Hyperbolic topology of normed linear spaces

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Dedicated to Professor Tsugunori Nogura on his sixtieth birthday

Abstract

In a previous paper [6], the authors introduced the hyperbolic topology on a metric space, which is weaker than the metric topology and naturally derived from the Lawson topology on the space of formal balls. In this paper, we characterize spaces $L_p(\Omega, \Sigma, \mu)$ on which the hyperbolic topology induced by the norm $\|\cdot\|_p$ coincides with the norm topology. We show the following.

(1) The hyperbolic topology and the norm topology coincide for $1 < p < \infty$.

(2) They coincide on $L_1(\Omega, \Sigma, \mu)$ if and only if $\mu(\Omega) = 0$ or Ω has a finite partition by atoms.

(3) They coincide on $L_\infty(\Omega, \Sigma, \mu)$ if and only if $\mu(\Omega) = 0$ or there is an atom in Σ.

Keywords: Normed linear space, L_p, uniformly rotund (convex), locally uniformly rotund (convex), atom, metric space, hyperbolic topology, norm topology, formal ball, Lawson topology.

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1 Introduction

The hyperbolic topology of a metric space (X, d) is the topology generated by sets of the form $\{z : d(z, x) - d(z, y) < t\}$ for $x, y \in X$ and $-d(x, y) < t$.

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Hyperbolic topology was introduced in [6] as a subspace topology of the Lawson topology of the poset of generalized formal balls. Let $\mathbb{R}$ and $\mathbb{R}_+$ denote the sets of real numbers and non-negative real numbers respectively. For a metric space (X, d), we call an element of $\mathbf{B}^+X = X \times \mathbb{R}_+$ a *formal ball* in (X, d). Formal balls are firstly introduced by Weihrauch and Schreiber in [7] to represent a metric space in a domain, and Edalat and Heckmann [3] investigated further properties of $\mathbf{B}^+X$ as a computational model for (X, d). We can generalize the notion of formal balls so that a ball with a negative radius is allowed, and call an element of $\mathbf{B}X = X \times \mathbb{R}$ a *generalized formal ball* in (X, d). The set $\mathbf{B}X$ is endowed with the partial order $\sqsubseteq$ defined as $(x, r) \sqsubseteq (y, s)$ if $d(x, y) \leq r - s$. Therefore, we can consider the Lawson topology on $\mathbf{B}X$, which is a topology defined on a partially ordered set. The Lawson topology of $\mathbf{B}X$ is generated by sets of the form $\{(y, s) : d(x, y) < r - s\}$ and $\{(y, s) : d(x, y) > r - s\}$ for $(x, r) \in \mathbf{B}X$. Then, Lawson topology restricted to the set $\mathrm{Bd}(x, r) = \{(y, s) \in \mathbf{B}X : d(x, y) = r - s\}$ derives a topology on X, because there is a one-to-one correspondence between $\mathrm{Bd}(x, r)$ and X. In [6], it is shown that this topology does not depend on the choice of x and r, and coincides with the hyperbolic topology defined above.

Now, our interest is how different is the hyperbolic topology from the metric topology. In [6], it is proved that the hyperbolic topology and the metric topology coincide on X if and only if the Lawson topology and the product topology coincide on $\mathbf{B}X$. Through this property, we can derive conditions for the hyperbolic topology and the metric topology to coincide on a metric space X from the conditions given in [6] for the Lawson topology and the product topology to coincide on $\mathbf{B}X$. In particular, we can derive an example of a metric space for which the hyperbolic topology and the metric topology do not coincide, from an example in [6]. However, the metric topology of the example is the discrete topology and one may think of it as an artificial example from a mathematical point of view. Our concern in this paper is whether they differ also in more natural spaces which appear in many branches of mathematics.

In this paper, we study the relation between the hyperbolic topology and the metric topology for normed linear spaces, in particular, spaces $L_p(\Omega, \Sigma, \mu)$ for Σ a σ-algebra of subsets of a set Ω, and μ a positive measure on Σ. We show the followings.

