Representations of complete uniform spaces via uniform domains

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Abstract

In this paper, we show that complete uniform spaces can be represented domain-theoretically. We introduce the notion of a uniform domain, which is an ω-algebraic domain with some uniform structure on the set $K(D)$ of finite elements of D. It is proved that when (X, μ) is a complete uniform space of countable weight, there is a uniform domain D such that X is the retract of the set $L(D)$ of limit elements of D. On the other hand, in every uniform domain D, there exists a minimal subspace $M(D)$ of $L(D)$ on which $K(D)$ induces a uniformity structure. Thus, a uniform domain can be considered as a set with a particular kind of base of a uniformity. Since every infinite increasing sequences in $K(D)$ identifies one element of $M(D)$, through a labelling of edges of $K(D)$, we obtain an admissible representation of a uniform space in a uniform domain. We also show that such a representation is a proper representation.

1 Introduction

In order to define computation over a separable topological space X, it is natural to select a countable base $\mathcal{B}$, ordered by set inclusion, assign a finite representation to each element of $\mathcal{B}$ so that it reflects the order structure of $\mathcal{B}$, and define a computation as a program which inputs/outputs infinite increasing sequences in $\mathcal{B}$ through the representation. In this setting, since each point of X is represented as an infinite increasing sequence of $\mathcal{B}$, we can consider it as a domain structure D such that the set of finite elements is the poset $\mathcal{B}$, and the set of limit elements includes X. Then, we can define computation directly through an effective structure of the domain D [Smy77], or consider its expansion as infinite sequences and define computation via

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Type-2 machines [Wei00]. For this purpose, it is important to select a suitable base so that it induces a domain structure with some good properties.

With this motivation, we study the domain-theoretic properties of a uniform space of countable weight. The theory of uniform spaces was developed by Weil, and for the case of a countably based case, a uniform space is just a metrizable space. Thus, the topological structure is uniformly defined just as a metric space, but, it is defined not through a metric function but through a filter of open coverings (or relations). There are two different formalizations of uniform spaces. One is based on entourages of the diagonal [Bou65],[Jam90], [Eng89], and the other is based on uniform coverings [Tuk40],[Isb64]. This paper uses the latter one, because it fits very well with our domain theoretical development.

In this paper, we introduce the notion of a uniform domain, which is an ω-algebraic domain with some uniformity condition. In a uniform domain D, the set $L(D)$ of limit elements has the set $M(D)$ of minimal elements, which becomes a Hausdorff space with the subspace topology of the Scott topology on D and, in addition, $M(D)$ is a retract of $L(D)$. When D is a uniform domain, the space $M(D)$ comes to have a complete uniformity structure through a uniformity base endowed by $K(D)$. On the other hand, when a complete uniform space (X, μ) with a uniformity base composed of a sequence of open coverings is given, we can form a uniform domain D such that $M(D)$ is homeomorphic to X.

In addition, when X is compact, we can take D as a finite-branching domain, for which we have natural representation as infinite sequences of Γ, when a labelling of each edge of the poset $K(D)$ in Γ is given.

In a finite-branching domain, every infinite increasing sequence in $K(D)$ can be expressed as a sequence of edges. Therefore, when a uniform domain D with $M(D)$ homeomorphic to a uniform space X is given, through a labelling of each edge of the poset $K(D)$, a representation of the space X can be introduced. Since $M(D)$ is a retract of $L(D)$, we can consider that every infinite path in $K(D)$ identifies an element of $M(D)$ through the retract function. We will show that this representation is a proper representation.

2 Uniform spaces

Let us start with some notation and terminology on coverings. Let $\{U_\alpha\}$ and $\{V_\beta\}$ be coverings of a set X. $\{U_\alpha\}$ is called a *refinement* of $\{V_\beta\}$ if each U_α is a subset of at least one V_β, and we write $\{V_\beta\} \succ \{U_\alpha\}$. $\succ$ is a preorder on the set of coverings of X. When $\{U_\alpha\}$ and $\{V_\beta\}$ are coverings of X, they have a coarsest common refinement defined as $\{U_\alpha \cap V_\beta\}$, which is denoted by $\{U_\alpha\} \cap \{V_\beta\}$.

When $\mathcal{U}$ is a covering and A is a subset of X, define $St(A, \mathcal{U})$ as $\cup\{V \in \mathcal{U} \mid V \cap A \neq \emptyset\}$. Then, the collection $\{St(U, \mathcal{U}) \mid U \in \mathcal{U}\}$ is also a covering, which is called the star of $\mathcal{U}$ and denoted by $\mathcal{U}^*$. Obviously, $\mathcal{U}^* \succ \mathcal{U}$. When

$\mathcal{V} \succ \mathcal{U}^*$, we call that $\mathcal{U}$ is a *star-refinement* of $\mathcal{V}$ and we write $\mathcal{V} \succ^* \mathcal{U}$.

