ is minimal among convex imaginary cubes of $C$.
4. $A$ is an *MCI* if it is an MCI of a cube.

It is clear that a convex object $A$ is an imaginary cube of $C$ if and only if each edge of $C$ contains at least one point of $A$. Therefore, an MCI of $C$ is a convex hull of some points of the edges of $C$, and thus it is a polytope.

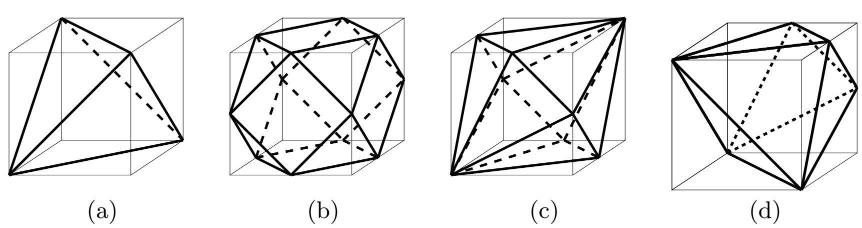



**Fig. 2.** Imaginary cubes in Figure 1 placed in cubes.

Let $A$ be an MCI of a cube $C$. The vertices of $A$ are divided into two categories: *v-vertices*, which are also vertices of $C$, and *e-vertices*, which are not vertices of $C$. We denote by $V(A)$ the set of v-vertices of $A$.

**Definition 2.** A *0/0.5/1 MCI of* $C$ is an MCI with its e-vertices at middle points of the edges of $C$.

Each object in Fig. 1 is a 0/0.5/1 MCI. Note that a regular tetrahedron has only v-vertices and a cuboctahedron has only e-vertices.

For a polytope $A$, a subset of $\mathrm{vert}(A)$ is called a star if it is composed of a vertex and all of its adjacent vertices.

**Theorem 3 (Theorem 3 and Corollary 4 of [1]).** *There is one-to-one correspondence between 0/0.5/1 MCIs of* $C$ *and subsets of* $\mathrm{vert}(C)$ *that do not contain any star as their subset. There are 15 0/0.5/1 MCI shapes.*

*Proof.* For an MCI $A$ of $C$, $V(A)$ does not contain any star because of its minimality. On the other hand, from a subset $S \subset \mathrm{vert}(C)$ without a star, we obtain an MCI by selecting its e-vertices on middle points of the edges of $C$ both of whose endpoints are not in $S$.

There are 15 equivalence classes of subsets of $\mathrm{vert}(C)$ without a star. Here, two subsets of $\mathrm{vert}(C)$ are equivalent if one is transformed to the other by an isometry which fixes $C$. We can easily check that every pair of them induces non-similar 0/0.5/1 MCIs. Therefore, there are 15 0/0.5/1 MCI shapes. □

We say that two MCIs $A$ and $A'$ of $C$ are *v-equivalent* if $V(A)$ can be transformed to $V(A')$ by an isometry which fixes $C$. There is a representative 0/0.5/1 MCI in each v-equivalence class. The list of all 0/0.5/1 MCIs is given in [1].

We define a *double imaginary cube* as an imaginary cube of two different cubes. As Fig. 3 shows, an H (Fig. 1(c)) is the intersection of two cubes and is a double imaginary cube. It is shown that all the convex double imaginary cubes are intersections of two cubes of the same size which share a diagonal and thus they are MCIs v-equivalent to H [1, Proposition 5].

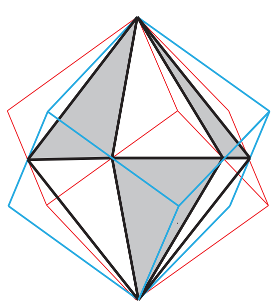


**Fig. 3.** H as the intersection of two cubes.

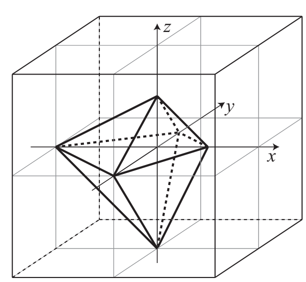



**Fig. 4.** T as a weak polytope.

We call an $n$-dimensional polytope with $2n$ vertices a *weak cross-polytope* if its vertices are on the positive and the negative sides of a set of axes of coordinates, and call it a *cross-polytope* if the distances from the origin to the vertices are the same. As Fig. 4 shows, a T (Fig. 1(d)) is a three-dimensional weak cross-polytope as well as an imaginary cube.

Hs and Ts form a tiling of three-dimensional Euclidean space and this tiling is closely related to the properties that H is a double imaginary cube shape and T is a weak cross-polytope imaginary cube shape. We explain this tiling in Section 3.4 together with another tiling by imaginary hypercubes.

# 3 Imaginary Hypercubes

## 3.1 Minimal convex imaginary $n$-cubes

We extend the theory of imaginary cubes to higher-dimensional cases. For $n \geq 2$, we say that an object $A$ is an *imaginary $n$-cube of an $n$-cube $C$* if $A$ has $(n-1)$-cube projections in $n$ orthogonal directions, just like $C$ has. An *imaginary $n$-cube* is defined as an imaginary $n$-cube of some $n$-cube.