(1) The hyperbolic topology coincides with the norm topology for every locally uniformly rotund (uniformly convex) normed space $(X, \|\cdot\|)$, and thus the two topologies coincide on $L_p(\Omega, \Sigma, \mu)$ for $1 < p < \infty$.

(2) They coincide on $L_1(\Omega, \Sigma, \mu)$ if and only if $\mu(\Omega) = 0$ or Ω has a finite

partition $\{A_1, \ldots, A_n\}$ by atoms.

(3) They coincide on $L_\infty(\Omega, \Sigma, \mu)$ if and only if $\mu(\Omega) = 0$ or there is an atom $A \in \Sigma$.

As special cases, they coincide in ℓ_p for $1 < p \leq \infty$, but do not coincide on ℓ_1.

Notation.

For each point x of a metric space (X, d) and each $r > 0$, we denote the *r-open ball* of x by $S_r(x) = \{y \in X : d(x, y) < r\}$ and the *r-closed ball* of x by $B_r(x) = \{y \in X : d(x, y) \leq r\}$.

2 Hyperbolic topology of a metric space

For a metric space (X, d), we call the topology $\mathcal{T}_H$ generated by those sets $\theta(x, y, t) = \{z : d(z, x) - d(z, y) < t\}$ for $x, y \in X$ and $-d(x, y) < t$ the *hyperbolic topology* of (X, d). We denote by $\mathcal{T}_M$ the metric topology induced by the original metric d. Obviously, $\mathcal{T}_H$ is weaker than $\mathcal{T}_M$.

We put $\theta_x(y, t) = \theta(x, y, t)$ for $x, y \in X$ and $-d(x, y) < t$. Then, $x \in \theta_x(y, t)$ and $\theta_x(y, s) \subset \theta_x(y, t)$ when $-d(x, y) < s < t$. The following proposition was proved in [6] via Lawson topology of the space of formal balls. Here, we give a simple and direct proof.

Proposition 2.1 *For each point $x \in X$, $\{\theta_x(y, t) : y \in X$ and $-d(x, y) < t\}$ generates a base for the $\mathcal{T}_H$-neighborhood system at x.*

Proof. Suppose that $-d(a, y) < s$ and $x \in \theta(a, y, s)$. We show $x \in \theta_x(y, t) \subset \theta(a, y, s)$ for $t = s - d(a, x)$. First, since $d(x, a) - d(x, y) < s$, we have $-d(x, y) < s - d(x, a) = t$. Therefore, $\theta_x(y, t)$ is well defined.

Suppose that $z \in \theta_x(y, t)$. Since $d(z, x) - d(z, y) < t = s - d(a, x)$, we have $d(z, a) - d(z, y) < d(z, a) + s - d(x, a) - d(z, x) \leq s$. Therefore, $z \in \theta(a, y, s)$. ∎

The hyperbolic topology $\mathcal{T}_H$ is Hausdorff because for $x, y \in X$, $\theta_x(y, 0)$ and $\theta_y(x, 0)$ are separating the two points. Moreover, we have $\mathrm{Cl}\ \theta_x(y, s) \subset \{z \in X : d(z, x) - d(z, y) \leq s\} \subset \theta_x(y, t)$ for $-d(x, y) < s < t$. Therefore, $\mathcal{T}_H$ is regular.

In [6], we studied the Lawson topology of the partial ordered set $\mathbf{B}X$ of formal balls in X, and proved the following.

Theorem 2.2 *For a metric space (X, d), the following are equivalent:*
(1) The Lawson topology and the product topology coincide on $\mathbf{B}X$.
(2) The hyperbolic topology and the metric topology coincide on X.

Here, we refer the reader to [3, 4, 6] for the poset of formal balls and Lawson topology of partially ordered sets. With this theorem, we can derive some conditions on X so that the hyperbolic topology and the metric topology coincide from the conditions given in [6] on X so that the Lawson topology and the product topology coincide on $\mathbf{B}X$. Here, we present them with direct proofs to make this paper self-contained.

