Uniform Spaces and Uniform Domains

--- Representing a Uniform space + a Base for Computation ---

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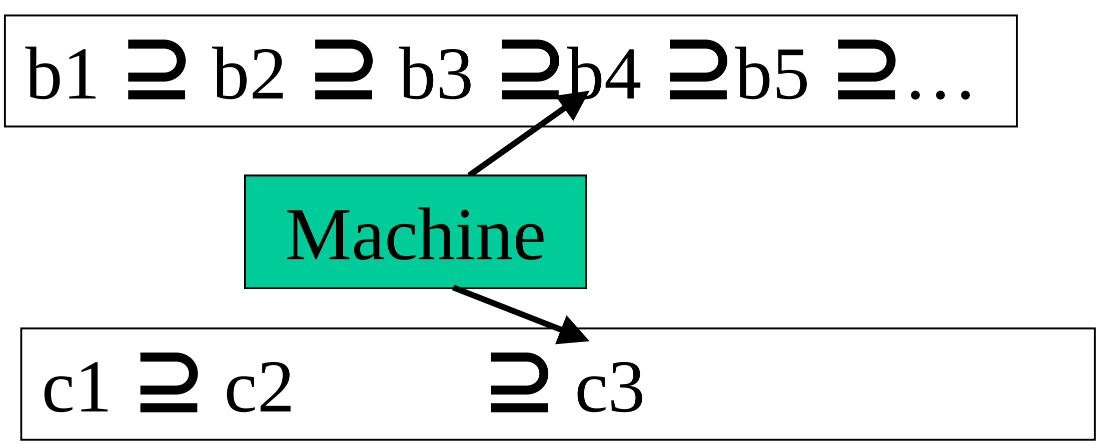
Computation over a Topological Space

[2nd-countable, T0 space]

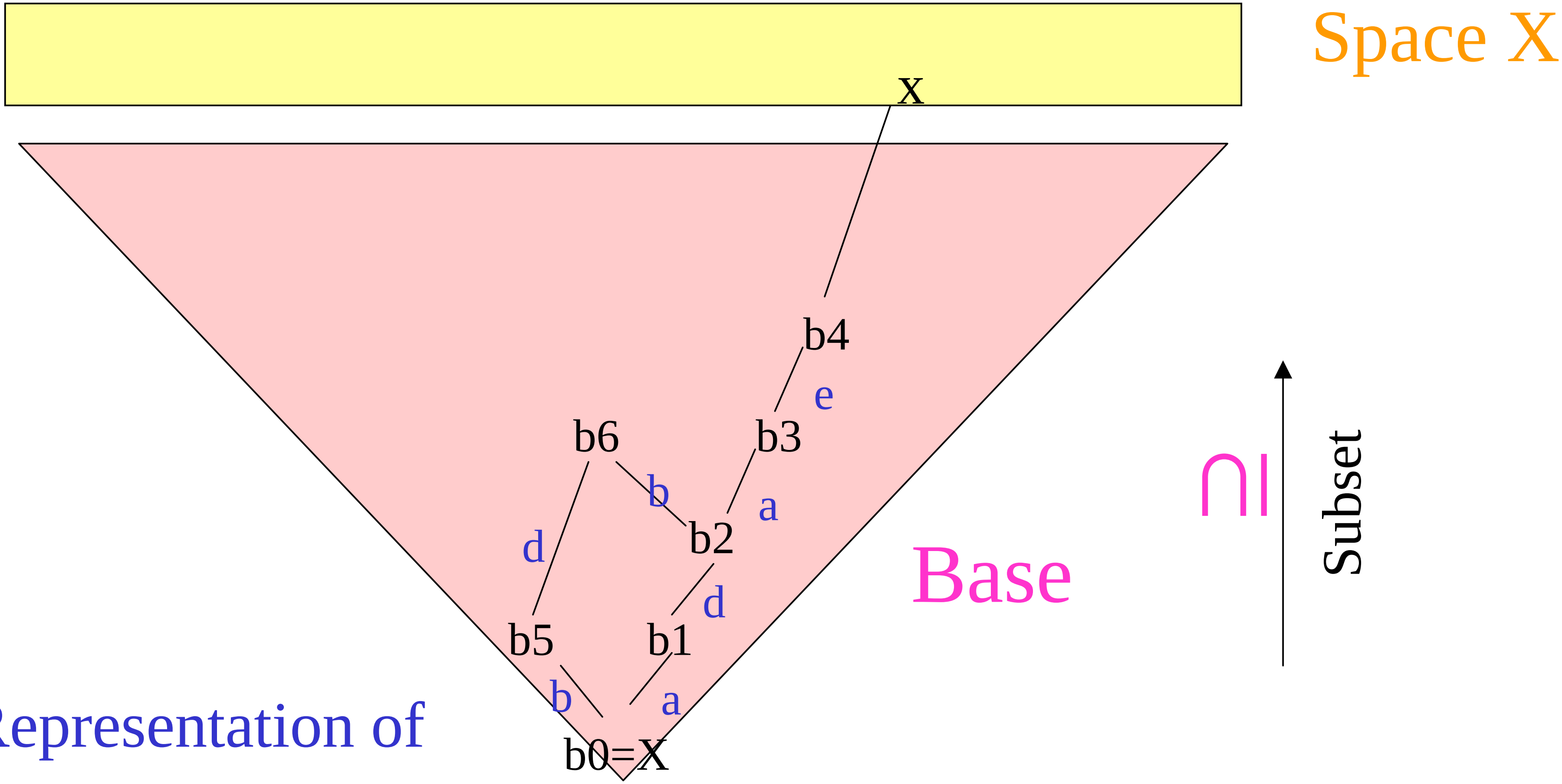
- Fix a base and a representation of the base.
- Computation with respect to approximations.
- approximation … given as a sequence of base elements

$$b1 \supseteq b2 \supseteq b3 \supseteq b4 \supseteq b5 \supseteq \ldots \rightarrow x$$

- A machine which inputs/outputs infinite sequences of (representation of the) base.