We define the following as we did in the three-dimensional case: a minimal convex imaginary $n$-cube ($n$-MCI for short) (of $C$), a 0/0.5/1 MCI (of $C$), v-vertices and e-vertices of an n-MCI (of $C$), and v-equivalence on $n$-MCIs of $C$. We omit the dimension when it is obvious from the context, and an $n$-MCI is called an MCI, for example.

**Theorem 4.** *For an $n$-cube $C$ with $n \geq 2$, there is a one-to-one correspondence between 0/0.5/1 MCIs of $C$ and subsets of* $\mathrm{vert}(C)$ *without a star.*

The proof of this theorem is the same as that of Theorem 1 and is omitted.

Also in higher-dimensional cases, some objects are 0/0.5/1 MCIs of two different $n$-cubes. However, the set of v-vertices of such an object does not depend on the choice of the cube as we will show in Theorem 9. Therefore, we only have to enumerate equivalence classes of subsets of $\mathrm{vert}(C)$ without a star in order to enumerate 0/0.5/1 MCI shapes. We calculated these numbers for the case $n \leq 5$ with a computer program.

| $n$ | 2 | 3 | 4 | 5 |
|---|---|---|---|---|
| shapes | 4 | 15 | 269 | 829036 |
| modulo orientation-preserving isometries | 4 | 16 | 338 | 1544164 |

Note that there is a 0/0.5/1 3-MCI that cannot be transformed to its mirror image by any orientation-preserving isometry. The second line is the enumeration modulo orientation-preserving isometries.

### 3.2 16-cells

A 16-cell is a four-dimensional cross-polytope. It is a four-dimensional regular polytope with 16 regular tetrahedron facets. See [8], for example, about properties of regular polytopes.

Let $A_1$ be a 16-cell given by $\mathrm{vert}(A_1) = V_1 = \{(\pm 1, 0, 0, 0), (0, \pm 1, 0, 0), (0, 0, \pm 1, 0), (0, 0, 0, \pm 1)\}$, and let $C_1 = \mathrm{conv}(\{-1/2, 1/2\}^4)$ be a 4-cube. Let $V_2$ and $V_3$ be the subsets of $\mathrm{vert}(C_1)$ with even and odd numbers of $1/2$-coordinates, respectively. One can see that $C_2 = \mathrm{conv}(V_1 \cup V_3)$ and $C_3 = \mathrm{conv}(V_1 \cup V_2)$ are also 4-cubes. Since $V_1$ does not contain any star of $C_2$ (resp. $C_3$), and every edge of $C_2$ (resp. $C_3$) contains a point in $V_1$, we can find that $A_1$ is an imaginary cube of $C_2$ (resp. $C_3$) that has no e-vertices. Thus, $A_1$ is a double imaginary 4-cube. Note that $A_2 = \mathrm{conv}(V_2)$ and $A_3 = \mathrm{conv}(V_3)$ are also 16-cells.

As we described in Section 2, T is a weak cross-polytope imaginary cube shape and H is a double imaginary cube shape in the three-dimensional case. We show that 16-cell is the only weak cross-polytope imaginary cube shape as well as the only double imaginary cube shape in four and higher-dimensional cases. First, we study weak cross-polytope imaginary cubes.

**Lemma 5.** *A convex imaginary $n$-cube polytope has at least $2^{n-1}$ vertices.*

*Proof.* An $n$-cube has $n2^{n-1}$ edges and a convex imaginary $n$-cube polytope contains a vertex on each of these edges. Since a vertex is on at most $n$ edges of the cube, we have the result. $\square$

**Proposition 6.** *For $n \geq 3$,* T *and 16-cell are the only weak cross-polytope imaginary hypercube shapes.*

*Proof.* Since an $n$-dimensional weak cross-polytope has $2n$ vertices, $n$ must satisfy $2n \geq 2^{n-1}$ by Lemma 5, and hence $n \leq 4$.

For $n = 4$, any weak cross-polytope has eight vertices. If a weak cross-polytope $A$ is an imaginary cube polytope of a 4-cube $C$, then $A$ is a MCI with no e-vertices, and each edge of $C$ contains one vertex of $A$ from the proof of Lemma 5. Thus $A$ is a 16-cell.

For $n = 3$, assume that a weak cross-polytope $A$ is an imaginary cube of a 3-cube $C$. Note that $A$ may not be an MCI of $C$. We set $V(A) := \mathrm{vert}(C) \cap A$. Since a 3-cube has 12 edges and $A$ has 6 vertices, we have $3\#V(A) + (6 - \#V(A)) \geq 12$, and get $\#V(A) \geq 3$.

If $\#V(A) = 3$, $A$ has three e-vertices and they must be on the edges of $C$ both of whose endpoints are not in $V(A)$. Thus, $A$ is an MCI of $C$, and we can find that T is the only such polytope.