Proposition 2.3 *If (X, d) is a totally bounded metric space (in particular, (X, d) is a compact metric space), then the hyperbolic topology and the metric topology coincide.*

Proof. Let $x \in X$ and $\varepsilon > 0$. It is immediate when X has at most one point and we may assume that $S_\varepsilon(x) \neq X$. Since d is totally bounded, there are finitely many points $x_1, x_2, \ldots, x_n$ of X such that $\cup_{i=1}^n S_{\varepsilon/2}(x_i) = X$. Suppose that $x \in B_{\varepsilon/2}(x_i)$ for $1 \leq i \leq k$ and $x \notin B_{\varepsilon/2}(x_i)$ for $k+1 \leq i \leq n$. We have $\cup_{i=1}^k S_{\varepsilon/2}(x_i) \subset S_\epsilon(x)$ and therefore $k < n$. For $k+1 \leq i \leq n$, we have $d(x_i, x) > \varepsilon/2$ and thus we can consider the $\mathcal{T}_H$-neighborhood $\theta_x(x_i, -\varepsilon/2)$ of x. For $z \in \theta_x(x_i, -\varepsilon/2)$, $d(z, x) - d(z, x_i) < -\varepsilon/2$ and therefore $d(z, x_i) > d(z, x) + \varepsilon/2 \geq \varepsilon/2$. Therefore, $\theta_x(x_i, -\varepsilon/2) \cap S_{\varepsilon/2}(x_i) = \emptyset$ and thus $\cap_{i=k+1}^n \theta_x(x_i, -\varepsilon/2) \cap \cup_{i=k+1}^n S_{\varepsilon/2}(x_i) = \emptyset$. Since $\cup_{i=1}^k S_{\varepsilon/2}(x_i) \cup \cup_{i=k+1}^n S_{\varepsilon/2}(x_i) = X$, $\cap_{i=k+1}^n \theta_x(x_i, -\varepsilon/2) \subset \cup_{i=1}^k S_{\varepsilon/2}(x_i) \subset S_\epsilon(x)$. ∎

Let $(X, \|\cdot\|)$ be a normed linear space and $\mathcal{T}_H$ the hyperbolic topology on X induced by the norm $\|\cdot\|$. Then we easily have the following.

Proposition 2.4 *Let $(X, \|\cdot\|)$ be a normed linear space, $a \in X$ and $\alpha > 0$. Then $\varphi_a : (X, \mathcal{T}_H) \to (X, \mathcal{T}_H)$ defined by $\varphi_a(x) = x + a$ and $\psi_\alpha : (X, \mathcal{T}_H) \to (X, \mathcal{T}_H)$ defined by $\psi_\alpha(x) = \alpha x$ are homeomorphisms.*

Proposition 2.5 *Let $(X, \|\cdot\|)$ be a normed linear space and d the metric induced by the norm $\|\cdot\|$. If the restriction of d on the unit ball $B_1(\mathbf{0})$ is totally bounded, then the hyperbolic topology and the metric topology coincide on X.*

Proof. By Proposition 2.4, we only need to show $U \subset B_{1/2}(\mathbf{0})$ for some $\mathcal{T}_H$-neighborhood U of $\mathbf{0}$. Since the metric space $(B_1(\mathbf{0}), d)$ is totally bounded, by Proposition 2.3, we have $\cap_{i=1}^k \theta_{\mathbf{0}}(x_i, r_i) \cap B_1(\mathbf{0}) \subset B_{1/2}(\mathbf{0})$

for some $x_i \in B_1(\mathbf{0})$ and $-\|x_i\| < r_i$ $(1 \le i \le k)$. Then, for every $y \in X$ such that $\|y\| > 1$, $z = y/\|y\| \in B_1(\mathbf{0})$ and $z \notin B_{1/2}(\mathbf{0})$, and therefore, $z \notin \theta_{\mathbf{0}}(x_i, r_i)$ for some $1 \le i \le k$. Therefore, $\|z\| - \|z - x_i\| \ge r_i$. Then, $\|y\| - \|y - x_i\| = \|y - z\| + \|z\| - \|y - x_i\| = \|z\| - \|z - x_i\| + \|z - x_i\| + \|y - z\| - \|y - x_i\| \ge \|z\| - \|z - x_i\| \ge r_i$. Therefore, $y \notin \theta_{\mathbf{0}}(x_i, r_i)$. Thus, $\cap_{i=1}^{k} \theta_{\mathbf{0}}(x_i, r_i) \subset B_{1/2}(\mathbf{0})$. ∎

The following lemma is a well-known property of a normed linear space (cf. Lemma 17 of [6]).