Definition 2.1 A family μ of coverings of X is a *uniformity* if μ satisfies the followings:
(1) when $\mathcal{U}$ and $\mathcal{V}$ are in μ, $\mathcal{U} \cap \mathcal{V}$ is in μ,
(2) when $\mathcal{V} \succ \mathcal{U}$ and $\mathcal{U} \in \mu$, $\mathcal{V}$ is in μ,
(3) every element of μ has a star-refinement in μ, and
(4) for each x and $y \in X$, there is a covering $\mathcal{U} \in \mu$ no element of which contains both x and y.

A uniform space is a pair (X, μ) where μ is a uniformity on X.

Definition 2.2 A subset ν of a uniformity μ is a *base* of μ if for all $\mathcal{U} \in \mu$, there is a $\mathcal{V} \in \nu$ such that $\mathcal{U} \succ^* \mathcal{V}$.

In order that a family ν of coverings is a base of an uniformity, ν must satisfy (1) when $\mathcal{U}$ and $\mathcal{V}$ are in ν, there is $\mathcal{W} \in \nu$ such that $\mathcal{U} \cap \mathcal{V} \succ \mathcal{W}$,
(2) when $\mathcal{U}$ is in ν, there is $\mathcal{W}$ in ν such that $\mathcal{U} \succ^* \mathcal{W}$,
(3) for each x and $y \in X$, there is a covering $\mathcal{U} \in \nu$ no element of which contains both x and y.

Definition 2.3 Let $\mathcal{U}(x) = \cup\{U \in \mathcal{U} \mid x \in U\}$ for a covering $\mathcal{U}$. The topology on X induced by a uniformity μ is the one in which $\{\mathcal{U}(x) | \ \mathcal{U} \in \mu\}$ becomes the neighbourhood filter of x.

We say that a topology on X is compatible with a uniformity μ when it coincides with the one induced by μ. When μ is a uniformity, one can choose a base which is composed of open coverings. The smallest cardinality of a base of μ is called the weight of the uniformity μ.

When (X, ρ) is a metric space, there is an induced uniformity on X with the base $\mathcal{B} = \{V_n \mid n = 1, 2, \ldots\}$ where V_n is the covering with all the $1/2^n$-balls. It is known that a uniformity μ on a set X is induced by a metric iff the weight of μ is countable.

Definition 2.4 A uniformity μ is *totally bounded* if every $\mathcal{U} \in \mu$ has a finite uniform refinement. A filter F on a uniform space (X, μ) is a *Cauchy filter* if F contains at least one element from each $\mathcal{U} \in \mu$. A uniform space μ is *complete* if every Cauchy filter on (X, μ) is convergent.

It is known that the set of Cauchy filters of a uniform space has minimal elements, and the set of minimal Cauchy filters form a completion of (X, μ).

When the topology on X induced by the uniformity μ is compact, we say that (X, μ) is a compact uniform space. It is known that a uniform space (X, μ) is compact iff it is both totally bounded and complete.

When μ is a uniformity on X and the weight of μ is countable, we can select, as a base, an infinite sequence $\mathcal{U}_0 = \{X\} \succ \mathcal{U}_1 \succ \mathcal{U}_2 \succ \ldots$ of open coverings. This fact, with the fact that the set of minimal Cauchy filters of a complete uniform space (X, μ) is homeomorphic to X, leads to the domain theoretic account of uniformity in the next section.

3 A Uniform Domain

When D is an ω-algebraic cpo, we write $K(D)$ for the set of finite elements of D, and $L(D)$ for the set of limit elements (i.e., non-finite elements) of D. We write $K(x)$ for the set $\{d \in K(D) \mid d < x\}$.

Let $(P, \leq)$ be a partial order. We write $a < b$ iff $a \leq b$ and $a \neq b$. The *level* of an element d of P is defined as the maximal length of a chain $\bot_P = a_0 < a_1 < \ldots < a_n = d$.

Definition 3.1 When each element of a poset P has a finite level and no maximal element exists in P, we say that P is a *ω-type* poset. A *ω-type domain* is an ω-algebraic cpo in which $K(D)$ is a ω-type poset.

Note that each element of a *ω-type* poset is finite. Therefore, when an *ω-type* poset P of countable cardinality is given, by taking the ideal completion, we have a ω-type domain D with $K(D) = P$. We write $K_n(P)$ for the set of level-n finite elements of P. We simply write $K_n(D)$ for $K_n(K(P))$, and, when $x \in D$, call $K_n(x) = K_n(D) \cap K(x)$ the set of *level-n approximations* of x.