If $\#V(A) \geq 4$, there is a pair $\{\boldsymbol{v}_1, \boldsymbol{v}_2\} \subset V(A)$ such that $\mathrm{vert}(A) \setminus \{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ is on a plane that is orthogonal to the line segment $[\boldsymbol{v}_1, \boldsymbol{v}_2]$. Suppose that $C = \mathrm{conv}\{0, 1\}^3$. Since $[\boldsymbol{v}_1, \boldsymbol{v}_2]$ contains an interior point of $C$, we can put $\boldsymbol{v}_1 = (0, 0, 0)$ and $\boldsymbol{v}_2 = (1, 1, 1)$ without loss of generality. Suppose that the other four vertices of $A$ are on a plane defined as $\{(x, y, z) \mid x + y + z = a\}$ $(a \in \mathbb{R})$. Since $A$ has four or more v-vertices, we get $a = 1, 2$. If $a = 1$, $\mathrm{vert}(A)$ must contain

$(1,0,0),(0,1,0)$ and $(0,0,1)$. However, no line passes through two of them and the origin $(1/3,1/3,1/3)$ at the same time. Therefore, we have no weak cross-polytope in this case. The case $a=2$ is similar to the case $a=1$. □

Next, we study double imaginary $n$-cubes. A convex object can be an imaginary 2-cube of two or more squares. For example, a square is an imaginary cube of infinitely many squares. In the three-dimensional case, there are many convex double imaginary cubes, and H is the only 0/0.5/1 MCI among them as we mentioned in Section 2.1. For $n \geq 4$, we show that 16-cell is the only convex double imaginary $n$-cube shape. We prepare two lemmas, whose proofs are omitted.

**Lemma 7.** *For $n \geq 3$, the dimension of the affine hull of an imaginary $n$-cube is $n$.*

Note that this lemma does not hold for $n=2$ because a line segment is an imaginary 2-cube.

For an $n$-dimensional hyperplane $G$, we denote by $r(G)$ he distance of $G$ from the origin.

**Lemma 8.** *Let $C = \mathrm{conv}(\{-1,1\}^n)$ and $G$ be an $n$-dimensional hyperplane.*
*(1) If $n > 4$ and one of the open half spaces defined by $G$ contains only one vertex of $C$, then $r(G) > 1$.*
*(2) If $n = 4$ and one of the open half spaces defined by $G$ contains only one vertex $v$ of $C$, then $r(G) \geq 1$. If $r(G) = 1$, in addition, then the four adjacent vertices of $v$ are on $G$.*

**Theorem 9.** *16-cell is the only convex double imaginary 4-cube shape. For $n > 4$, there is no double imaginary $n$-cube.*

*Proof.* Let $n \geq 4$. Suppose that $B$ is a double imaginary cube of two $n$-cubes $C_1$ and $C_2$. One can see that $A = C_1 \cap C_2$ is a convex double imaginary cube because we have $B \subset A$. We consider the double imaginary cube $A$.

We can assume without loss of generality that $C_1 = \mathrm{conv}(\{-1,1\}^n)$ and that the edge length of $C_2$ is less than or equal to the edge length of $C_1$, that is, 2. Let $P$ be a facet of $C_2$ and $G$ be the hyperplane containing $P$. All the edges of $P$ must intersect with $C_1$ because $A$ is an imaginary cube of $C_2$. Hence $P \cap C_1$ is an imaginary cube of an $(n-1)$-cube $P$. Since $n \geq 4$, the dimension of its affine hull is $n-1$ by Lemma 7. On the other hand, it is immediate to show that each facet of $C_1$ is not on $G$. Therefore, there exists a vertex $v$ of $C_1$ in the open half-space defined as the opposite side of $C_2$ with respect to $G$. Such a vertex of $C_1$ is unique because every edge of $C_1$ must intersect with $C_2$. Therefore, if $n > 4$, then $r(G) > 1$ by Lemma 8. Since it also holds for the facet $P'$ which is parallel to $P$, the edge length of $C_2$ is greater than 2, contradicting the assumption. Therefore, we have $n = 4$. By Lemma 8, the two 4-cubes have the same size and $P$ contains all the four adjacent vertices of $v$. Therefore, $P \cap C_1$ is a regular tetrahedron. Since $C_1$ and $C_2$ have the same size, it holds for all the facets of $C_1$ and $C_2$. Therefore, $A$ is a 16-cell. Since a 16-cell is a minimal convex imaginary 4-cube, it is the only convex double imaginary 4-cube. □

### 3.3 Higher dimensional extensions of H and T

In this subsection, we make $n$-dimensional extensions of the four 0/0.5/1 MCIs in Fig. 1 in each $n \geq 2$. We regard the 0/0.5/1 $n$-MCI which has no v-vertices as an imaginary $n$-cube corresponding to a cuboctahedron.

As an $n$-dimensional counterpart of a regular tetrahedron, we define $S^n$ and $S'^n$ as follows:

$$V(S^n) = \{\boldsymbol{x} \in \{0,1\}^n \mid \boldsymbol{x} \cdot \mathbf{1} \equiv 0 \pmod 2\},$$
$$V(S'^n) = \{\boldsymbol{x} \in \{0,1\}^n \mid \boldsymbol{x} \cdot \mathbf{1} \equiv 1 \pmod 2\}.$$

Let $\boldsymbol{x}, \boldsymbol{y} \in \{0,1\}^n$ be two vertices of $C = \mathrm{conv}(\{0,1\}^n)$. If $\boldsymbol{x}$ and $\boldsymbol{y}$ are the two endpoints of an edge of $C$, we get $\boldsymbol{x} \cdot \mathbf{1} = \boldsymbol{y} \cdot \mathbf{1} \pm 1$. Therefore, every edge of $C$ contains points of both $S^n$ and $S'^n$. Therefore, $S^n$ and $S'^n$ are imaginary cubes of $C$ that have no e-vertices. Moreover, since both $V(S)$ and $V(S')$ contain no star of $C$, $S$ and $S'$ are MCIs of $C$. Note that $S^n$ and $S'^n$ have the same shape which is denoted by $\mathrm{S}^n$. The shape $\mathrm{S}^4$ is 16-cell.