Lemma 2.6 *Let $(X, \|\cdot\|)$ be a normed linear space and d the metric induced by the norm $\|\cdot\|$. The restriction of d on the unit ball $B_1(\mathbf{0})$ is totally bounded if and only if $(X, \|\cdot\|)$ is finite-dimensional.*

Thus, we have the following.

Theorem 2.7 *The hyperbolic topology and the norm topology coincide for finite-dimensional normed linear spaces.*

Here is an example in [6] of a metric space (X_0, d_s) on which the two topologies differ. Let X_0 be an infinite set with a fixed point $x_0 \in X_0$, and d_s the following metric function on X_0.

$$d_s(x, y) = \begin{cases} 0, & \text{if } x = y, \\ 1, & \text{if } x_0 \in \{x, y\} \text{ and } x \ne y, \\ 2, & \text{otherwise.} \end{cases}$$

The metric topology of (X_0, d_s) is the discrete topology where the hyperbolic topology of (X_0, d_s) is generated by those sets $\{x\}$ for $x \in X_0 - \{x_0\}$, and $X_0 - A$, where A ranges over finite subsets of X_0 which do not contain x_0.

3 The hyperbolic topology in $L_p(\Omega, \Sigma, \mu)$

In this section, we study the relation between the hyperbolic topology and the metric topology for normed linear spaces, especially, $L_p(\Omega, \Sigma, \mu)$ for $1 \le p \le \infty$. In this section, we denote the norm in $L_p(\Omega, \Sigma, \mu)$, $1 \le p \le \infty$, by $\|\cdot\|$ instead of $\|\cdot\|_p$.

3.1 Locally uniformly rotund (convex) spaces

First, we consider the case $1 < p < \infty$. In this case, we have a general theorem. We consider the uniformly rotund spaces (uniformly convex spaces), which are introduced by J. A. Clarkson [1].

Definition 3.1 A normed linear space $(X, \|\cdot\|)$ is said to be *uniformly rotund* (*uniformly convex*) if for every $\varepsilon > 0$ there is $\delta(\varepsilon) > 0$ such that for each $x, y \in X$ with $\|x\| = \|y\| = 1$ and $\|x - y\| \geq \varepsilon$, $\frac{\|x+y\|}{2} < 1 - \delta(\varepsilon)$.

A normed linear space $(X, \|\cdot\|)$ is said to be *locally uniformly rotund* (or *locally uniformly convex*) if for each $x \in X$ with $\|x\| = 1$ and $\varepsilon > 0$ there is $\delta(x, \varepsilon) > 0$ such that for each $y \in X$ with $\|y\| = 1$ and $\|x - y\| \geq \varepsilon$, $\frac{\|x+y\|}{2} < 1 - \delta(x, \varepsilon)$.

Uniform rotundity means that when x and y are points on the unit sphere with the distance greater than ε, then the middle point is in the ball $B_{1-\delta_X(\varepsilon)}(\mathbf{0})$ and therefore the distance from the unit sphere is greater than $\delta_X(\varepsilon)$.

The sum norm and the max norm on $\mathbb{R}^2$ are not locally uniformly rotund because the unit sphere has the form of a square with each of these two norms. It is known that $L_p(\Omega, \Sigma, \mu)$ for $1 < p < \infty$ are uniformly rotund ([1]) and hence locally uniformly rotund. We notice that if $(X, \|\cdot\|)$ is locally uniformly rotund and $x \in X$ with $\|x\| = 1$, then the real number $\delta(x, \varepsilon)$ works for the point $-x$, i.e., we may assume that $\delta(-x, \varepsilon) = \delta(x, \varepsilon)$.