Base as a poset, points as limits



Representation of

b0 ⊇ b1 ⊇ b2 ⊇…

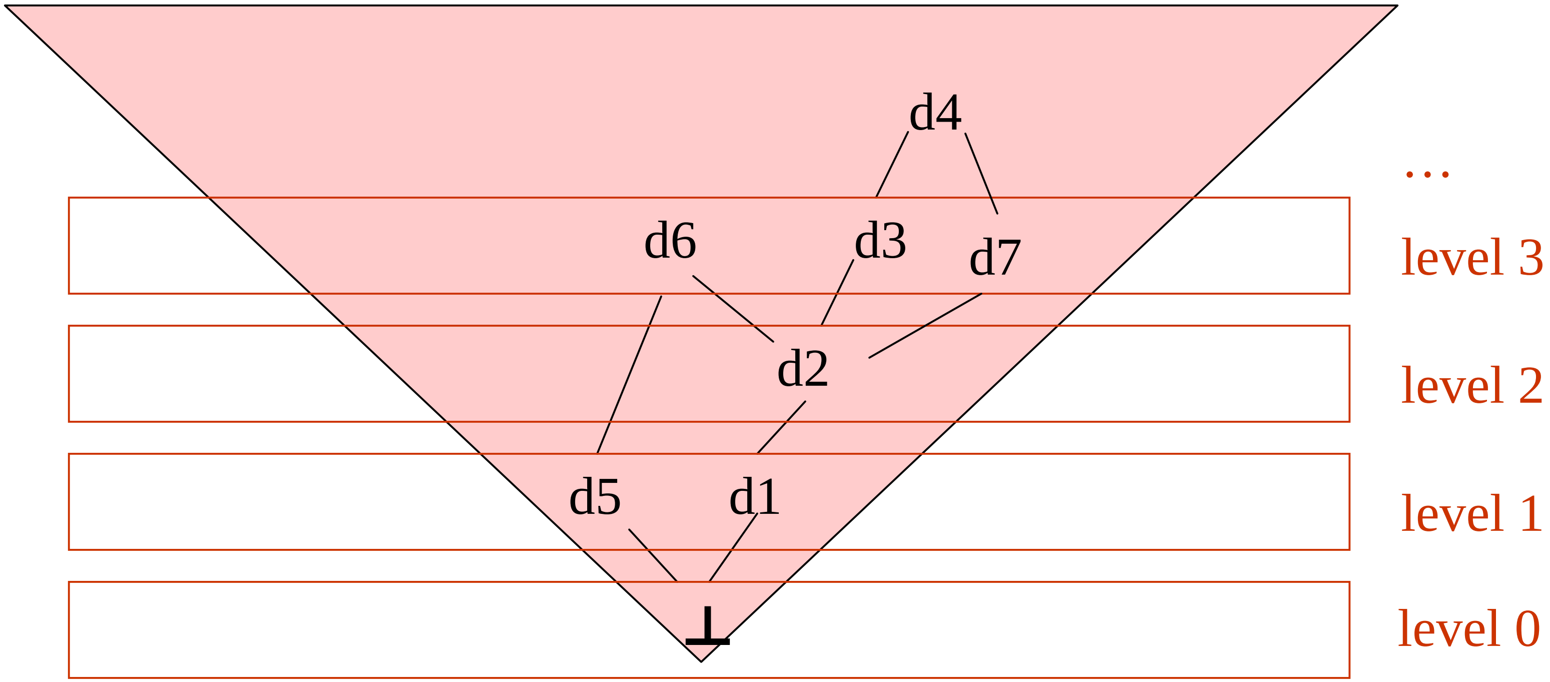
as a infinite string adae…

⇒ Domain Representation of a uniform space

ω-type poset

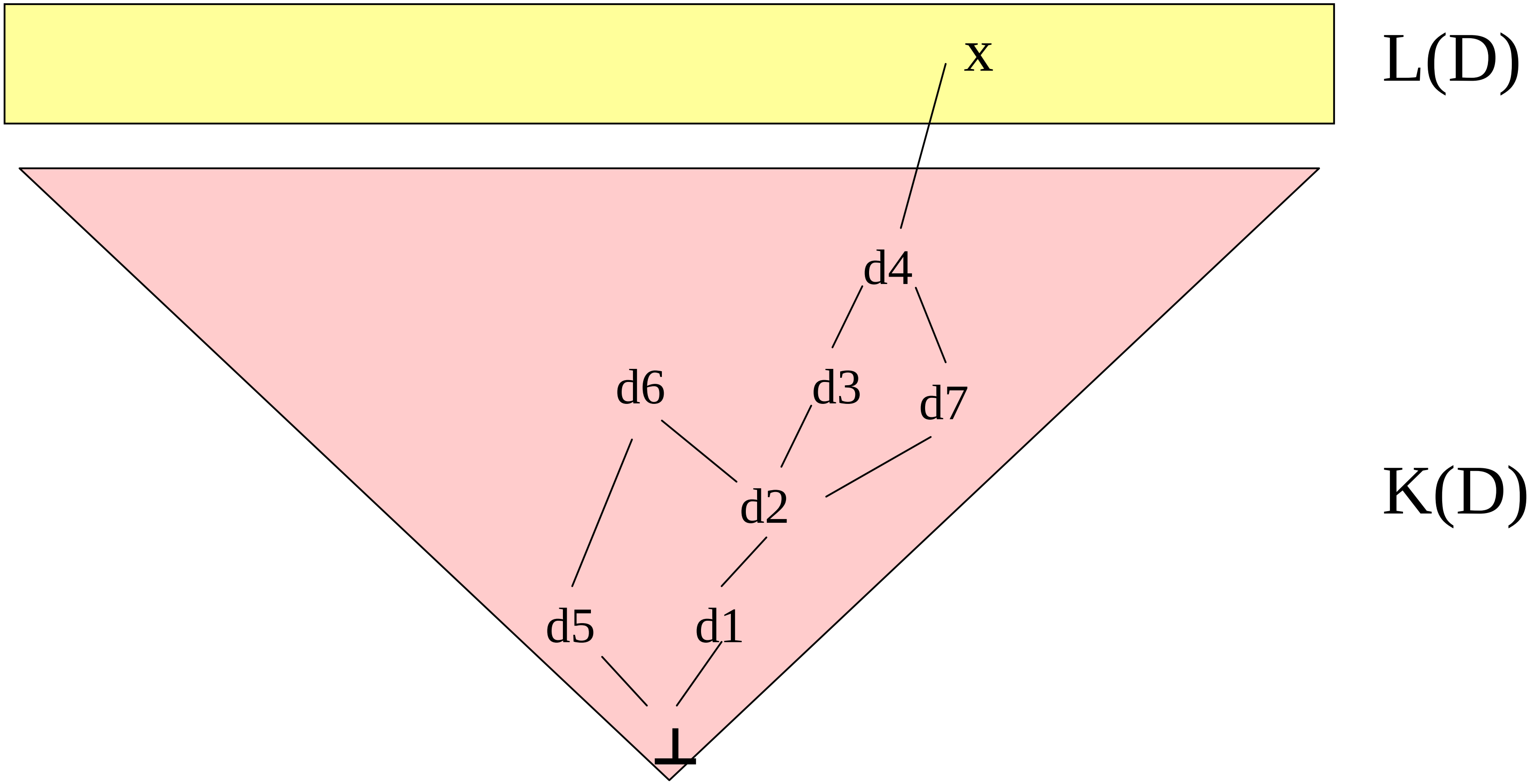
- the level of d ∈ P: the maximal length of a chain

  $\bot < d1 < d2 < \ldots < d$
- ω-type poset: every element d ∈ P has a finte level.



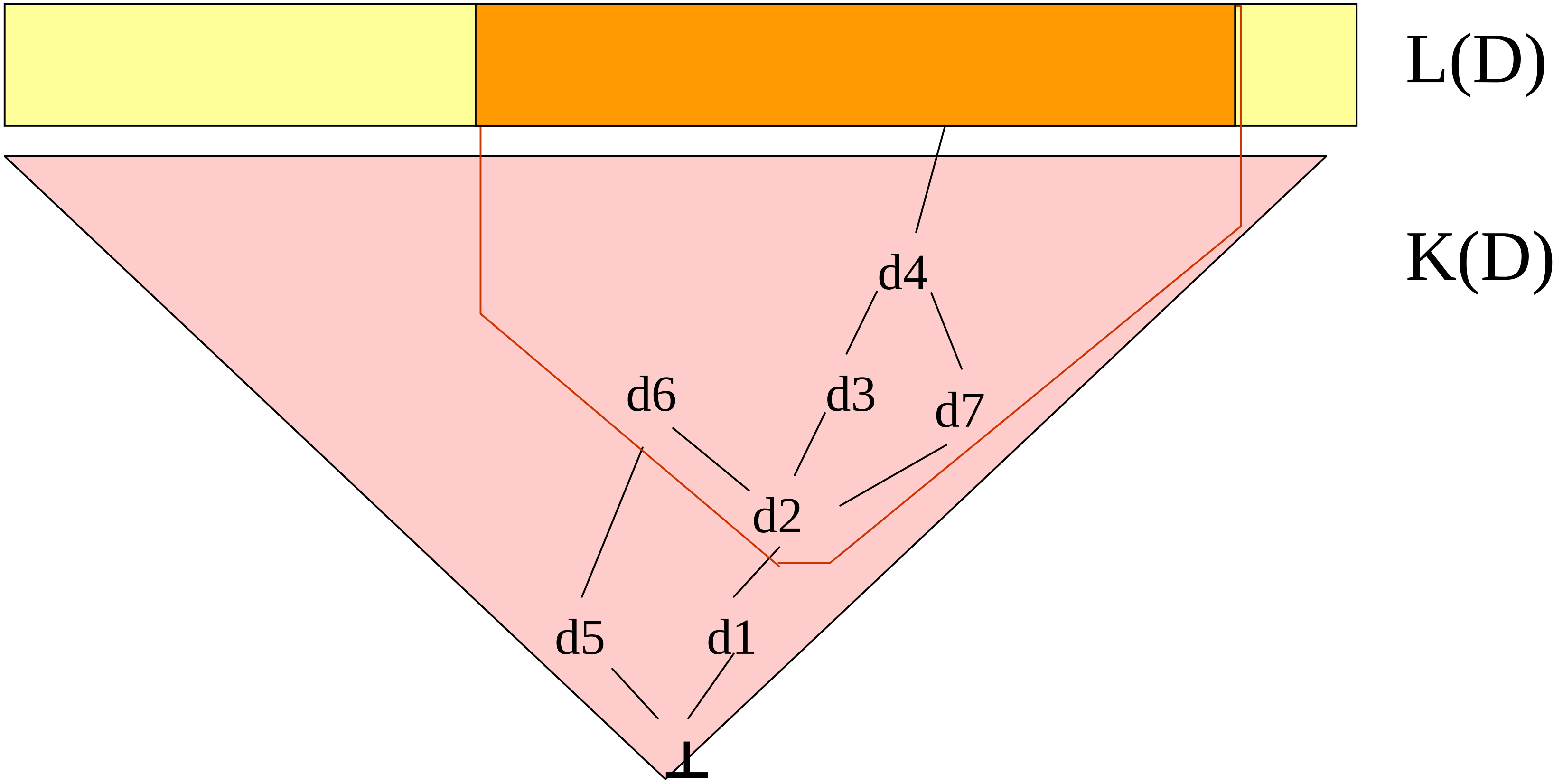
ω-type domain D

- Ideal completion of a ω-type poset
  (the set of increasing sequences / equivalence)
- It is a ω-algebraic cpo.