Concerning H and T, we define three 0/0.5/1 MCIs of an $n$-cube $C = \mathrm{conv}(\{-1,1\}^n)$ as follows:

$$\begin{aligned} V(H^n) &= \{\boldsymbol{x} \in \{-1,1\}^n \mid \boldsymbol{x} \cdot \mathbf{1} \equiv 0 \pmod 3\}, \\ V(T^n) &= \{\boldsymbol{x} \in \{-1,1\}^n \mid \boldsymbol{x} \cdot \mathbf{1} \equiv -1 \pmod 3\}, \\ V(T'^n) &= \{\boldsymbol{x} \in \{-1,1\}^n \mid \boldsymbol{x} \cdot \mathbf{1} \equiv 1 \pmod 3\}. \end{aligned} \tag{1}$$

By a similar argument, one can see that they define 0/0.5/1 MCIs. Note that $T^n$ and $T'^n$ are similar because we have $T^n = -T'^n$. We denote by $\mathrm{H}^n$ and $\mathrm{T}^n$ the shapes of $H^n$ and $T^n$, respectively.

These sets of vertices satisfy the following equations. We have

$$\begin{aligned} V(H^{n+1}) &= V(T'^n) \times \{-1\} \cup V(T^n) \times \{1\}, \\ V(T^{n+1}) &= V(H^n) \times \{-1\} \cup V(T'^n) \times \{1\}, \\ V(T'^{n+1}) &= V(T^n) \times \{-1\} \cup V(H^n) \times \{1\}. \end{aligned} \tag{2}$$

One can see from (1) that each of $H^n$, $T^n$ and $T'^n$ is mapped to itself by a permutation of the $n$ coordinates. Therefore, one can derive from equation (2) that for $n \geq 4$, $\mathrm{H}^n$ has $2n$ copies of $\mathrm{T}^{n-1}$ facets. The other facets are $(n-1)$-simplexes because each vertex figure of an $n$-cube is a simplex. On the other hand, $\mathrm{T}^n$ has $n$ copies of $\mathrm{H}^{n-1}$ facets, $n$ copies of $\mathrm{T}^{n-1}$ facets and some $(n-1)$-simplex facets for $n \geq 4$. In the case $n = 3$, the six 2-simplex facets of $\mathrm{H}^3$ coincide with $\mathrm{T}^2$ and the three $\mathrm{H}^2$ facets of $\mathrm{T}^3$ degenerate to line segments. Thus, $\mathrm{H}^3$ has twelve $\mathrm{T}^2$ faces and $\mathrm{T}^3$ has eight faces.

One can see that the set of e-vertices of $H^n$, $T^n$ and $T'^n$ are the sets

$$\begin{aligned} &\{\boldsymbol{x} \in \{-1,0,1\}^n \mid \boldsymbol{x} \cdot \mathbf{1} \equiv 0 \pmod 3,\ \boldsymbol{x} \cdot \boldsymbol{x} = n-1\}, \\ &\{\boldsymbol{x} \in \{-1,0,1\}^n \mid \boldsymbol{x} \cdot \mathbf{1} \equiv -1 \pmod 3,\ \boldsymbol{x} \cdot \boldsymbol{x} = n-1\}, \text{ and} \\ &\{\boldsymbol{x} \in \{-1,0,1\}^n \mid \boldsymbol{x} \cdot \mathbf{1} \equiv 1 \pmod 3,\ \boldsymbol{x} \cdot \boldsymbol{x} = n-1\}, \end{aligned} \tag{3}$$

respectively.

### 3.4 Tilings by imaginary cubes

As we mentioned above, Hs and Ts form a tiling of three-dimensional Euclidean space, and 16-cells form a tiling of four-dimensional Euclidean space. We explain these tilings from the viewpoints of weak cross-polytope imaginary cubes and double imaginary cubes.

We set positive integers $n \geq 3$ and $k \geq 2$. Consider a subset $Z$ of the $n$-dimensional cubic lattice

$$Z = \{\boldsymbol{x} \in \mathbb{Z}^n \mid \boldsymbol{x} \cdot \mathbf{1} \equiv 0 \pmod{k}\}.$$

We call a cube $\mathrm{conv}(\{0,1\}^n) + \{\boldsymbol{v}\}$ ($\boldsymbol{v} \in \mathbb{Z}^n$) a *lattice-cube*. In each lattice-cube $C$, take an MCI of $C$ whose set of v-vertices is $Z \cap \mathrm{vert}(C)$. Such an MCI is a translation of one of $M_r$ for $0 \leq r < k$ defined as

$$V(M_r) = \{\boldsymbol{x} \in \{0,1\}^n \mid \boldsymbol{x} \cdot \mathbf{1} \equiv r \pmod{k}\}.$$