Definition 3.2 Let D be a ω-type domain and $d \in K_m(D)$. We denote by $d^* \subset K_m(D)$ the set of elements of $K_m(D)$ which are compatible with d. If, for each $d \in K_m(D)$, there exists a lower bound of d^* in $K_n(D)$, we define that $n <^* m$.

Here, a is compatible with b means that a and b have an upper bound in D (which also implies that a and b have an upper bound in $K(D)$).

Lemma 3.3 *Let D be a ω-type domain.*
1) When $m <^* n$, we have $m \leq n$.
2) When $m \leq n$, $n <^* o$, and $o \leq p$, we have $m <^* p$.

Definition 3.4 When D be a ω-type domain and for each $n \in \mathbb{N}$, there is a $m \in \mathbb{N}$ such that $n <^* m$, we say that D is a *uniform domain*.

As we will show, a uniform domain corresponds to a complete uniform space of countable weight. When D is a uniform domain and $n \in \mathbb{N}$, there is a maximum number m such that $m <^* n$. We will denote this number by $mub(n)$. $mub(n)$ diverges to ∞ as n increases to ∞.

Example 3.5 Let $\Sigma = \{0, 1\}$ and $\Sigma^*_{\bot,1}$ and $\Sigma^\omega_{\bot,1}$ be the sets of finite and infinite sequences of Σ in which at most one undefined cell ($\bot$) is allowed to exist, respectively. By considering $\Sigma^*_{\bot,1}$ as a subset of $\{0, 1, \bot\}^\omega$ with an assignment of $\bot$ to those cells after the length of a sequence, we can define a partial order on $BD_1 = \Sigma^*_{\bot,1} \cup \Sigma^\omega_{\bot,1}$. Here, the order on $\{0, 1, \bot\}^\omega$ is the product order of the ordering on $\{0, 1, \bot\}$ defined as $\bot < 0$ and $\bot < 1$. BD_1 is a ω-type domain with $K(D) = \Sigma^*_{\bot,1}$ and $L(D) = \Sigma^\omega_{\bot,1}$, where the level of $d \in \Sigma^*_{\bot,1}$ is the number of non-$\bot$ elements of d. BD_1 is not a uniform domain;

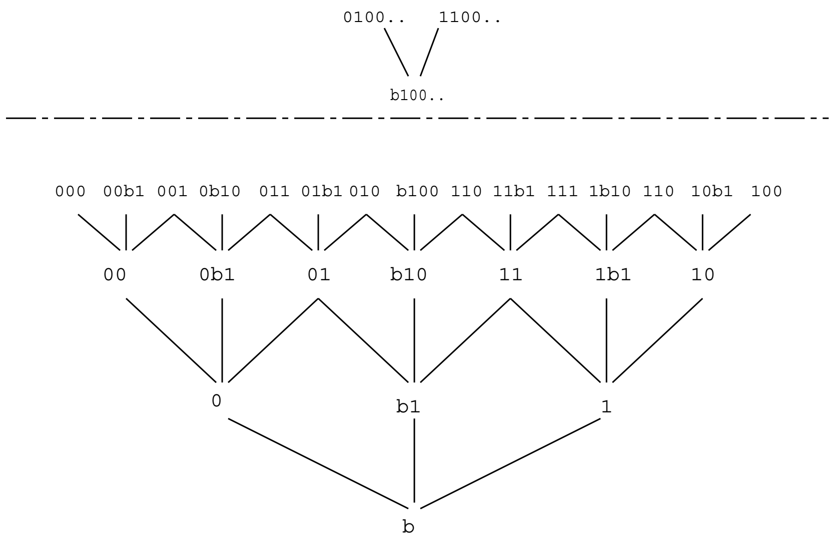

Fig. 1. The uniform domain RD.

we have $\perp 1^n < 1^{n+1}$ and $1\perp 1^{n-1} < 1^{n+1}$, and therefore, $1\perp 1^{n-1} \in (\perp 1^n)^*$ in $K_n(BD_1)$. However, a lower bound of $1\perp 1^{n-1}$ and $\perp 1^n$ is the bottom of BD_1.

Example 3.6 Next, consider a subdomain RD of BD_1 where the sequences after $\perp$ is restricted to 10000.... More precisely, $L(RD) = \Sigma^\omega \cup \Sigma^* \perp 10^\omega$ and $K(RD) = \Sigma^* \cup \Sigma^* \perp 10^*$. RD is a uniform domain; when $n > 1$, $(\perp 10^{n-1})^*$ is the three elements set $\{110^{n-2}, 010^{n-2}, \perp 10^{n-1}\}$ which has a lower bound $\perp 10^{n-2} \in K_{n-1}(D)$.