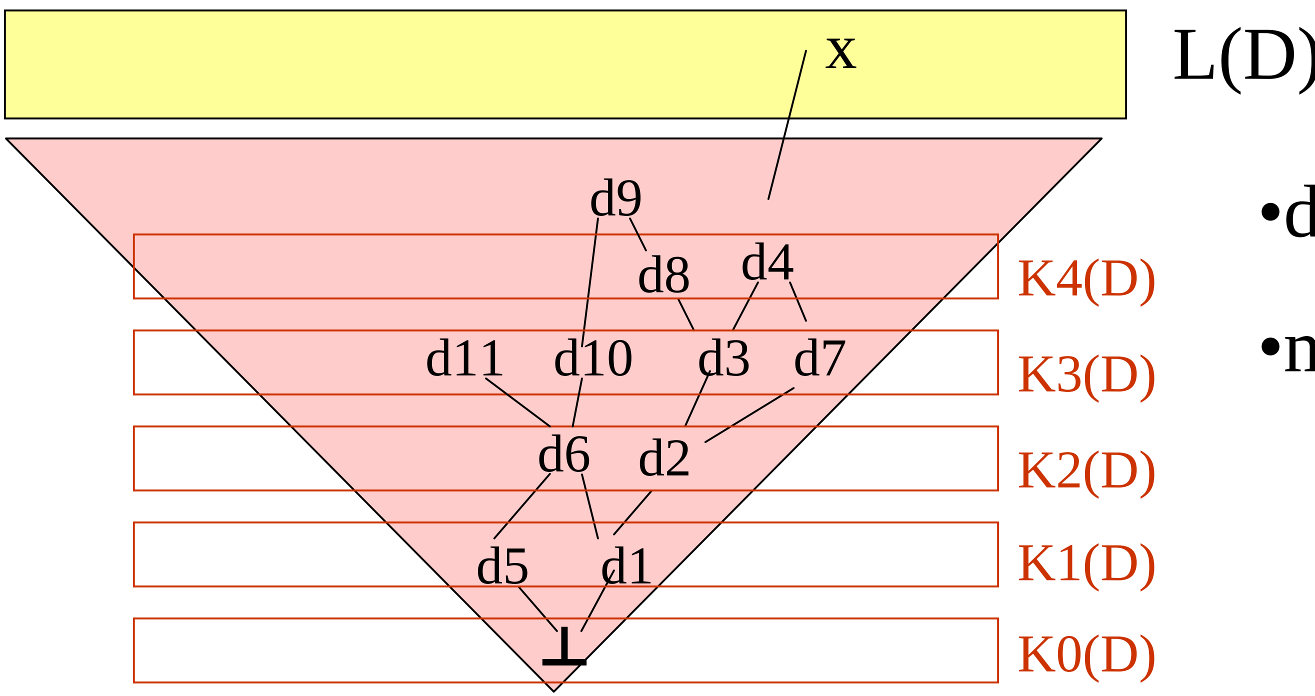
Scott Topology on a ω-type domain D

- (relative)Scott topology on L(D)
- { ↑d | d ∈ K(D)} as a base.
- T0 space



Uniform Domain

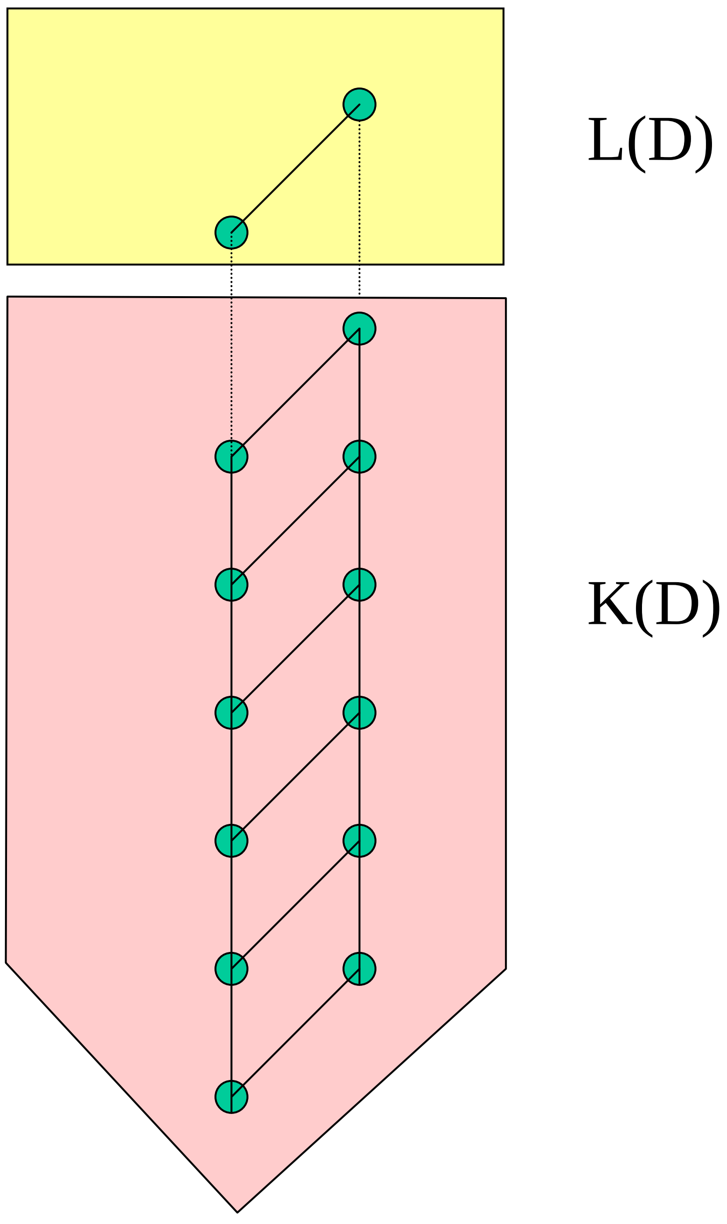
- For d ∈ Kn(D), define d* = {e ∈ Kn(D) | e ↑ d }
- Define mlb(n) as the maximum level of a lower bound of d* for d ∈ Kn(D).
- Uniform domain is a ω-type domain D such that mlb(n) → ∞ as n → ∞.