Note that every pair of these MCIs which are placed in adjacent $n$-cubes share the faces on their intersection. After placing such MCIs, there remain holes around lattice points

$$\{\boldsymbol{x} \in \mathbb{Z}^n \mid \boldsymbol{x} \cdot \mathbf{1} \not\equiv 0 \pmod{k}\}.$$

These holes are weak cross-polytopes because all of the vertices are on the lattice edges. Therefore, for every $n$ and $k$, we have a tiling of $n$-dimensional space by translations of $M_r$ for $0 \leq r < k$ and $n$-dimensional weak cross-polytopes of several shapes. In the case $n = 3$ and $k = 2$, this tiling is the three-dimensional tiling by regular tetrahedra and regular octahedra. In the case $n = 3$ and $k = 3$, $M_r$ ($r = 0, 1, 2$) are $H$, $T$, and $T'$, respectively, and each hole is a T. Therefore, we have the three-dimensional tiling by Hs and Ts. In the case $n = 4$ and $k = 2$, not only MCIs placed in lattice-cubes but also the holes are 16-cells, and we get the four-dimensional tiling by 16-cells. Since T and 16-cell are the only weak cross-polytope imaginary $n$-cube shapes for $n \geq 3$ (Proposition 6), among these tilings, there are only two tilings by imaginary cubes.

These two tilings are related to the fact that H and 16-cell are double imaginary cubes. The three-dimensional tiling by Hs and Ts can be characterized as follows [1]. Let $\sigma^3$ be the isometry on three-dimensional Euclidean space to rotate by 180 degrees around the axis $x = y = z$. Then, the tiling is a Voronoi tessellation of the union $\mathbb{Z}^3 \cup \sigma^3(\mathbb{Z}^3)$ of the two cubic lattices such that Voronoi cells of points in $\mathbb{Z}^3 \cap \sigma^3(\mathbb{Z}^3)$ have the shape H and those of other points have the shape T. See [6], for example, for the notion of Voronoi tessellations.

This construction can be extended to higher-dimensional cases. In the $n$-dimensional Euclidean space, let $\sigma^n$ be the orthogonal transformation on $\mathbb{R}^n$ that satisfies $\sigma^n(\mathbf{1}) = \mathbf{1}$ and $\sigma^n(\boldsymbol{v}) = -\boldsymbol{v}$ for $\boldsymbol{v} \in \mathbb{R}^n$ with $\boldsymbol{v} \cdot \mathbf{1} = 0$. Then, take the Voronoi tessellation of $\mathbb{Z}^n \cup \sigma^n(\mathbb{Z}^n)$. The Voronoi cell of the origin is the intersection of two $n$-cubes $\mathrm{conv}(\{-1/2, 1/2\}^n)$ and $\sigma^n(\mathrm{conv}(\{-1/2, 1/2\}^n))$, and Voronoi cells of points in $\mathbb{Z}^n \cap \sigma^n(\mathbb{Z}^n)$ are its translations.

In the case $n = 4$, $\sigma^4$ maps the set $V_1$ to $V_3$, $V_3$ to $V_1$, and $V_2$ to itself, where the sets $V_1, V_2$, and $V_3$ are defined in Section 3.2. Therefore, the cube $\mathrm{conv}(\{-1/2, 1/2\}^4)) = \mathrm{conv}(V_2 \cup V_3)$ is mapped to the cube $\mathrm{conv}(V_2 \cup V_1)$ and their intersection $\mathrm{conv}(V_2)$ is the Voronoi cell at the origin. One can show that the other Voronoi cells are also 16-cells, and therefore this tiling is the four-dimensional tiling by 16-cells.

For $n \geq 3$, if the intersection $E^n$ of two cubes $\mathrm{conv}(\{-1/2, 1/2\}^n)$ and $\sigma^n(\mathrm{conv}(\{-1/2, 1/2\}^n))$ is an imaginary cube of an $n$-cube $C$, then it must also be an imaginary cube of $\sigma^n(C)$. It is easy to show that $C$ and $\sigma^n(C)$ are different $n$-cubes and thus $E^n$ is a double imaginary cube. Since H and 16-cell are the only double imaginary $n$-cube shapes for $n \geq 3$ (Theorem 9), among these Voronoi tessellations there are only two tilings by imaginary cubes.

## 4 Fractal imaginary hypercubes

### 4.1 Fractal imaginary cubes

From a regular tetrahedron, one can form a fractal (i.e., self-similar) object known as a Sierpinski tetrahedron (Fig. 5(a)). It has similarity dimension two and it is also an imaginary cube.