Definition 3.7 A subset S of a poset P is called the *minimal subspace* iff (1) $y \leq x$ implies $y = x$ for all $x \in S$ and $y \in P$, and (2) for each $y \in P$ there is an element $x \in S$ such that $x \leq y$. A subset S of a domain D is called the *minimal subspace* of D (denoted by $M(D)$) when it is the minimal subspace of $L(D)$.

Many of the domains studied in computer science, for example, $P_\omega = \{a \mid a \subseteq N\}$ and Plotkin's T^ω do not have such minimal subspaces. However, uniformity ensures the existence of the minimal subspace. The proof is analogous to that of the existence of a minimal Cauchy filter in a uniform space.

Theorem 3.8 *Let D be a uniform domain.*
1) D has a minimal subspace $M(D)$.
2) $M(D)$ is a retract of $L(D)$.
3) $M(D)$ is a Hausdorff space.

Proof. 1) Let $x \in L(D)$. For each element $d \in K(x)$, we take the set S_d of lower bounds of d^*, and take their union $S_x = \cup_{d \in K(x)} S_d$. We show that S_x is a directed set. Let p, q be lower bounds of d^* and e^*, respectively, for $d \in K_n(x)$ and $e \in K_m(x)$. Since $K(x)$ is directed, we have an upper bound

$f \in K_l(x)$ of d and e for some $l \geq max(n, m)$. Let $r \in f^*$. Since r and f have an upper bound, r and d also have an upper bound, which implies that each level n approximation of r belongs to d^* and thus lower bounded by p. Therefore, we have $p \leq r$. In the same way, we also have $q \leq r$. Therefore, each element of f^* is an upper bound of p and q. Thus, we only need to show that $f^* \cap S_x$ is not empty. Let k be a level such that $l <^* k$ and let g be an element of $K_k(x)$ such that $f < g$. Then, g^* has a lower bound in $K_l(D)$. Actually, it must be less than g and thus belongs to f^*.

This directed set S_x is infinite because $mub(n)$ diverges to ∞ as n increases to ∞. Therefore, the least upper bound y of S_x belongs to $L(D)$. We have $y \leq x$ because each element of S_x is upper bounded by an element of $K(x)$.

To show the minimality, we prove that we have $z \geq y$ when $x \geq z \in L(D)$. Let $e \in K_n(z)$ be a level n approximation of z. Take some $d \in K_n(x)$. Then, since $e \in K(x)$, d and e have an upper bound. It means that $e \in d^*$ and therefore, there is an element $f \in S_x \cap K_{mub(n)}(D) = K_{mub(n)}(y)$ such that $f \leq e$. Since $mub(n)$ diverges to ∞ as n increases, we have $z \geq y$.

2) The last paragraph of the above proof shows that y is the unique minimal element smaller than x.

3) Let r be the retract map from $L(D)$ to $M(D)$. Suppose that $x \in M(D)$ and $y \in M(D)$ are not separated by open sets in $M(D)$. Since $r(x) = x$, we have $K(x) = S_x$. Therefore, for each $c \in K(x)$, there exists a $d \in K(x)$ such that c is a lower bound of d^*. Let $e \in K(y)$ have the same level as d. Then, since $\uparrow d$ and $\uparrow e$ intersect in $M(D)$, we have $e \in d^*$. Therefore, we have $c \leq e$. In the same way, for each $e \in K(y)$, there is a $c \in K(y)$ such that $c \geq e$. Thus, we have $x = y$. □

Example 3.9 $M(RD)$ is homeomorphic to the unit interval [0,1]. See [Tsu01] for the detail.

Definition 3.10 A poset P is *totally bounded* if P is ω-type and $K_n(P)$ is a finite set for all n. A domain D is *totally bounded* if $K(D)$ is totally bounded.

We say that a topological space is quasi-compact iff each open cover has a finite subcover, and compact if it is quasi-compact and Hausdorff.

Proposition 3.11 *When D is a totally bounded domain,*
1) $L(D)$ is quasi-compact.
2) $M(D)$ is quasi-compact.