Theorem 3.2 *If $(X, \|\cdot\|)$ is a locally uniformly rotund normed linear space, then the hyperbolic topology coincides with the norm topology on X.*

Proof. The two topologies coincide for the case $X = \{\mathbf{0}\}$. We shall show that $\mathcal{T}_M \subset \mathcal{T}_H$. By Proposition 2.4, it suffices to show that the $\mathcal{T}_M$-neighborhood $S_1(\mathbf{0}) = \{x \in X : \|x\| < 1\}$ of $\mathbf{0}$ contains a $\mathcal{T}_H$-neighborhood V of $\mathbf{0}$. Fix a point $x \in X$ with $\|x\| = 1$. Let $\delta(x, 1) > 0$ be a real number defined in Definition 3.1. We put $t = \max\{1 - 2\delta(x, 1), 0\}$ and $V = \theta_{\mathbf{0}}(x, -t) \cap \theta_{\mathbf{0}}(-x, -t)$. Then V is a $\mathcal{T}_H$-neighborhood of $\mathbf{0}$. To show that $V \subset S_1(\mathbf{0})$, we assume that there is $z \in V - S_1(\mathbf{0})$. Then we have $\|x - z\| - \|z\| > t$ and $\|-x - z\| - \|z\| > t$. Let $z' = z/\|z\|$. It follows that $\|z' - x\| \geq \|z - x\| - \|z' - z\| > \|z\| + t - (\|z\| - 1) = t + 1 \geq 1 - 2\delta(x, 1) + 1 = 2 - 2\delta(x, 1)$. Hence we have

$$\|\frac{z' + (-x)}{2}\| > 1 - \delta(x, 1) = 1 - \delta(-x, 1).$$

Since $\|z'\| = 1$, by the choice of $\delta(-x, 1)$, we have that $\|z' - (-x)\| < 1$. Similarly, we can see that $\|z' - x\| < 1$. Hence, $2 = 2\|x\| = \|x - (-x)\| \leq \|x - z'\| + \|z' - (-x)\| < 1 + 1 = 2$. This is a contradiction. Hence $V \subset S_1(\mathbf{0})$ and hence $\mathcal{T}_M \subset \mathcal{T}_H$. This completes the proof. ∎

Corollary 3.3 *Let $1 < p < \infty$, Σ a σ-algebra of subsets of a set Ω, and μ a positive measure on Σ. The hyperbolic topology and the norm topology coincide on $L_p(\Omega, \Sigma, \mu)$.*

The following is a special case of Corollary 3.3.

Corollary 3.4 *If $1 < p < \infty$, then the hyperbolic topology coincides with the norm topology on ℓ_p and $L_p[0, 1]$.*

3.2 The case $p = 1$

Let (Ω, Σ, μ) be a measure space. A set $A \in \Sigma$ is said to be an *atom* if $\mu(A) > 0$ and for each $B \subset A$ with $B \in \Sigma$ we have $\mu(B) = 0$ or $\mu(B) = \mu(A)$. We say that a measure space (Ω, Σ, μ) *has a finite partition by atoms* if there are finitely many atoms $A_i \in \Sigma$, $i = 1, \ldots, n$ such that $\Omega = A_1 \cup \cdots \cup A_n$ and $A_i \cap A_j = \emptyset$ if $i \neq j$.

We notice that the measure space $(\mathbb{N}, 2^{\mathbb{N}}, \mu)$, where μ is the counting measure, contains atoms, but it does not have a finite partition by atoms.