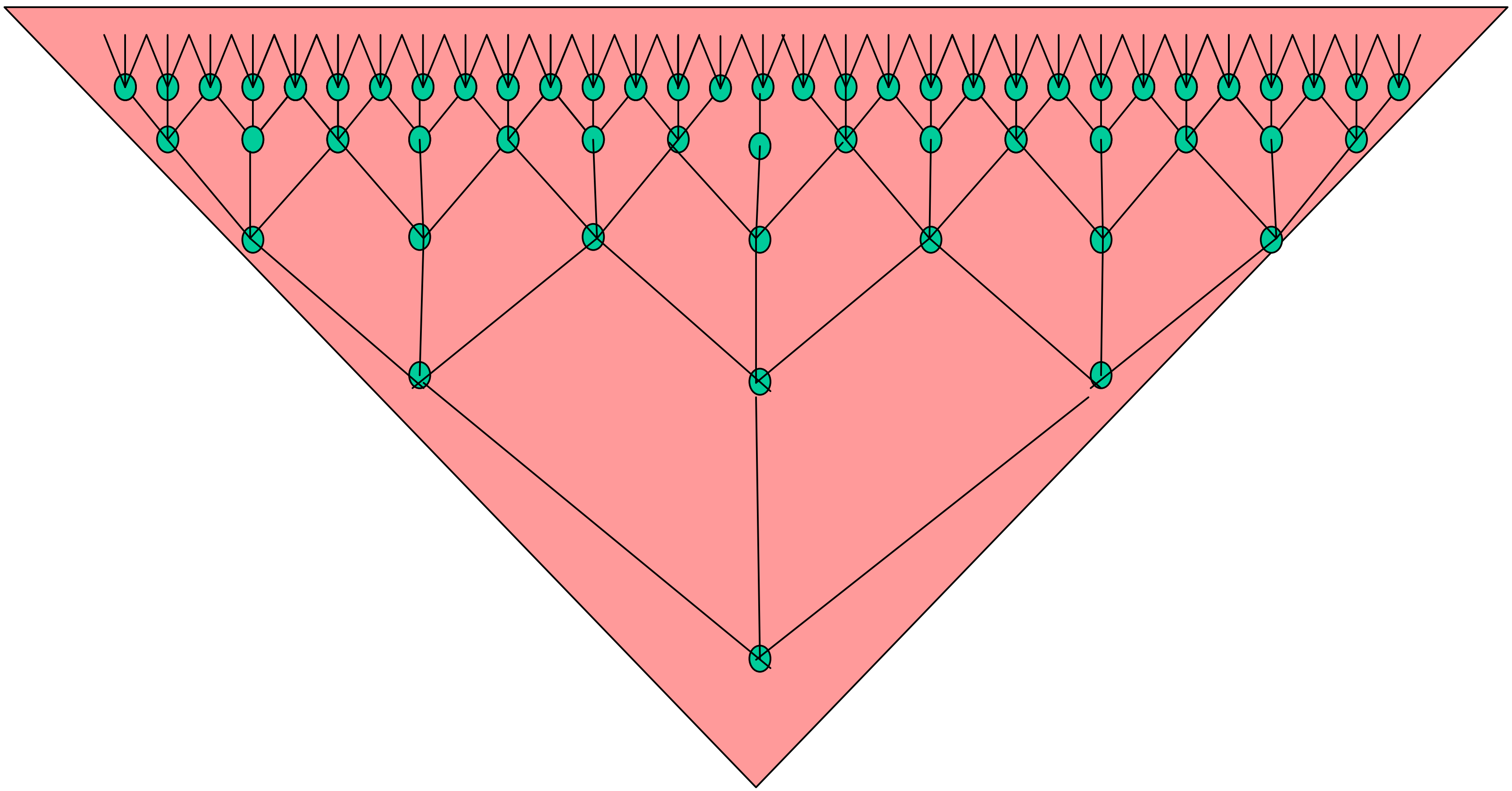
•d3*={d3,d7,d10}

•mub(3) = 1

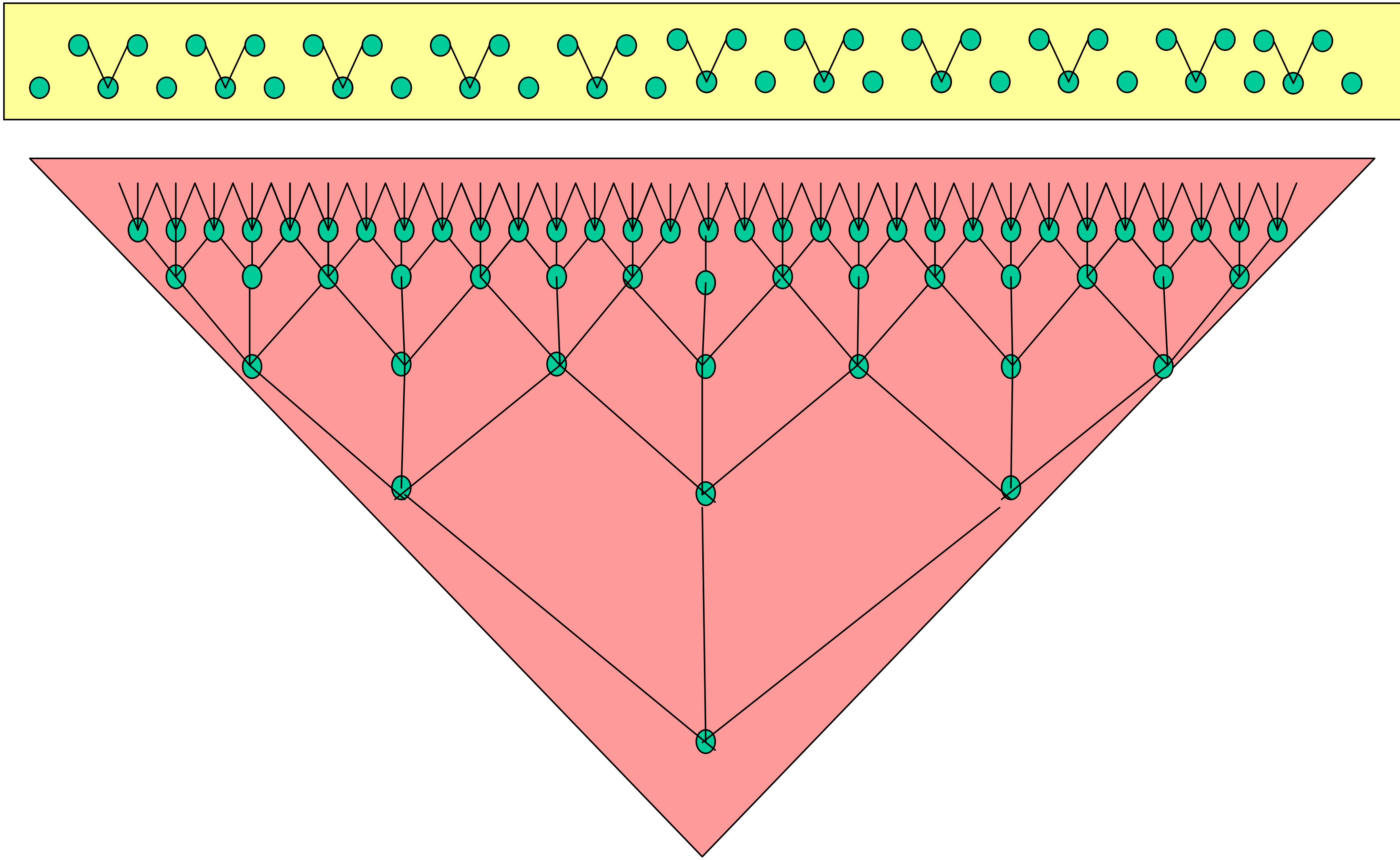
An example of a uniform domain



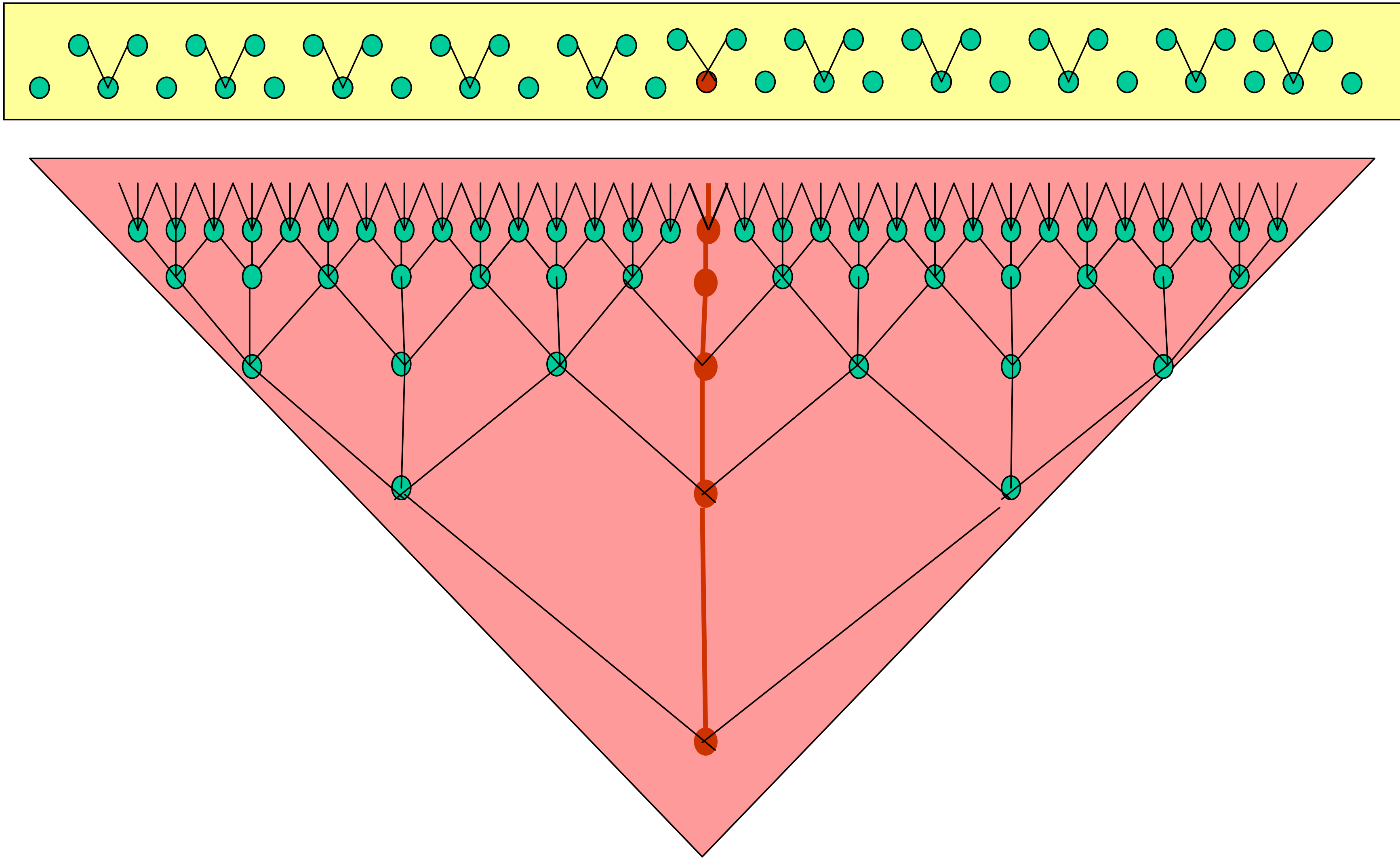
uniform domain RD



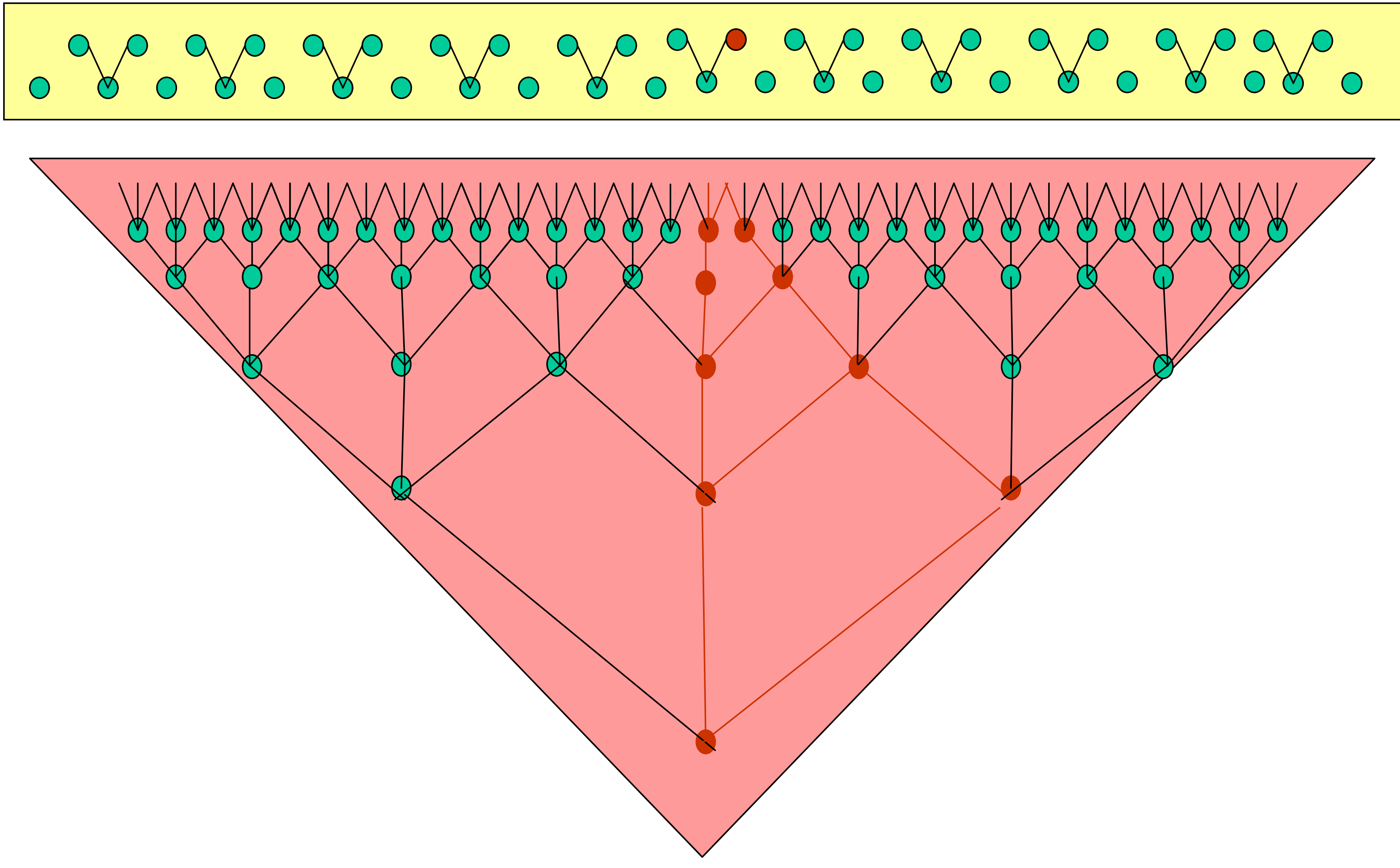
uniform domain RD



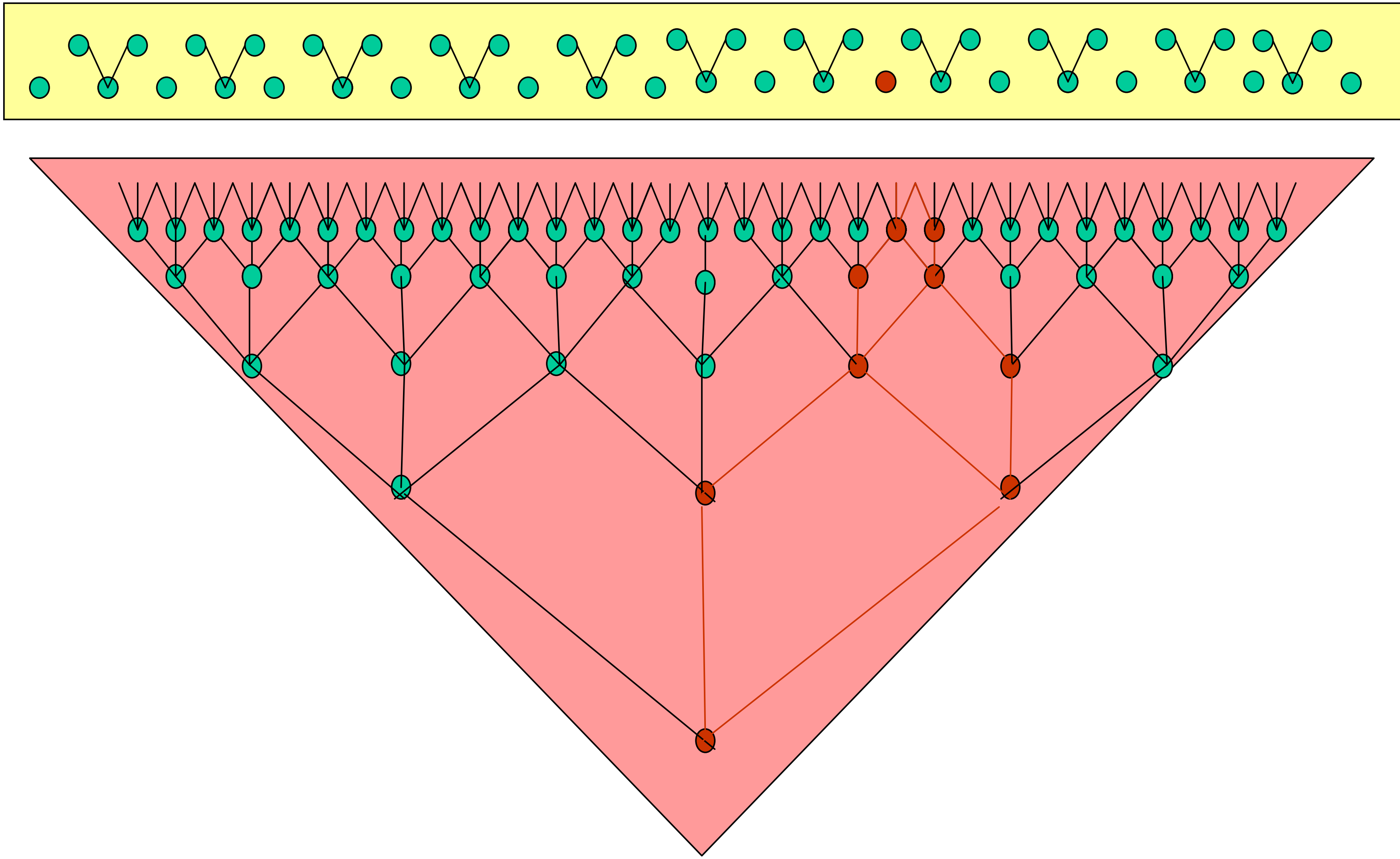
uniform domain RD



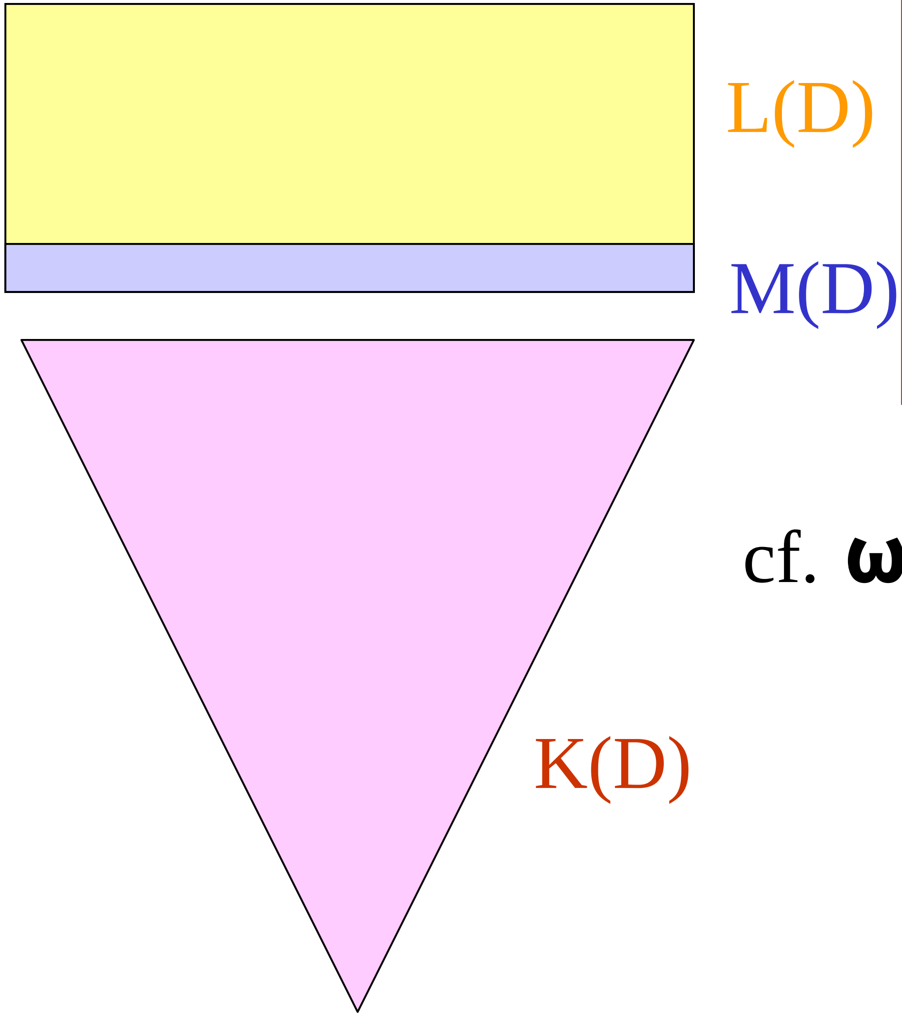
uniform domain RD



uniform domain RD



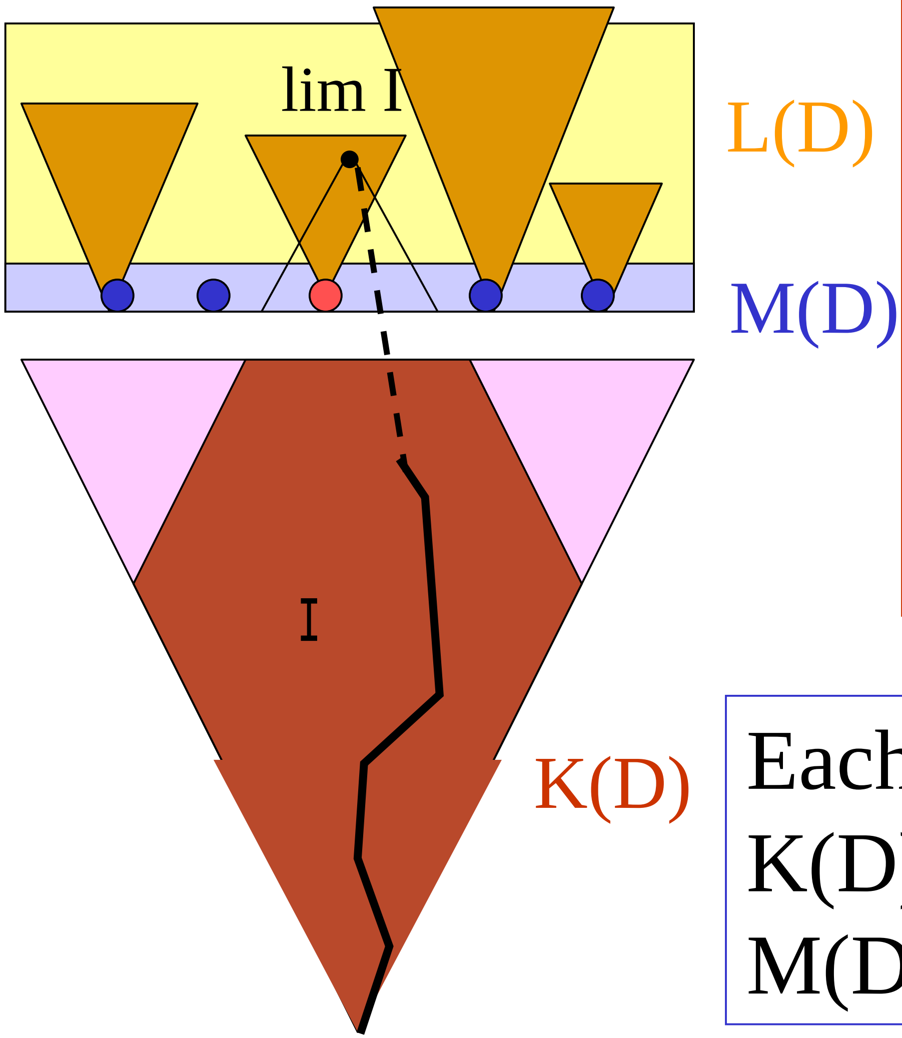
Minimal subspace of a uniform domain