Proof. 1) Suppose that $\{\uparrow d \mid d \in S\}$ forms an open cover of $L(D)$ for $S \subset K(D)$. Let T be the subset of S which consists of minimal elements of S. Then, $\{\uparrow d \mid d \in T\}$ is also an open covering of $L(D)$. Suppose that T is an infinite set. Let $J = \{j \in K(D) \mid \ \uparrow j \cap T \text{ is infinite}\}$. J is a downward-closed set. Let $J_n = J \cap K_n$. We consider a subset of the order relation on J which is composed from the order relations between adjoining levels. That is, when $d \in J_n$ and $e \in J_m$ with $n \leq m$, we define $d << e$ if $d = d_n < d_{n+1} < \ldots < d_m = e$ such that $d_i \in J_i$ for $i = n, \ldots, m$. For each $d \in J_{n+1}$, there is at least one element

$e \in J_n$ such that $e << d$. Therefore, for each $d \in J$, there is at least one element $e \in J_1$ such that $e << d$. Since each J_i is not empty, J is an infinite set. Since J is infinite and J_1 is finite, at least one element j_1 of J_1 satisfies $j_1 << d$ for infinitely many $d \in J$. By repeating this for the upper subspace of j_1 with respect to $<<$, we can form an infinite sequence $j_1 < j_2 < \ldots$ so that $j_i \in J_i$. When x is the least upper bound of this sequence, x is covered by some $t \in T$ and thus $t < j_i$ for some i. Since infinitely many elements of T are greater than j_i, this contradicts with the fact that T is an anti-chain.

2) It is immediate because every open covering of $M(D)$ is an open covering of $L(D)$. □

Corollary 3.12 *When D is a totally bounded uniform domain, $M(D)$ is compact.*

Proof. By Proposition 3.11 and Theorem 3.8 (3). □

Theorem 3.13 *A totally bounded domain D has a minimal subspace.*

First consider the following lemma.

Lemma 3.14 *Each co-directed subset S of $L(D)$ has a lower bound in $L(D)$.*

Proof. (Lemma 3.14) For each n, since $K_n(x)$ is a non-empty finite set for each $x \in S$ and $\{K_n(x) \mid x \in S\}$ is co-directed with respect to set inclusion, $J_n = \cap\{K_n(x) \mid x \in S\}$ is a finite non-empty set. As in Proposition 3.11, we consider restricted order $<<$ on $\cup_n J_n$ and form an infinite increasing sequence $\perp < p_1 < p_2 < \ldots$ such that $p_i \in J_i$. The limit $p \in L(D)$ of this sequence is a lower bound of S. □

Proof. (Theorem 3.13) Since the union of a family of co-directed subsets of $L(D)$ containing x is also co-directed and contains x, we can apply Zorn's lemma to form a maximal co-directed set containing x. A lower bound y of such a set is a minimal element of $L(D)$ such that $y \leq x$. □

Definition 3.15 A ω-type poset P is *finite-branching* if P is of type ω and each element has finite number of adjacent elements. Here, c is *adjacent* from d means that $d < c$ and there is no element b such that $d < b < c$. A domain D is *finite-branching* if $K(D)$ is finite-branching.

Proposition 3.16 *A finite-branching poset is totally bounded.*

Example 3.17 Figure 2 shows an example of a non-finite-branching totally bounded uniform domain. In addition to the points and arcs in this figure, for each d_i $(i = 0, 1, 2, \ldots)$, we consider that an infinite increasing chain starting from d_i exists in $K(D)$. Therefore, the set of limit elements $L(D)$ is $\{y, x_0, x_1, x_2, \ldots\}$, with the topology $\{x_i\}$ and $\{y, x_i, x_{i+1}, \ldots\}$ for $i = 0, 1, 2 \ldots$ open . At the same time, if we remove the node e, we have a finite-branching uniform domain with the same set of limit elements. Thus for each space X,

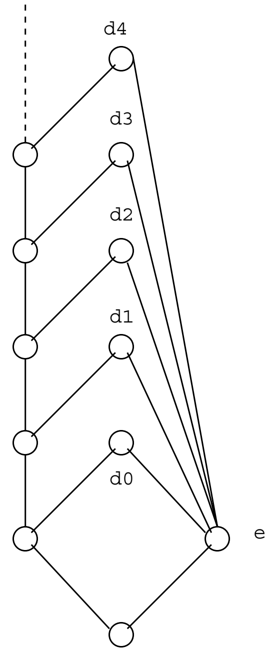

Fig. 2. A non-finite-branching totally bounded domain.

there may be several uniform domains D such that $X = L(D)$ with different properties.

When we consider the finite-branching case, the proofs of Proposition 3.11 and Theorem 3.13 become much simpler. In addition, in this case, our motivation to use uniform domains as an intermediate structure to represent topological spaces as infinite sequences works very well, as we show in Section 5.

Definition 3.18 A *path* to d is a chain $\bot = a_0 < a_1 < \ldots < a_n = d$ with a_i an adjacent element of a_{i-1} for $i = 1, \ldots, n$. We define that a ω-type poset P is *coherent* if all the paths to d have the same length at every $d \in P$. We call a ω-type domain D coherent if $K(D)$ is coherent. We call an uniform coherent finite-branching domain a *ufb-domain* in short.