Let $\mathcal{H}^n$ be the metric space of non-empty compact subsets of $\mathbb{R}^n$ with the Hausdorff metric. According to the theory of IFS (iterated function system) fractals developed by Hutchinson [7], for contractions $f_i : \mathbb{R}^n \to \mathbb{R}^n$ $(i = 1, \ldots, m)$, an IFS $I = \{f_i \mid i = 1, 2, \ldots, m\}$ defines a fractal object as the fixedpoint of the following contraction map on $\mathcal{H}^n$:

$$F_I(X) = \bigcup_{i=1}^{m} f_i(X). \tag{4}$$

As for a Sierpinski tetrahedron, let $S$ be a regular tetrahedron and let $I_S = \{f_i : \mathbb{R}^3 \to \mathbb{R}^3 \mid i = 1, 2, 3, 4\}$ be an IFS where $f_i (i = 1, 2, 3, 4)$ are homothetic transformations (i.e., similitudes that perform no rotations) with the scale 1/2

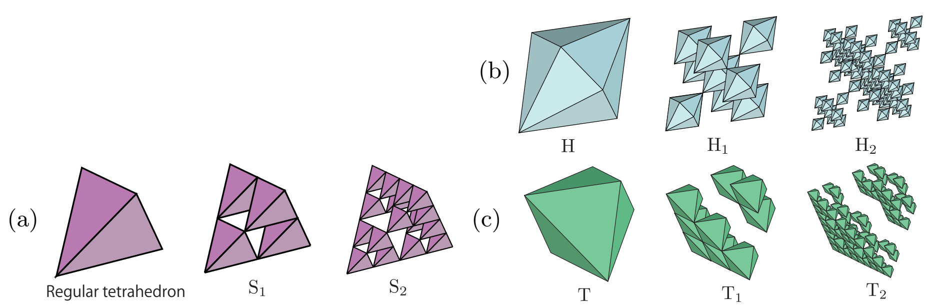



**Fig. 5.** The first two approximations of (a) Sierpinski tetrahedron, (b) $H_\infty$, and (c) $T_\infty$.

whose centers are vertices of $S$. The induced fractal is a Sierpinski tetrahedron. It is an imaginary cube of the cube $C$ of which $S$ is an imaginary cube. Note that this fractal object is minimal among imaginary cubes of $C$.

As generalizations of a Sierpinski tetrahedron, fractal imaginary cubes such that an IFS that induces the fractal is composed of $k^2$ homothetic transformations of scale $1/k$ are studied [2]. Sierpinski tetrahedron is the only such shape for $k = 2$. In the case $k = 3$, there are two such fractal shapes $\mathrm{H}_\infty$ and $\mathrm{T}_\infty$ whose convex hulls are H and T, respectively (Fig. 5(b,c)). In particular, $\mathrm{H}_\infty$ is a double imaginary cube. In the following, we explain these fractal imaginary cubes and their higher-dimensional counterparts.

For $k \geq 2$, let $I = \{f_i : \mathbb{R}^n \to \mathbb{R}^n \mid i = 1, 2, \ldots, k^{n-1}\}$ be an IFS such that $f_i (i = 1, 2, \ldots, k^{n-1})$ are homothetic transformations with the scale $1/k$. Let $X_I$ be the fractal object obtained as the fixedpoint of the contraction map $F_I$ on $\mathcal{H}^n$ defined by (4). Since $X_I$ is the fixedpoint of $F_I$, for any $B \in \mathcal{H}^n$, the sequence $B, F_I(B), F_I^2(B), \ldots$ converges to $X_I$ with respect to the Hausdorff metric. Here, $f^m$ is the $m$-times repetition of $f$.

**Lemma 10.** *Let $C$ be an $n$-cube and let $(A_i; i = 0, 1, \ldots)$ be a sequence of imaginary $n$-cubes of $C$. If the sequence $(A_i; i = 0, 1, \ldots)$ converges to $A$ with respect to the Hausdorff metric, then $A$ is also an imaginary cube of $C$.*

*Proof.* For each projection $p$ from $C$ to a hyperplane containing a facet of $C$, $p(A_i)$ for $i = 0, 1, \ldots$ are the same $(n-1)$-cube $p(C)$. Since $p$ induces a continuous map from $\mathcal{H}^n$ to $\mathcal{H}^{n-1}$, $p(A)$ is also equal to $p(C)$. □

**Proposition 11.** *Let $I$ be an IFS as above. The limit $X_I$ is an imaginary cube of an $n$-cube $C$ if and only if $F_I(C)$ is an imaginary cube of $C$.*

*Proof.* Suppose that $X_I$ is an imaginary cube of $C$. We have $C \supset X_I$ and the sequence $C \supset F_I(C) \supset F_I^2(C) \cdots$ converges to $X_I$. Therefore, all of $F_I^i(C)$ are imaginary cubes of $C$. In particular, $F_I(C)$ is an imaginary cube of $C$. Conversely, if $F_I(C)$ is an imaginary cube of $C$, then all of $F_I^i(C)$ are imaginary cubes of $C$ by induction, and the limit $X_I$ is also an imaginary cube of $C$ from Lemma 10. □

The fractal object $X_I$ has the similarity dimension $n - 1$. Note that $F_I(C)$ is an imaginary cube of $C$ if and only if $f_i(C)$ $(i = 1, 2, \ldots, k^{n-1})$ are $n$-cubes obtained by cutting $C$ into $k^n$ $n$-cubes of the same size and selecting $k^{n-1}$ of them so that they form an imaginary $n$-cube. Such a selection of $k^{n-1}$ cubes corresponds to an $(n-1)$-dimensional Latin hypercube of order $k$. See, for example, [9] for the notion of a Latin hypercube.