Theorem When D is a uniform domain, L(D) has a minimal subspace M(D).

cf. **ω**-type domain of P(N)
--- K(D):finite set,
L(D):infinite set

Minimal subspace of a uniform domain (2)

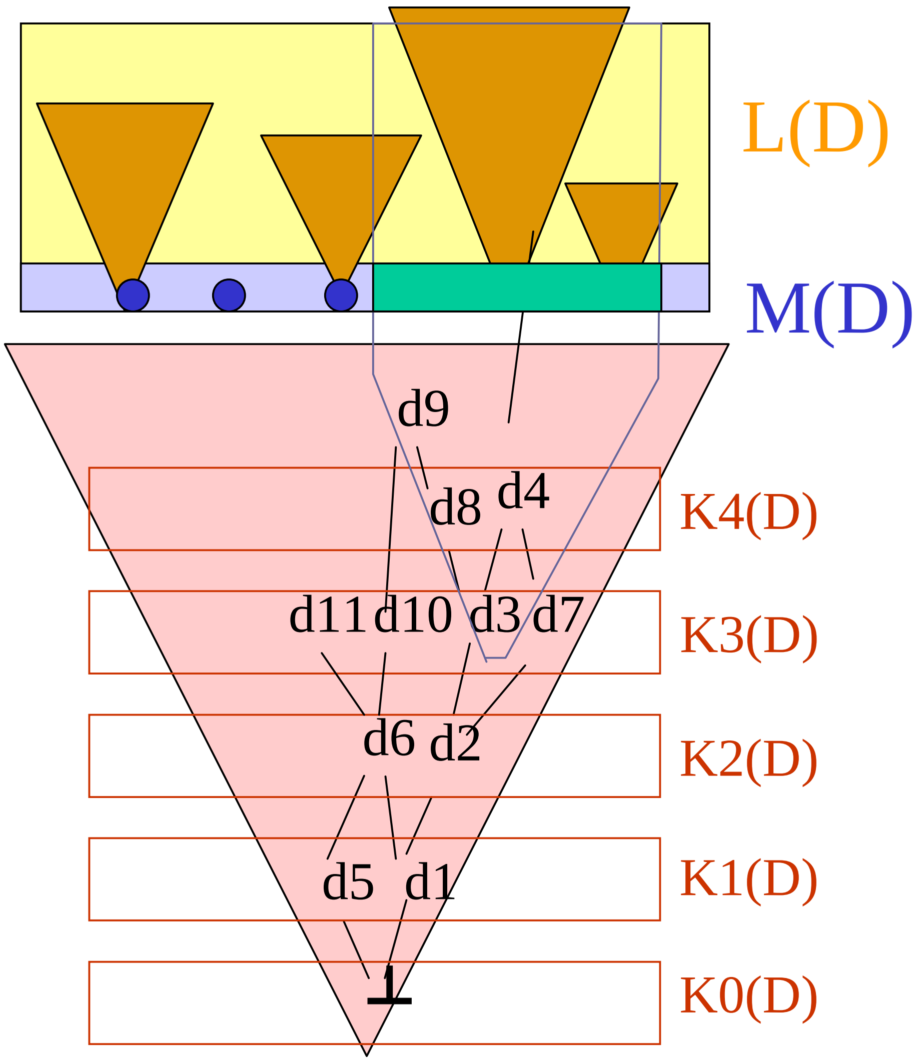


Theorem When D is a uniform domain,

- M(D) is a retract of L(D).
- M(D) is a Hausdorff space.

Each increasing sequence in K(D) specifies one point of M(D).

Uniform domain and uniform space.



- $[d] = \uparrow d \cap M(D)$
  open set of M(D)
- $Vn = \{[d] \mid d \in Kn(D)\}$
  open covering of M(D)