Lemma 3.5 *Let (Ω, Σ, μ) be a measure space and $1 \leq p \leq \infty$. If $\mu(\Omega) = 0$, or (Ω, Σ, μ) has a finite partition by atoms, then $L_p(\Omega, \Sigma, \mu)$ is finite dimensional.*

Proof. If $\mu(\Omega) = 0$, then $L_p(\Omega, \Sigma, \mu) = \{\mathbf{0}\}$. We suppose that $\mu(\Omega) > 0$. Let $\{A_1, \ldots, A_n\}$ be a finite partition of Ω by atoms, $f \in L_p(\Omega, \Sigma, \mu)$ and $i \leq n$. Let $f_i : A_i \to \mathbb{R}$ be the restriction of f over A_i. We define mappings $g_i : \Omega \to \mathbb{R}$, $i = 1, \ldots, n$, as follows:

$$g_i(x) = \begin{cases} 1, & \text{if } x \in A_i, \\ 0, & \text{if } x \notin A_i. \end{cases}$$

Since A_i is an atom, it follows that f_i is a constant mapping a.e. Hence, $L_p(\Omega, \Sigma, \mu)$ is generated by $g_1, \ldots, g_n$ and hence $L_p(\Omega, \Sigma, \mu)$ is n-dimensional. ∎

The following is a direct consequence of Theorem 2.7 and Lemma 3.5.

Corollary 3.6 *Let (Ω, Σ, μ) be a measure space and $1 \leq p \leq \infty$. If $\mu(\Omega) = 0$, or (Ω, Σ, μ) has a finite partition by atoms, then the hyperbolic topology coincides with the norm topology on $L_p(\Omega, \Sigma, \mu)$.*

Now we consider the case $p = 1$. We can easily show the following lemma.

Lemma 3.7 *Let (Ω, Σ, μ) be a measure space such that $\mu(\Omega) > 0$. If (Ω, Σ, μ) does not have a finite partition by atoms, then there is a countable set $\{A_1, A_2, \dots\} \subset \Sigma$ such that $A_i \cap A_j = \emptyset$ if $i \neq j$ and $\mu(A_i) > 0$ for each i.*

Theorem 3.8 *Let (Ω, Σ, μ) be a measure space. The hyperbolic topology coincides with the norm topology on $L_1(\Omega, \Sigma, \mu)$ if and only if $\mu(\Omega) = 0$ or (Ω, Σ, μ) has a finite partition by atoms.*

Proof. Corollary 3.6 proves the "if" part. To prove the "only if" part, we suppose that $\mu(\Omega) > 0$ and (Ω, Σ, μ) does not have a finite partition by atoms. By Lemma 3.7, there is a countable set $\{A_1, A_2, \dots\} \subset \Sigma$ such that $A_i \cap A_j = \emptyset$ if $i \neq j$ and $\mu(A_i) > 0$ for each i. We may assume that $\Omega = \cup_{i=1}^{\infty} A_i$. Let $\mathbf{0} \in L_1(\Omega, \Sigma, \mu)$ be the constant mapping and $S_1(\mathbf{0})$ the 1-open neighborhood of $\mathbf{0}$ in the norm topology. Let $f_1, \dots, f_n \in L_1(\Omega, \Sigma, \mu)$ and $t_1, \dots, t_n \in \mathbb{R}$, where $-\|f_i\| < t_i$. Put $U = \cap_{i=1}^{n} \theta_{\mathbf{0}}(f_i, t_i)$. It suffices to show that $U - S_1(\mathbf{0}) \neq \emptyset$. For each $i \leq n$ we put $\delta_i = \|f_i\| + t_i > 0$ and $\delta = \min\{\delta_1, \dots, \delta_n\}$. Since $\sum_{k=1}^{\infty} \int_{A_k} |f_i| d\mu = \int_{\Omega} |f_i| d\mu < \infty$, there is $k(i)$ such that $\sum_{k=k(i)}^{\infty} \int_{A_k} |f_i| d\mu < \delta/2$. Let $K = \max\{k(1), \dots, k(n)\}$. We define a function $g : \Omega \to \mathbb{R}$ by

$$g(x) = \begin{cases} \frac{1}{\mu(A_K)}, & \textit{if } x \in A_K, \\ 0, & \textit{otherwise.} \end{cases}$$