### 4.2 Higher-dimensional extensions of the Sierpinski tetrahedron

Let $C$ be the $n$-cube $\mathrm{conv}(\{0,1\}^n)$. We set $P^n_{\boldsymbol{a}} = \frac{1}{2}(C + \{\boldsymbol{a}\})$ for $\boldsymbol{a} \in \{0,1\}^n$. There are the following two ways of selecting $2^{n-1}$ $n$-cubes from $\{P^n_{\boldsymbol{a}} \mid \boldsymbol{a} \in \{0,1\}^n\}$ to form an imaginary cube.

$$\hat{S}^n = \cup\{P^n_{\boldsymbol{a}} \mid \boldsymbol{a} \in \{0,1\}^n,\, \boldsymbol{a} \cdot \mathbf{1} \equiv 0 \pmod 2\},$$
$$\hat{S}'^n = \cup\{P^n_{\boldsymbol{a}} \mid \boldsymbol{a} \in \{0,1\}^n,\, \boldsymbol{a} \cdot \mathbf{1} \equiv 1 \pmod 2\}.$$

These two imaginary cubes have the same shape which we denote by $\hat{\mathrm{S}}^n$ (Fig. 6). Let $I_S$ be the IFS that consists of $2^{n-1}$ homothetic transformations with the scale $1/2$ that map $C$ to the cubes in $\hat{S}^n$, and let $S^n_\infty$ be the fractal induced by $I_S$. $S^n_\infty$ is a fractal imaginary $n$-cube with the similarity dimension $n-1$ by Proposition 11. We denote by $\mathrm{S}^n_\infty$ the shape of $S^n_\infty$. The shape $\mathrm{S}^3_\infty$ is the Sierpinski tetrahedron. Since all the components of $I_S$ are homothetic transformations, the convex hull of $S^n_\infty$ is equal to the convex hull of the centers of $I_S$, which is $S^n$ defined in Section 3.3.

It is immediate to show that $\hat{S}^n$ and $\hat{S}'^n$ are the only two ways of selecting $2^{n-1}$ $n$-cubes from $\{P^n_{\boldsymbol{a}} \mid \boldsymbol{a} \in \{0,1\}^n\}$ to form an imaginary cube. Therefore, $\mathrm{S}^n_\infty$ is the only fractal imaginary cube shape obtained as the limit of an IFS that is composed of $2^{n-1}$ homothetic transformations with the scale $1/2$.

### 4.3 Fractal imaginary cubes $H_\infty$ and $T_\infty$ and their higher-dimensional extensions.

We study the case $k=3$. Let $C$ be the $n$-cube $\mathrm{conv}(\{-1,1\}^n)$. We define $Q^n_{\boldsymbol{a}} \subset C$ ($\boldsymbol{a} \in \{-1,0,1\}^n$) as $\frac{1}{3}(C+\{2\boldsymbol{a}\})$. There are the following three ways of selecting $3^{n-1}$ $n$-cubes from $\{Q^n_{\boldsymbol{a}} \mid \boldsymbol{a} \in \{-1,0,1\}^n\}$ to form an imaginary cube of $C$.

$$\hat{H}^n = \cup\{Q^n_{\boldsymbol{a}} \mid \boldsymbol{a} \in \{-1,0,1\}^n,\ \boldsymbol{a}\cdot\mathbf{1} \equiv 0 \pmod 3\},$$
$$\hat{T}^n = \cup\{Q^n_{\boldsymbol{a}} \mid \boldsymbol{a} \in \{-1,0,1\}^n,\ \boldsymbol{a}\cdot\mathbf{1} \equiv -1 \pmod 3\},$$
$$\hat{T}'^n = \cup\{Q^n_{\boldsymbol{a}} \mid \boldsymbol{a} \in \{-1,0,1\}^n,\ \boldsymbol{a}\cdot\mathbf{1} \equiv 1 \pmod 3\}.$$

$\hat{T}^n$ and $\hat{T}'^n$ have the same shape which we denote by $\hat{\mathrm{T}}^n$. We denote by $\hat{\mathrm{H}}^n$ the shape of $\hat{H}^n$ (Fig. 6).

Let $I_H$ (resp. $I_T$) be the IFS that consists of $3^{n-1}$ homothetic transformations with the scale $1/3$ that map $C$ to the cubes in $\hat{H}^n$ (resp. $\hat{T}^n$), and let $H^n_\infty$ (resp. $T^n_\infty$) be the fractal induced by $I_H$ (resp. $I_T$). $H^n_\infty$ and $T^n_\infty$ are fractal imaginary $n$-cubes with the similarity dimension $n-1$ by Proposition 11. We write $\mathrm{H}^n_\infty$ and $\mathrm{T}^n_\infty$ for their shapes.