Theorem When D is a uniform domain, M(D) is a countable complete uniform space with Vn uniform coverings.

Cor. M(D) is metrizable.

Countable complete uniform space with a base (X, μ)

- X: topological space,
- $\mu = U0 \gg U1 \gg U2 \gg \ldots$

  A sequence of uniform open coverings s.t.

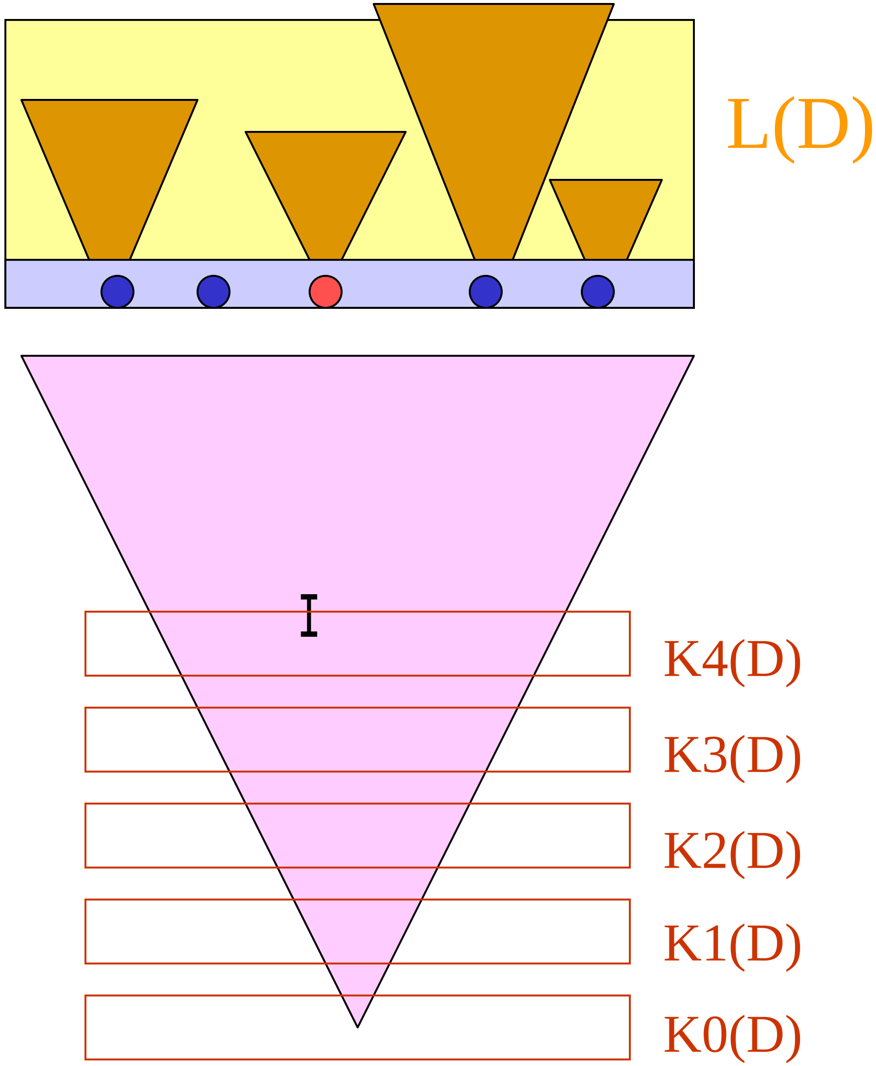
  $\gg$: refinement,

  $\forall n \exists m.\ Un \gg^* Um$
- $U \gg^* V$ if $U \gg \{ St(v, V) \mid v \in V\}$
- $St(A, V)$ is $\bigcup\{u \in V \mid u \cap A \neq \phi\}$