In a ufb-domain, every adjacent element of $K_n(D)$ belongs to K_{n+1}. Therefore, we can view each element of $K_n(x)$ as a level-n approximation of x, and consider each infinite path $\bot = a_0 < a_1 < \ldots < a_n \ldots$ as a convergent sequence of approximations.

4 Uniform spaces induced by uniform domains

When $d \in K(D)$, we define $\hat{d}$ as the subset $\uparrow d \cap M(D)$ of $M(D)$.

Theorem 4.1 *When D is a uniform domain, D induces a complete uniformity μ of countable weight on $M(D)$, defined through the base consisting of the coverings $\mathcal{B} = \{\mathcal{V}_0, \mathcal{V}_1, \ldots\}$ defined as $\mathcal{V}_i = \{\hat{d} \mid d \in K_i(D)\}$.*

Proof. We first show that $\mathcal{B}$ satisfies the three conditions of a base of a uniformity. (1) is satisfied because, when $j < i$ and $d \in K_i(D)$, there is a $e \in K_j(D)$ such that $e < d$, and thus $\mathcal{V}_j \succ \mathcal{V}_i$. For (2), we prove that for each n and k such that $n \leq mub(k)$, $\mathcal{V}_n \succ^* \mathcal{V}_k$ holds. First, when $d \in K_k(D)$, $St(\hat{d}, \mathcal{V}_k) = \cup\{\hat{e} | \ e \in K_k(D) \text{ and } \hat{e} \cap \hat{d} \neq \emptyset\} = \cup\{\hat{e} \mid e \in d^*\}$. Since there exists a lower bound of d^* in $K_{mub(k)}(D)$, a lower bound b of d^* also exists in $K_n(D)$. Therefore, we have $\cup\{\hat{e} \mid e \in d^*\} \subset \hat{b}$, where $\hat{b} \in \mathcal{V}_{mub(k)}$. Thus, we have $\mathcal{V}_n \succ^* \mathcal{V}_k$. For (3), since $M(D)$ is Hausdorff, let $d, e \in K_n$be finite elements such that $\uparrow d$ and $\uparrow e$ separate x and y. Since x and y are in $M(D)$, as in the proof of Theorem 3.8 (3), there are $b \in K_m(x)$ and $c \in K_m(y)$ such that $d < b^*$ and $e < c^*$. Thus, the level-m approximations of x and y do not intersect.

We need to show that the two topological structures on $M(D)$, one is the subspace topology of the Scott topology of D and the other is the topology induced by the uniformity coincide, which is immediate.

Let F be a Cauchy filter of μ. Then, since F must include one element of each element of $\mathcal{B}$, F induces an infinite ideal of D, which converges to an element of $L(D)$. Since $M(D)$ is a retract of $L(D)$, F identifies one element of $M(D)$ as its limit. Thus, μ is complete. Obviously, μ has a countable weight. □

Since the weight of the uniformity constructed in Theorem 4.1 is countable, we have the following.

Corollary 4.2 *When D is a uniform domain, $M(D)$ is metrizable.*

Proposition 4.3 *When D is a totally bounded uniform domain, the uniformity on $M(D)$ constructed in Theorem 4.1 is totally bounded, and thus it is compact.*

Proof. Since every member of the base $\mathcal{B}$ is a finite covering. □

Theorem 4.4 *Let (X, μ) be a complete uniform space with a countable weight, and $\mathcal{U}_0 \succ \mathcal{U}_1 \succ \dots$ be a sequence of open coverings which forms a base of μ. From this sequence, we can form a coherent uniform domain D such that X is homeomorphic to $M(D)$.*

Proof. From this sequence, we define a new sequence $\mathcal{V}_0 \succ \mathcal{V}_1 \succ \dots$ as $\mathcal{V}_0 = \mathcal{U}_0$, $\mathcal{V}_{i+1} = (\mathcal{V}_i \cap \mathcal{U}_{i+1}) \cup \mathcal{U}_{i+1}$. Note that it is also a sequence of finite open coverings. Note also that $\mathcal{V}_i$ is a union of $\mathcal{U}_i$ and a refinement of $\mathcal{U}_i$. In this case, the uniformity induced by the new sequence is the same as μ, and we have $\mathcal{U}_i \succ^* \mathcal{U}_j$ iff $\mathcal{V}_i \succ^* \mathcal{V}_j$.