It is clear that $g \in L_1(\Omega, \Sigma, \mu)$ and $\|g\| = 1$. On the other hand, for each $i \leq n$ we have

$$
\begin{aligned}
\|g\| - \|f_i - g\| &= 1 - \int_\Omega |f_i - g| d\mu \\
&= 1 - \sum_{k=1}^{\infty} \int_{A_k} |f_i - g| d\mu \\
&= 1 - (\sum_{k \neq K} \int_{A_k} |f_i| d\mu + \int_{A_K} |f_i - g| d\mu) \\
&< 1 - (\|f_i\| - \int_{A_K} |f_i| d\mu + \int_{A_K} |g| d\mu - \int_{A_K} |f_i| d\mu) \\
&< 1 - \|f_i\| + \frac{\delta}{2} - 1 + \frac{\delta}{2} \\
&= \delta - \|f_i\| \\
&\leq \delta_i - \|f_i\| \\
&= t_i.
\end{aligned}
$$

This implies that $g \in \theta_{\mathbf{0}}(f_i, t_i)$ and hence $g \in U - S_1(\mathbf{0})$. ∎

Since the measure space $(\mathbb{N}, 2^{\mathbb{N}}, \mu)$, where μ is the counting measure, does not have a finite partition by atoms, the following is a direct consequence of Theorem 3.8.

Corollary 3.9 *The hyperbolic topology is strictly weaker than the norm topology on* ℓ_1.

3.3 The case $p = \infty$

For the case $p = \infty$, we have the following.

Theorem 3.10 *Let* (Ω, Σ, μ) *be a measure space. The hyperbolic topology coincides with the norm topology on* $L_\infty(\Omega, \Sigma, \mu)$ *if and only if* $\mu(\Omega) = 0$ *or* (Ω, Σ, μ) *has an atom.*

Proof. If $\mu(\Omega) = 0$, then the hyperbolic topology coincides with the norm topology by Corollary 3.6. Now, we suppose that $\mu(\Omega) > 0$ and (Ω, Σ, μ) has an atom $A \in \Sigma$. Let $S_1(\mathbf{0})$ be the 1-open neighborhood of $\mathbf{0}$ in the norm topology. Let $f_1, f_2 \in L_\infty(\Omega, \Sigma, \mu)$ be defined by

$$
f_1(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A, \end{cases}
$$

$$f_2(x) = \begin{cases} -1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A. \end{cases}$$

Let $U = \theta_{\mathbf{0}}(f_1, -1/2) \cap \theta_{\mathbf{0}}(f_2, -1/2)$. By Proposition 2.4, it suffices to show that $\mathbf{0} \in U \subset S_1(\mathbf{0})$. It is obvious that $\mathbf{0} \in U$. Let $g \in U$. Then $1/2 < \|g - f_i\| - \|g\|$ for each $i = 1, 2$, because $g \in \theta_{\mathbf{0}}(f_i, -1/2)$. We have $\|g - f_i\| = \max\{\|g|_{\Omega-A} - f_i|_{\Omega-A}\|, \|g|_A - f_i|_A\|\} = \max\{\|g|_{\Omega-A}\|, \|g|_A - f_i|_A\|\}$. If $\|g|_{\Omega-A}\| \geq \|g|_A - f_i|_A\|$, then we have $1/2 < \|g - f_i\| - \|g\| = \|g|_{\Omega-A}\| - \|g\| \leq 0$. This is a contradiction. Hence we have $\|g|_{\Omega-A}\| < \|g|_A - f_i|_A\|$, and hence $\|g - f_i\| = \|g|_A - f_i|_A\|$ for each $i = 1, 2$. Furthermore, since A is an atom, $g|_A = t$ a.e. on A for some $t \in \mathbb{R}$. Since $1/2 < \|g - f_1\| - \|g\| = \|g|_A - f_1|_A\| - \|g\|$, we have

$$1/2 < \|g|_A - f_1|_A\| - \|g\| \leq \|g|_A - f_1|_A\| - \|g|_A\| = |t - 1| - |t|.$$