μ becomes a base of a uniformity on X

Theorem When (X, μ) is a countable complete uniform space with a base, we can form from μ a uniform domain D s.t. M(D) = X.

Totally bounded uniform domain



Kn(D) finite set for all n

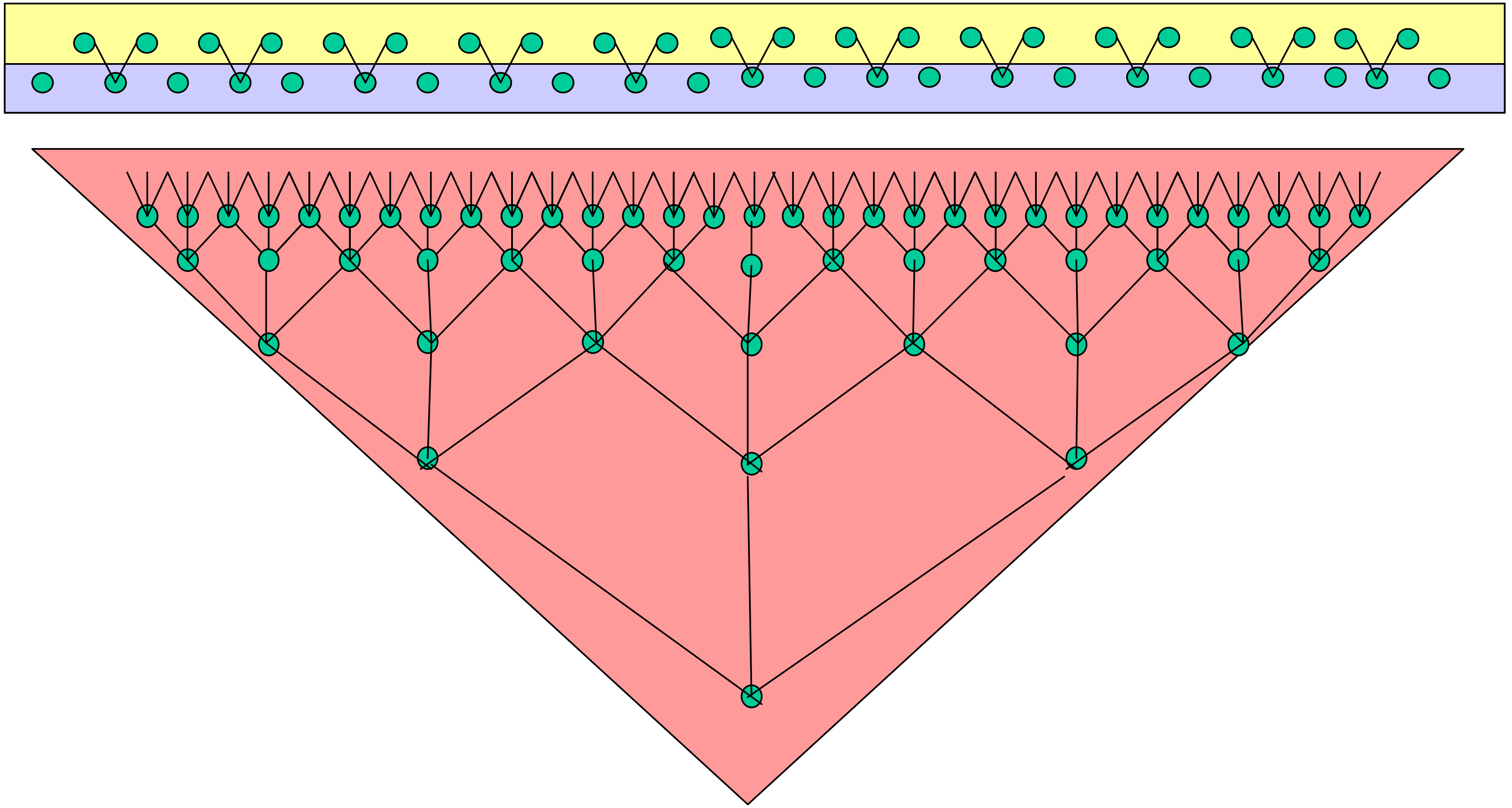
Theorem When D is a totally bounded uniform domain, M(D) is a compact uniform space.

Theorem When (X, **μ**) is a compact countable uniform space, the corresponding uniform domain is totally bounded.

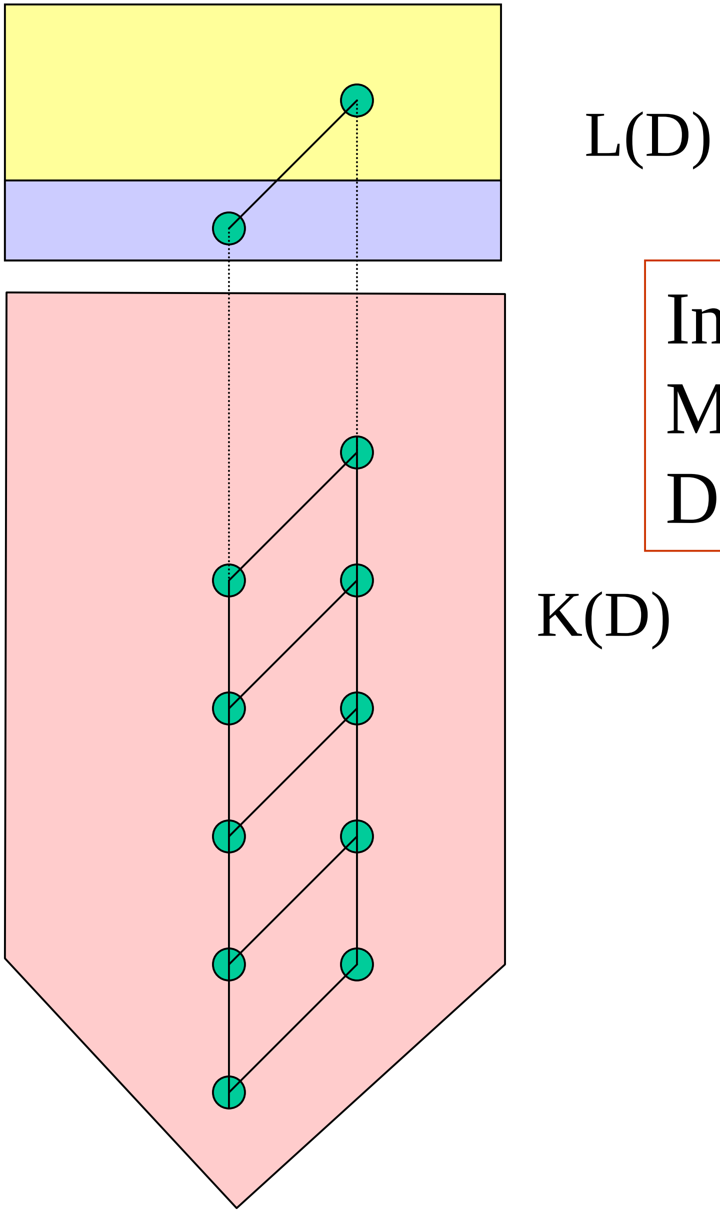
M(RD) is homeomorphic to I=[0,1]

Signed digit representation[Gianantonio]

Gray code [Tsuiki]