By considering each open set as a point and define an order relation $<$ between elements of $\mathcal{V}_i$ and $\mathcal{V}_{i+1}$ as set inclusion, we can form a partial order on $\cup_i \mathcal{V}_i$. Taking the ideal completion, we can form our domain D. It is immediate to see that D is a coherent uniform domain; $St(U, \mathcal{V})$ corresponds to d^* where d is an element in $K(D)$ corresponding to U, and we have $\mathcal{V}_i \succ^* \mathcal{V}_j$, iff $i <^* j$. □

Proposition 4.5 *When (X, μ) is a compact uniform domain and $\mathcal{U}_0 \succ \mathcal{U}_1 \succ \dots$ is a sequence of finite open coverings which forms a base of μ, the uniform domain D constructed in Theorem 4.4 is a ufb-domain.*

It is known that for every compact space X, there exists exactly one uniformity on X that induces the topology of X. Therefore, if we apply the construction of a uniformity in Theorem 4.1 to a uniform domain constructed in Theorem 4.4, we obtain the same uniformity. Thus, the diversity of uniform domains roughly corresponds to the choice of the variety of bases. One of the directions of further study is about properties of uniform domains which do not depend on the choice of the particular base.

5 Representations via uniform domains

Next, we study how can we obtain a representation of a space X from a uniform domain which includes X as its minimal subspace.

In this section, we do not need to assume the uniformity of D. Therefore, we assume that D is a finite-branching domain, whose minimal subspace $M(D)$ is a retract of $L(D)$. Note that a domain corresponding to a compact uniform space through Proposition 4.5 satisfies this.

In this case, each infinite path ϕ in $K(D)$ expresses an element of $L(D)$, and when it has a minimal subspace $M(D)$ which is a retract of $L(D)$, we can consider that ϕ is representing an element of $M(D)$. Therefore, in order to express such a path as an infinite sequence of characters, we introduce a labelling of edges of $K(D)$.

Definition 5.1 Let P be a finite-branching poset. A *labelling* of edges of P by an alphabet Γ is a set of injective functions $\rho = \{\rho_d : adj(d) \to \Gamma \mid d \in P\}$. When D is a finite-branching domain, a *labelling* of edges of D is a labelling of $K(D)$. We call the triple (D, Γ, ρ) a labelled fb-domain.

Though the set Γ may be infinite, the subset used for labelling $K_n(D)$ is finite for each n. Let (D, Γ, ρ) be a labelled fb-domain and $\bot = a_0 < a_1 < \dots < a_l = d$ be a path in $K(D)$ with c_i $(i = 1, \dots, l)$ the label of the edge (a_{i-1}, a_i). Then, this chain can be represented as the sequence $c_1 c_2 \dots c_l$ in Γ^*, and therefore, we can identify a path in $K(D)$ with a sequence in Γ^*. We consider the set of all the paths of all the elements of $K(D)$, which can be identified as a subset of Γ^*, and forms a subtree of Γ^* with the prefix order. By taking the ideal completion of this set, we define a cpo $Path(D)$. We denote by pr_D the function from $Path(D)$ to D defined for compact elements as taking the endpoint of a path and extended to limit elements continuously, and by $\mathrm{PATH}(D)$ the set $L(Path(D))$ of infinite sequences of $Path(D)$.

Definition 5.2 1) A representation of a set X with the alphabet Σ is a surjective partial function from Σ^ω to X.

2) A representation ρ is called fiber compact iff $\rho^{-1}(x)$ is a compact subset of Σ^ω for each $x \in x$.
3) A representation ρ is called proper iff $\rho^{-1}(K)$ is a compact subset of Σ^ω for each quasi-compact set $K \subset X$.

Fiber compact and proper representation have some good properties [Sch95].

When (D, Γ, ρ) is a labelled fb-domain, we have a surjective partial function $pr_D :\subseteq \Gamma^\omega \to L(D)$ which is defined for PATH$(D) \subset \Gamma^\omega$. Therefore, pr_D is a representation of $L(D)$. Since $M(D)$ is a subset of $L(D)$, pr_D restricted to $pr_D^{-1}(M(D))$ is a representation of $M(D)$, which we will denote by ρ_F.

In addition, when $M(D)$ is a retract of $L(D)$ with the retract function $r : L(D) \to M(D)$, we can consider all the paths as denoting an element of $M(D)$. Thus we have another representation $r \circ pr_D :\subseteq \Gamma^\omega \to M(D)$. This representation, which we will denote by ρ_P, has the domain PATH(D).

Proposition 5.3 *The two representations ρ_F and ρ_P are admissible.*

See [Wei00] for the definition and properties of admissible representations. In order to investigate properties of these two representations, we examine what kind of approximate information each finite element $d \in K(D)$ introduces on $M(D)$. When ρ is a representation and $a \in K(Path(D))$, we write $\rho(a)$ for the set $\{\rho(p) \mid a < p, p \in \text{PATH}(D)\}$. From the definition, the value of $\rho(a)$ does not depend on the path but only on $pr_D(a) \in K(D)$ for $\rho = \rho_F$ and $\rho = \rho_P$. Therefore, we also define $\rho(d)$ for a finite element $d \in K(D)$.