Hence we have $t < 1/4$. Similarly, we also have that

$$1/2 < \|g|_A - f_2|_A\| - \|g\| \leq \|g|_A - f_2|_A\| - \|g|_A\| = |t + 1| - |t|,$$

and hence $t > -1/4$. Hence, we have $-1/4 < t < 1/4$. Since $\|g|_{\Omega-A}\| < \|g|_A - f_i|_A\| = \|t - f_i|_A\|$ for each $i = 1, 2$, it follows that $\|g|_{\Omega-A}\| < \min\{|t - 1|, |t + 1|\} \leq 1$. Finally, we have $\|g\| = \max\{\|g|_{\Omega-A}\|, \|g|_A\|\} = \max\{\|g|_{\Omega-A}\|, |t|\} < 1$. Hence $g \in S_1(\mathbf{0})$ and hence $U \subset S_1(\mathbf{0})$. Therefore, $\mathcal{T}_H = \mathcal{T}_M$ in $L_\infty(\Omega, \Sigma, \mu)$.

Conversely, we suppose that $\mu(\Omega) > 0$ and (Ω, Σ, μ) does not have an atom. Let $f_1, \ldots, f_n \in L_\infty(\Omega, \Sigma, \mu)$ and $t_1, \ldots, t_n \in \mathbb{R}$ such that $f_i \neq 0$ a.e. and $-\|f_i\| < t_i$. It suffices to show that $\theta_{\mathbf{0}}(f_1, t_1) \cap \cdots \cap \theta_{\mathbf{0}}(f_n, t_n) - S_1(\mathbf{0}) \neq \emptyset$. Since $\theta_{\mathbf{0}}(f_i, s) \subset \theta_{\mathbf{0}}(f_i, t)$ if $-\|f_i\| < s < t$, we may assume that $t_i \leq 0$ for each $i \leq n$.

For each $i \leq n$ we put

$$A_i^+ = \{x \in \Omega : f_i(x) > -t_i\},$$

$$A_i^- = \{x \in \Omega : f_i(x) < t_i\}.$$

We have $\|f_i\| > -t_i$ and hence $\mu(A_i^+ \cup A_i^-) > 0$. Thus, $\mu(A_i^+) > 0$ or $\mu(A_i^-) > 0$. Let $I^+ = \{i \leq n : \mu(A_i^+) > 0\}$ and $I^- = \{1, \ldots, n\} - I^+$. Put

$$B_i = \begin{cases} A_i^+, & \text{if } i \in I^+, \\ A_i^-, & \text{if } i \in I^-. \end{cases}$$

We notice that $\mu(B_i) > 0$ for each $i \le n$. Since (Ω, Σ, μ) does not have an atom, it is easy to see that for each $i \le n$ there is $C_i \in \Sigma$ such that $C_i \subset B_i$, $\mu(C_i) > 0$ and $C_i \cap C_j = \emptyset$ if $i \ne j$. Define $g : \Omega \to \mathbb{R}$ by

$$g(x) = \begin{cases} 1, & \text{if } x \in C_i \text{ and } i \in I^-, \\ -1, & \text{if } x \in C_i \text{ and } i \in I^+, \\ 0, & \text{otherwise.} \end{cases}$$

Then $g \in L_\infty(\Omega, \Sigma, \mu)$ and $\|g\| = 1$ (and hence $g \notin S_1(\mathbf{0})$). Furthermore, it is easy to see that $\|g\| - \|g - f_i\| < t_i$ for each $i \le n$. Hence $g \in \cap_{i=1}^n \theta_{\mathbf{0}}(f_i, t_i)$. This implies that $\mathcal{T}_H \subsetneq \mathcal{T}_M$. ∎

The following is a direct consequence of the theorem above.

Corollary 3.11 *The hyperbolic topology coincides with the norm topology on ℓ_∞.*

By a similar argument to the proof of Theorem 3.10, we can show the following.

Corollary 3.12 *The hyperbolic topology does not coincide with the norm topology on $C([0, 1])$.*

Remark 3.13 Combining the results in this section and Theorem 2.2, we have the characterization of spaces $L_p(\Omega, \Sigma, \mu)$ such that the Lawson topology and the product topology coincide on $\mathbf{B}L_p(\Omega, \Sigma, \mu)$. In addition, we have examples of normed linear spaces for which the two topologies do not coincide on their spaces of formal balls, for instance, ℓ_1 and $C([0, 1])$.

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