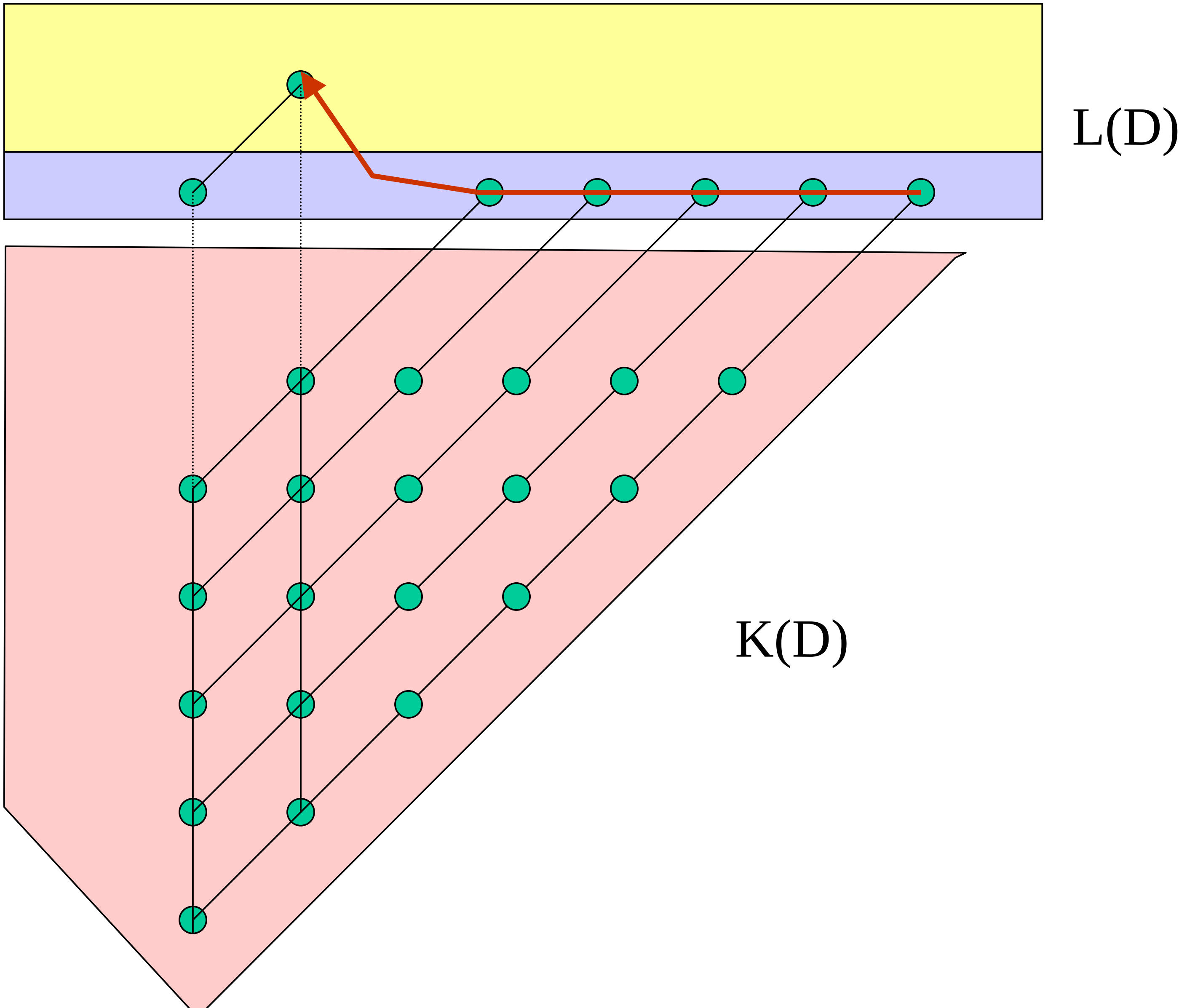


An example M(D) not dense in D

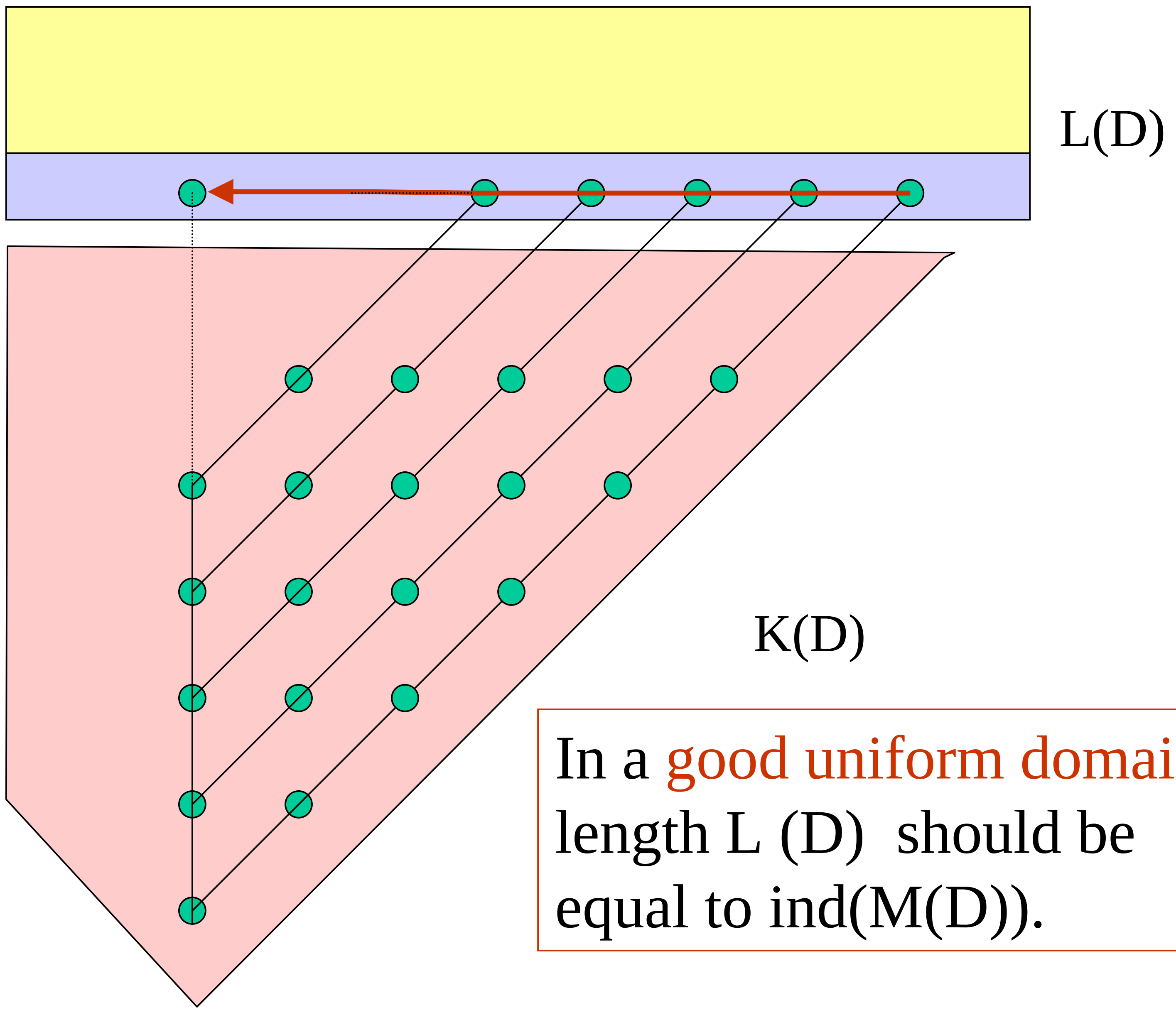


In a good uniform domain, M(D) should be dense in D

A uniform domain M(D) dense in D



Another uniform domain with the same M(D)



In a good uniform domain, length L (D) should be equal to ind(M(D)).

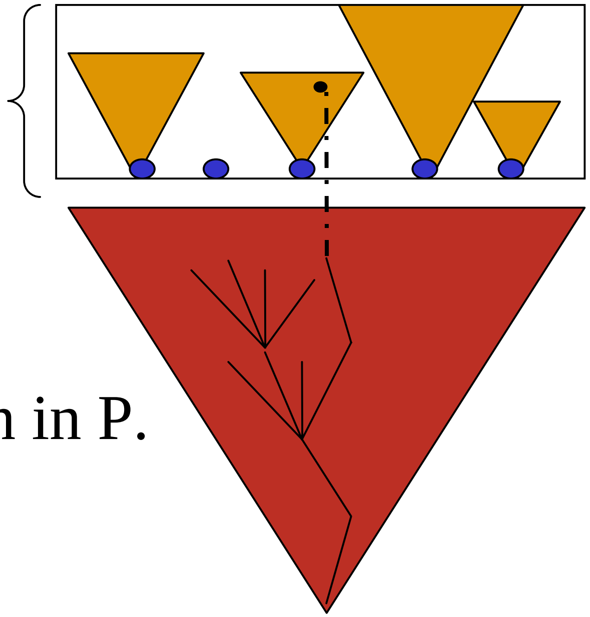
What is a good uniform domain?

- Uniform domain = uniform space + base.
- Which uniform domain is natural as a base of computation?
  - M(D) dense in D. $(\forall d.\ \exists e.\ d \in e^*)$
  - $d \leqq e$ iff $[d] \supseteq [e]$.
  - ind(M(D)) = length(L(D))

  we have such domain for each separable metric space X [tsuiki,2002]
- In general we have

  ind(L(D)) = length(L(D))

  ind(M(D)) $\leqq$ length(L(D))



length(P): the maximal length of a chain in P.

ind: Small inductive dimension.

Conclusion

•Computation ---- depends on the selection of a base

Uniform domain =
Uniform space + base

•Which property of a uniform domain is not dependent of the selection of a base?

•Are there any good conditions on a uniform domain which guarantees good computational properties?