Proposition 5.4 *1) $\rho_F(d) = \uparrow d \cap M(D)$,*
2) $\rho_P(d) = cl(\uparrow d \cap M(D))$.

Here, $cl(s)$ is the closure of s. When D is a labelled fb-domain, we call $pr_D^{-1}(x)$ the fiber of x.

Theorem 5.5 *When D is a labelled fb-domain and $x \in L(D)$, the followings are equivalent:*
1) The fiber of x is compact in PATH(D)*.*
2) x is a minimal element of $L(D)$.

Proof. 1) $\Rightarrow$ 2) Since the fiber $pr_D^{-1}(x)$ is a compact subset, it is a closed subset of Γ^ω. Therefore, consider its closure $P(x)$ in the domain Γ^∞. $P(x)$ is a subdomain of Γ^∞ which has, as finite elements, the set of finite prefixes of elements of $pr_D^{-1}(x)$. Since $pr_D^{-1}(x)$ is closed in Γ^ω, we have $P(x) \cap \Gamma^\omega = pr_D^{-1}(x)$.

Suppose that $y \leq x$ and $y \in L(D)$. Let ϕ be a path $\perp = d_0 < d_1 < \ldots < d_n < \ldots$ to y and ϕ_i $(i = 0, 1, \ldots)$ be the finite subpath $d_0 < d_1 < \ldots < d_i$. Then, since $d_i < x$, each ϕ_i can be extended to a path to x and therefore $\phi_i \in P(x)$ for $i = 0, 1, \ldots$. Therefore, the limit of ϕ also belongs to $P(x)$, which means that $y = x$.

2) $\Rightarrow$ 1) First, note that $L(Path(D))$ is a totally bounded metric space as a subspace of Γ^ω. Therefore, we only need to show that the fiber of x

is a closed subset of $Path(D)$, or equivalently, $P(x) \cap \mathrm{PATH}(D) = pr_D^{-1}(x)$. Suppose that $\phi_1 < \phi_2 < \ldots$ is a strictly increasing chain in $K(P(x))$ and ϕ be its limit. Then, $pr_D(\phi) \leq x$ because $pr_D(e_j) \leq x$. Since x is a minimal element of $L(D)$, it means that $pr_D(\phi) = x$. □

Corollary 5.6 *ρ_F is a fiber compact representation.*

Next, we consider ρ_P.

Lemma 5.7 *When D is a labelled fb-domain in which $M(D)$ is a retract of $L(D)$ and Y is a compact subset of $M(D)$, $\uparrow Y$ is closed in $L(D)$.*

Proof. Let $P(Y)$ be the subdomain of D such that the set of finite elements is $K(P(Y)) = \{d \in K(D) \mid d < x, x \in \uparrow Y\}$. We need to show that $L(P(Y)) = \uparrow Y$. Suppose that $d_1 < d_2 < \ldots$ is a strictly increasing sequence in $P(Y)$ whose limit is s. We show that $s \in \uparrow Y$. If infinitely many members d_i satisfy $d_i < y$ for some $y \in Y$, the limit is also in $\uparrow Y$ because Y is closed. Therefore, we assume that there are $z_i \in L(D)$ (i = 1,2,...) such that $d_i < z_i$, $z_i \in \uparrow Y, z_i \notin Y$. If infinitely many number of z_i are the same, we have $r(s) = r(z_i)$ and since $M(D)$ is a retract of $L(D)$, it means $s \in \uparrow Y$. Therefore, we assume that infinitely many z_i are different. Let $y_i = r(z_i)$. For the same reason, infinitely many y_i are different. Therefore, we have an infinite sequence (y_i) in the compact set Y. Thus, we can form a convergent subsequence of (y_i). Therefore, through reindexing, we assume that (y_i) converges to $y \in Y$. We show that $y \leq s$. Suppose that $e < y$. Then, there is a number k such that $e < y_i$ for all $i > k$. Since $y_i < z_i$, we have $e < z_i$. This means e and d_i have an upper bound. Then, y and s cannot be separated by open sets. Since $M(D)$ is Hausdorff, if $r(s) \neq y$, then s and y can be separated by open sets. Therefore, we can conclude $r(s) = y$ and therefore $y < s$. □

Theorem 5.8 *When D is a ufb-domain and Y is a compact subset of $M(D)$, $pr^{-1}(\uparrow Y)$ is compact.*

Proof. From the lemma above, $\uparrow Y$ is closed and thus $pr^{-1}(\uparrow Y)$ is also closed.□

Corollary 5.9 *ρ_P is a proper representation.*

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