The convex hull of $H^n_\infty$ is equal to the convex hull of the centers of the components of $I_H$ because they are homothetic transformations. It is the set

$$D^n = \{\boldsymbol{x} \in \{-1,0,1\}^n \mid \boldsymbol{x}\cdot\mathbf{1} \equiv 0 \pmod 3\}.$$

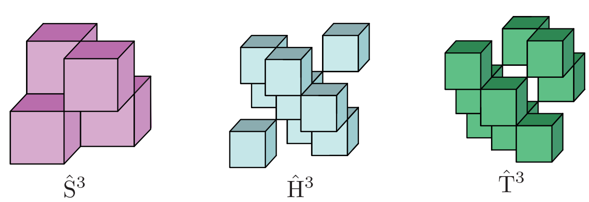



**Fig. 6.** The three shapes $\hat{\mathrm{S}}^3$, $\hat{\mathrm{H}}^3$, and $\hat{\mathrm{T}}^3$.

From (3), the set of vertices of the polytope $H^n$ defined in Section 3.3 is the intersection of $D^n$ with the edges of $C$. Therefore, the convex hull of $D^n$ coincides with $H^n$. Similarly, we can show that the convex hull of $T^n_\infty$ is $T^n$.

**Theorem 12.** *For $n \geq 3$, $\mathrm{H}^n_\infty$ and $\mathrm{T}^n_\infty$ are the only fractal imaginary cube shapes obtained as the limit of an IFS that is composed of $3^{n-1}$ homothetic transformations with the scale* $1/3$.

*Proof.* Suppose that $n \geq 2$ and that $U \subset \{-1, 0, 1\}^n$ satisfies $\#U = 3^{n-1}$ and $\cup\{Q^n_{\boldsymbol{a}} \mid \boldsymbol{a} \in U\}$ is an imaginary cube of $C$. We show that there exist $\boldsymbol{b} \in \{-1, 1\}^n$ and $r \in \{-1, 0, 1\}$ such that $U = U(\boldsymbol{b}, r)$ for $U(\boldsymbol{b}, r) = \{\boldsymbol{a} \in \{-1, 0, 1\}^n \mid \boldsymbol{a} \cdot \boldsymbol{b} \equiv r \pmod 3\}$. It is clear that such a selection $U$ is congruous to that of $\hat{H}^n$ or $\hat{T}^n$. We show this by induction on $n$, and it is true for $n = 2$.

Note that we have $U(\boldsymbol{b}, r) = U(\boldsymbol{b}', r')$ if and only if $(\boldsymbol{b}, r) = \pm(\boldsymbol{b}', r')$. Since simultaneous equations $a_1 + a_2 \equiv r_1$, $a_1 - a_2 \equiv r_2 \pmod 3$ always have a solution $(a_1, a_2) = 2(r_1 + r_2, r_1 - r_2)$, one can also find that if $\boldsymbol{b} \neq \pm\boldsymbol{b}'$, then we have $U(\boldsymbol{b}, r) \cap U(\boldsymbol{b}', r') \neq \emptyset$ for any choice of $r, r' \in \{-1, 0, 1\}$.

Suppose that $n \geq 3$. We divide $U$ into three parts

$$U = U_{-1} \times \{-1\} \cup U_0 \times \{0\} \cup U_1 \times \{1\},$$

where $U_i \subset \{-1, 0, 1\}^{n-1}$ satisfies $\#U_i = 3^{n-2}$ and that $\cup\{Q^{n-1}_{\boldsymbol{a}} \mid \boldsymbol{a} \in U_i\}$ is an imaginary $(n-1)$-cube for $i \in \{-1, 0, 1\}$. From the assumption, we can put $U_i = U(\boldsymbol{b}_i, r_i)$ for $i \in \{-1, 0, 1\}$. Considering the projection in the $n$-th direction, we have $U_i \cap U_j = \emptyset$ for $-1 \leq i < j \leq 1$. Therefore, we can assume that $\boldsymbol{b}_{-1} = \boldsymbol{b}_0 = \boldsymbol{b}_1 = (b_1, \ldots, b_{n-1})$, and we get $\{r_{-1}, r_0, r_1\} = \{-1, 0, 1\}$. In each case, there is $b_n \in \{-1, 1\}$ such that $r_0 \equiv r_{-1} - b_n \equiv r_1 + b_n \pmod 3$, and hence we obtain $U = U((b_1, \ldots, b_n), r_0)$. □

## References


1. Tsuiki, H.: Imaginary cubes and their puzzles. Algorithms **5** (2012) 273 – 288
2. Tsuiki, H.: Does it look square? — hexagonal bipyramids, triangular antiprismoids, and their fractals. In: Proceedings of Conference Bridges Donostia, Tarquin publications (2007) 277–286
3. Tsuiki, H.: Imaginary cubes—objects with three square projection images. In: Proceedings of Bridges 2010: Mathematics, Music, Art, Architecture, Culture, Tessellations Publishing, Phoenix, Arizona (2010) 159–166
4. Tsuiki, H.: Sudoku colorings of the hexagonal bipyramid fractal. In: Computational Geometry and Graph Theory, International Conference KyotoCGGT 2007 Revised Selected Papers. Volume 4535 of LNCS., Springer (2008) 224–235
5. Ziegler, G.M.: Lectures on Polytopes. Springer (1995)
6. Okabe, A., Boots, B., Sugihara, K.: Spatial tessellations Concepts and Applications of Voronoi Diagrams. John Wiley & Sons (1992)
7. Hutchinson, J.: Fractals and self similarity. Indiana Univ. Math. J. **30** (1981) 713–747
8. Coxeter, H.S.M.: Regular Polytopes. Dover (1973)
9. McKay, B.D., Wanless, I.M.: A census of small latin hypercubes. SIAM J. Discrete Math. **22** (2008) 719–736